
Sums of Squares, Moment Matrices & Optimization over Polynomials: Introduction

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Polynomial optimization problem

(P) *Minimize a polynomial function p over a basic closed semi-algebraic set K*

$$p_{\min} := \inf_{x \in K} p(x)$$

where

$$K := \{x \in \mathbb{R}^n \mid h_1(x) \geq 0, \dots, h_m(x) \geq 0\}$$

$p, h_1, \dots, h_m \in \mathbb{R}[x]$ are multivariate polynomials

Unconstrained polynomial minimization: $K = \mathbb{R}^n$

$$p_{\min} := \inf_{x \in \mathbb{R}^n} p(x)$$

$$p_{\min} \geq 0 \iff p \geq 0 \text{ on } \mathbb{R}^n$$

Example: The partition problem.

A sequence $a_1, \dots, a_n \in \mathbb{N}$ can be **partitioned**

if $\sum_{i \in I} a_i = \sum_{i \in [n] \setminus I} a_i$ for some $I \subseteq [n]$, i.e. if

$p_{\min} = 0$, where $p(x) = (\sum_{i=1}^n a_i x_i)^2 + \sum_{i=1}^n (x_i^2 - 1)^2$

E.g., the sequence **1, 1, 2, 2, 3, 4, 5** can be partitioned.

\rightsquigarrow **NP-complete problem**

Example: Testing matrix copositivity

$M \in \mathbb{R}^{n \times n}$ is **copositive** if $x^T M x \geq 0 \quad \forall x \in \mathbb{R}_+^n$

i.e. if $p_{\min} = 0$, where $p(x) = \sum_{i,j=1}^n M_{ij} x_i^2 x_j^2$

\rightsquigarrow **co-NP-complete problem**

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ is copositive}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix} \text{ is copositive}$$

0/1 Linear programming

$$\min c^T x \text{ s.t. } Ax \leq b, x_i^2 = x_i \ (i = 1, \dots, n)$$

Example: The **stability number** $\alpha(G)$ of a graph $G = (V, E)$ can be computed via any of the programs:

$$\alpha(G) = \max \sum_{i \in V} x_i \text{ s.t. } x_i + x_j \leq 1 \ (ij \in E), x_i^2 = x_i \ (i \in V)$$

$$\frac{1}{\alpha(G)} = \min x^T (I + A_G) x \text{ s.t. } \sum_{i \in V} x_i = 1, x_i \geq 0 \ (i \in V)$$

\rightsquigarrow (P) is **NP-hard** for **linear** objective and **quadratic** constraints,
or for **quadratic** objective and **linear** constraints

Strategy

Approximate (P) by a hierarchy of
convex (semidefinite) relaxations

Shor (1987), Nesterov, Lasserre, Parrilo (2000-)

Such relaxations can be constructed using

**representations of nonnegative polynomials as sums of
squares of polynomials**

and

the dual theory of moments

Underlying paradigm

Testing whether a polynomial p is nonnegative is **hard**

but

one can test whether p is a sum of squares of polynomials
efficiently via semidefinite programming

Plan of the lectures

- **SOS/Moment relaxations for (P):** Introduction and main properties (*convergence, optimality criterion, extraction of global minimizers*), and global overview of underlying results (about *SOS representations and the moment problem*).
- **Results for the moment problem:** some classical results (Haviland, bounded sequences, ..), results of Curto-Fialkow (flat extensions of moment matrices, finite rank theorem), eigenvalue method for complex roots.
- **Representation results for positive polynomials:**
 - density of sums of squares within positive polynomials
 - positive polynomials over their gradient variety
 - positive polynomials with some structure like convexity, sparsity, or equations, etc.

Some recent books

- A. Prestel & C. Delzell (Springer, 2001)

Positive Polynomials - From Hilbert's 17th Problem to Real Algebra

- S. Basu, R. Pollack & M.-F. Roy (Springer, 2003)

Algorithms in Real Algebraic Geometry

- M. Marshall (AMS, 2008)

Positive Polynomials and Sums of Squares

- J.-B. Lasserre (Imperial College Press, 2009)

Moments, Positive Polynomials and Their Applications

Some notation

- $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$: ring of polynomials in n variables
- $\mathbb{R}[\mathbf{x}]_d$: all polynomials with degree $\leq d$

$$p \in \mathbb{R}[\mathbf{x}]_d \rightsquigarrow p(\mathbf{x}) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq d}} p_\alpha \underbrace{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}_{\mathbf{x}^\alpha} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq d}} p_\alpha \mathbf{x}^\alpha$$

$$\rightsquigarrow p(\mathbf{x}) = \vec{p}^T [\mathbf{x}]_d$$

after setting $\vec{p} = (p_\alpha)_\alpha$: vector of coefficients

and $[\mathbf{x}]_d = (x^\alpha)_\alpha$: vector of monomials

Notation (continued)

If M is a matrix indexed by monomials of degree $\leq d$, then its kernel can be seen as a subset of $\mathbb{R}[x]_d$:

$$p \in \text{Ker}M \text{ if } M\vec{p} = 0$$

For instance, for $M = \begin{matrix} & \begin{matrix} 1 & x & x^2 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \end{matrix}$

$$p = 1 - x \in \text{Ker}M$$

Notation (continued)

- p is a **sum of squares of polynomials (SOS)** if

$$p(x) = \sum_{j=1}^m [u_j(x)]^2 \quad \text{for some } u_1, \dots, u_m \in \mathbb{R}[x]$$

Then, $\deg(p)$ is **even** and $\deg(u_i) \leq \deg(p)/2$

Example: $x^2 + y^2 + 2xy + z^6 = (x + y)^2 + (z^3)^2 \in \Sigma_3$

- $p = \sum_{\alpha} p_{\alpha} x^{\alpha} \in \mathbb{R}[x]$ with degree $d \rightsquigarrow$ its **homogenization** $\tilde{p} \in \mathbb{R}[x, t]$ defined by $\tilde{p}(x, t) = t^d p(x/t) = \sum_{\alpha} p_{\alpha} x^{\alpha} t^{d-|\alpha|}$

$$p \geq 0 \text{ on } \mathbb{R}^n \iff \tilde{p} \geq 0 \text{ on } \mathbb{R}^{n+1}$$

$$p \text{ is SOS} \iff \tilde{p} \text{ is SOS}$$

Linear/Semidefinite Programming: Short Recap

Given: $c, a_1, \dots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$

Variables: $x \in \mathbb{R}^n$ (primal variable), $y \in \mathbb{R}^m$ (dual variable)

Primal LP:

$$p^* := \max c^T x \text{ s.t. } a_j^T x = b_j \ (j = 1, \dots, m), \ x \geq 0$$

Dual LP:

$$d^* := \min y^T b \text{ s.t. } \sum_{j=1}^m y_j a_j - c \geq 0$$

Strong duality:

$$p^* = d^*$$

Semidefinite Programming

Semidefinite Programming (SDP) is the analogue of LP where:

vector variable

$$x \in \mathbb{R}^n$$

$$x \succeq 0$$

matrix variable

$$\rightsquigarrow X \in \text{Sym}_n$$

$$\rightsquigarrow X \succeq 0$$

$X \in \text{Sym}_n$: X is a symmetric $n \times n$ matrix

$X \succeq 0$: X is a **positive semidefinite** matrix

$$X \succeq 0 \iff u^T X u \geq 0 \text{ for all } u \in \mathbb{R}^n$$

$$\iff X = U U^T \text{ for some } U \in \mathbb{R}^{n \times n}$$

$$\iff \text{all eigenvalues are nonnegative}$$

Semidefinite Programming (continued)

Given: $C, A_1, \dots, A_m \in \text{Sym}_n, b \in \mathbb{R}^m$

Variables: $X \in \text{Sym}_n$ (primal), $y \in \mathbb{R}^m$ (dual)

(P) Primal SDP:

$$p^* := \sup \langle C, X \rangle \text{ s.t. } \langle A_j, X \rangle = b_j \ (j = 1, \dots, m), \ X \succeq 0$$

(D) Dual SDP:

$$d^* := \inf \sum_{j=1}^m b_j y_j \text{ s.t. } \sum_{j=1}^m y_j A_j - C \succeq 0$$

• Weak duality:

$$p^* \leq d^*$$

• If (P) is *strictly feasible* ($\exists X \succ 0$ feasible) and (D) is feasible, then $p^* = d^*$ and (D) *attains its infimum*.

Complexity

- LP can be solved in polynomial time
- SDP can be solved in polynomial time (to *arbitrary precision*)
 - ↪ (theoretically) with the **ellipsoid method** [Nemirovskii, Shor, ..] [since one can test in polynomial time whether $X \succeq 0$]
 - ↪ (practically) with **interior-point methods**

Cf. Vandenberghe & Boyd [SIAM Review 1996], Handbook by Wolkowicz, Saigal & Vandenberghe [2000]

A geometric property of SDP

SDP: $\sup \langle C, X \rangle$ s.t. $\langle A_j, X \rangle = b_j$ ($j = 1, \dots, m$), $X \succeq 0$

Lemma: If X is an optimum solution with **maximum rank** then

$\text{Ker } X \subseteq \text{Ker } X'$ for any other optimum solution X' .

Proof: $X'' := \frac{1}{2}(X + X')$ is also an optimum solution with

$$\begin{aligned} \text{Ker } X'' &= \text{Ker } X \cap \text{Ker } X' \subseteq \text{Ker } X \\ \implies \text{Ker } X &= \text{Ker } X'' \subseteq \text{Ker } X' \end{aligned}$$

Note: Interior-point algorithms return (*often*) maximum rank optimum solutions

A small example of SDP

$$\begin{aligned} \max (X_{13} + X_{31})/2 \quad \text{such that} \quad & X \succeq 0, \quad X \in \mathbb{R}^{3 \times 3} \\ & X_{11} = 1, \quad X_{12} = 1 \\ & X_{23} = 1, \quad X_{33} = 2 \\ & 2X_{13} + X_{22} = 3 \end{aligned}$$

$$\max c \quad \text{such that} \quad X = \begin{pmatrix} 1 & 1 & c \\ 1 & 3 - 2c & 1 \\ c & 1 & 2 \end{pmatrix} \succeq 0$$

One can check that $\max c = 1$ and $\min c = -1$

The Gram-matrix method to recognize sums of squares [cf. Powers-Wörmann 1998]

Write $p(x) = \sum_{|\alpha| \leq 2d} \mathbf{p}_\alpha x^\alpha \in \mathbb{R}[x]_{2d}$ as a sum of squares:

$$p(x) = \sum_{j=1}^k (\mathbf{u}_j(x))^2 = \sum_{j=1}^k [x]_d^T \mathbf{u}_j \mathbf{u}_j^T [x]_d$$

$$= [x]_d \left(\underbrace{\sum_{j=1}^k \mathbf{u}_j \mathbf{u}_j^T}_{=: \mathbf{U} \succeq 0} \right) [x]_d = \sum_{|\beta|, |\gamma| \leq d} x^\beta x^\gamma \mathbf{U}_{\beta, \gamma}$$

$$= \sum_{|\alpha| \leq 2d} x^\alpha \left(\underbrace{\sum_{\substack{|\beta|, |\gamma| \leq d \\ \beta + \gamma = \alpha}} \mathbf{U}_{\beta, \gamma}}_{=\mathbf{p}_\alpha} \right)$$

Recognize sums of squares via SDP

$$p(x) = \sum_{|\alpha| \leq 2d} p_\alpha x^\alpha \text{ is a sum of squares of polynomials}$$



The following semidefinite program is feasible:

$$\left\{ \begin{array}{l} U \succeq 0 \\ \sum_{\substack{|\beta|, |\gamma| \leq d \\ \beta + \gamma = \alpha}} U_{\beta, \gamma} = p_\alpha \quad (|\alpha| \leq 2d) \end{array} \right.$$

Example: Is $p = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 2y^4$ SOS ?

Solution: Try to write

$$p(x, y) \equiv \underbrace{(x^2 \ xy \ y^2)}_U \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix} \quad \text{with } U \succeq 0$$

Equating coefficients:

$$\begin{array}{ll} x^4 = x^2 \cdot x^2 & 1 = a \\ x^3y = x^2 \cdot xy & 2 = 2b \\ x^2y^2 = xy \cdot xy = x^2 \cdot y^2 & 3 = d + 2c \\ xy^3 = xy \cdot y^2 & 2 = 2e \\ y^4 = y^2 \cdot y^2 & 2 = f \end{array}$$

Example continued

$$\text{Hence } U = \begin{pmatrix} 1 & 1 & c \\ 1 & 3 - 2c & 1 \\ c & 1 & 2 \end{pmatrix} \succeq 0 \iff -1 \leq c \leq 1$$

$$\bullet \text{ For } c = -1, U = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\rightsquigarrow p = (x^2 + xy - y^2)^2 + (y^2 + 2xy)^2$$

$$\bullet \text{ For } c = 0, U = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \end{pmatrix}$$

$$\rightsquigarrow p = (x^2 + xy)^2 + \frac{3}{2}(xy + y^2)^2 + \frac{1}{2}(xy - y^2)^2$$

Which nonnegative polynomials are SOS ?

Hilbert [1888] classified the pairs (n, d) for which every nonnegative polynomial of degree d in n variables is SOS:

- $n = 1$
- $d = 2$
- $n = 2, d = 4$

$\Sigma_{n,d} \subset \mathcal{P}_{n,d}$ for all other (n, d)

Motzkin: $1 + x^4y^2 + x^2y^4 - 3x^2y^2 \in \mathcal{P}_{2,6} \setminus \Sigma_{2,6}$

Homogeneous Motzkin: $z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \geq 0$,
not SOS

$x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 \geq 0$, but not SOS

$x^2y^2 + y^2z^2 + z^2x^2 + w^4 - 4xxzw \geq 0$, but not SOS

How many nonnegative polynomials are sums of squares ?

[Blekherman 2003]: **Very few !**

At **fixed degree $2d$** and large number n of variables, there are significantly more nonnegative polynomials than sums of squares:

$$c \cdot n^{\frac{d-1}{2}} \leq \left(\frac{\text{vol}(\hat{\mathcal{P}}_{n,2d})}{\text{vol}(\hat{\Sigma}_{n,2d})} \right)^{\frac{1}{D}} \leq C \cdot n^{\frac{d-1}{2}}$$

$$\hat{\mathcal{P}}_{n,2d} := \{p \in \mathcal{P}_{n,2d} \mid p \text{ homogeneous, } \deg(p) = 2d, \\ \int_{S^{n-1}} p(x) \mu(dx) = 1\}$$

$$D := \binom{n+2d-1}{2d} - 1, \text{ the dimension of the ambient space}$$

How many nonnegative polynomials are sums of squares ?

Many !

*The SOS cone is dense in the cone of nonnegative polynomials (allowing **variable degrees**):*

[Lasserre-Netzer 2006]: If $p \geq 0$ on $[-1, 1]^n$, then

$$\forall \epsilon > 0 \quad \exists k \in \mathbb{N} \quad \text{s.t.} \quad p + \epsilon \left(1 + \sum_{i=1}^n x_i^{2k} \right) \text{ is SOS}$$

Artin [1927] solved Hilbert's 17th problem [1900]

$$p \geq 0 \text{ on } \mathbb{R}^n \implies p = \sum_i \left(\frac{p_i}{q_i} \right)^2, \text{ where } p_i, q_i \in \mathbb{R}[x]$$

That is, $p \cdot q^2$ is SOS for some $q \in \mathbb{R}[x]$

Sometimes, the shape of the common denominator is known:

Pólya [1928] + Reznick [1995]: For p *homogeneous*

$$p > 0 \text{ on } \mathbb{R}^n \setminus \{0\} \implies p \cdot \left(\sum_{i=1}^n x_i^2 \right)^r \text{ SOS for some } r \in \mathbb{N}$$

An example [Parrilo 2000]

$$M := \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

$$p := \sum_{i,j=1}^5 M_{ij} x_i^2 x_j^2$$

p is not SOS

But $(\sum_{i=1}^5 x_i^2)p$ is SOS

This is a **certificate** that $p \geq 0$ on \mathbb{R}^5 , i.e., that M is copositive

SOS certificates for positivity on a semi-algebraic set K

Let $K = \{x \in \mathbb{R}^n \mid h_1(x) \geq 0, \dots, h_m(x) \geq 0\}$

Set $h_0 := 1$

Quadratic module: $M(h) := \left\{ \sum_{j=0}^m s_j h_j \mid s_j \in \Sigma_n \right\}$

Preordering: $T(h) := \left\{ \sum_{e \in \{0,1\}^m} s_e h_1^{e_1} \cdots h_m^{e_m} \mid s_e \in \Sigma_n \right\}$

$$p \in M(h) \implies p \in T(h) \implies p \geq 0 \text{ on } K$$

The Positivstellensatz [Krivine 1964] [Stengle 1974]

Not an equivalence:

$$K = \{x \in \mathbb{R} \mid (1 - x^2)^3 \geq 0\}$$

$$p = 1 - x^2$$

Then, $p \geq 0$ on K , but $p \notin T(h)$

The Positivstellensatz characterizes equivalence:

$$p \geq 0 \text{ on } K \iff pf = p^{2m} + g \quad \text{for some } f, g \in T(h) \\ \text{and some } m \in \mathbb{N}$$

But this does not yield tractable relaxations for (P)!

The Positivstellensatz [Krivine 1964] [Stengle 1974]

$$K = \{x \in \mathbb{R}^n \mid h_1(x) \geq 0, \dots, h_m(x) \geq 0\}$$

$$(1) \quad p > 0 \text{ on } K \iff pf = 1 + g \text{ for some } f, g \in T(h)$$

$$(2) \quad p \geq 0 \text{ on } K \iff pf = p^{2m} + g \text{ for some } f, g \in T(h), m \in \mathbb{N}$$

$$(3) \quad p = 0 \text{ on } K \iff -p^{2m} \in T(h) \text{ for some } m \in \mathbb{N}$$

(2) \implies solution to **Hilbert's 17th problem**

(3) \implies **Real Nullstellensatz**

But this does not yield tractable relaxations for (P)!!!

SOS certificates of positivity on K compact

Schmüdgen [1991]: Assume K is compact.

$$p > 0 \text{ on } K \implies p \in T(h)$$

Putinar [1993]:

Assume the following *Archimedean condition* holds:

$$(A) \quad \forall p \in \mathbb{R}[x] \quad \exists N \in \mathbb{N} \quad N \pm p \in M(h)$$

$$\text{Equivalently: } \exists N \in \mathbb{N} \quad N - \sum_{i=1}^n x_i^2 \in M(h)$$

$$p > 0 \text{ on } K \implies p \in M(h)$$

$$p_{\min} = \inf_{x \in K} p(x) = \sup \lambda \text{ s.t. } p - \lambda \geq 0 \text{ on } K$$

Relax	$p - \lambda \geq 0 \text{ on } K$	[hard condition]
by	$p - \lambda \in M(h)$	[SOS but unbounded degrees ...]
by	$p - \lambda \in M(h)_{2t}$	[tractable SDP!]

$$M(h)_{2t} := \left\{ \sum_{j=0}^m s_j h_j \mid s_j \in \Sigma_n, \deg(s_j h_j) \leq 2t \right\}$$

\rightsquigarrow Relaxation (SOS_t):

$$p_t^{\text{SOS}} := \sup \lambda \text{ s.t. } p - \lambda \in M(h)_{2t} \leq p_{t+1}^{\text{SOS}} \leq p_{\min}$$

Asymptotic convergence

If (A) holds for K , then $\lim_{t \rightarrow \infty} p_t^{\text{SOS}} = p_{\min}$

Proof: $p - p_{\min} + \epsilon > 0$ on K
 $\implies \exists t \ p - p_{\min} + \epsilon \in M(h)_{2t}$
 $\implies p_t^{\text{SOS}} \geq p_{\min} - \epsilon$

Note: A representation result valid for “ $p \geq 0$ on K ” gives

finite convergence: $p_t^{\text{SOS}} = p_{\min}$ for some t

[Nie-Schweighofer 2007]: $p_{\min} - p_t^{\text{SOS}} \leq \frac{c'}{\sqrt[c]{\log(t/c)}}$ for t big,
where $c = c(h)$ and $c' = c(p, h)$

$$p_{\min} = \inf_{x \in K} p(x) = \inf_{\substack{\mu \text{ probability} \\ \text{measure on } K}} \int_K p(x) d\mu(x)$$

$$= \inf_{L \in \mathbb{R}[x]^*} L(p) \text{ s.t. } L \text{ comes from a probability measure } \mu \text{ on } K$$

$$= \inf_{L \in \mathbb{R}[x]^*} L(p) \text{ s.t. } L(1) = 1 \text{ \& } L(f) \geq 0 \forall f \geq 0 \text{ on } K$$

[Haviland] The following are equivalent for $L \in \mathbb{R}[x]^*$:

- L comes from a nonnegative measure on K , i.e.,

$$L(f) = \int_K f(x) d\mu(x) \quad \forall f \in \mathbb{R}[x]$$

- $L(p) \geq 0$ if $p \geq 0$ on K

$$p_{\min} = \inf_{L \in \mathbb{R}[x]^*} L(p) \quad \text{s.t.} \quad L(f) \geq 0 \quad \forall f \geq 0 \text{ on } K$$

Relax $L(f) \geq 0 \quad \forall f \geq 0 \text{ on } K$

by $L(f) \geq 0 \quad \forall f \in M(h)$

by $L(f) \geq 0 \quad \forall f \in M(h)_{2t}$

\rightsquigarrow Relaxation (MOMt):

$$p_t^{\text{mom}} := \inf_{L \in \mathbb{R}[x]_{2t}^*} L(p) \quad \text{s.t.} \quad L \geq 0 \text{ on } M(h)_{2t}$$

Weak duality:

$$p_t^{\text{sos}} \leq p_t^{\text{mom}} \leq p_{\min}$$

Equality: $p_t^{\text{sos}} = p_t^{\text{mom}}$ e.g. if $\text{int}(K) \neq \emptyset$

The dual relaxation (MOMt) is again an SDP

$$L \in \mathbb{R}[x]_{2t}^* \rightsquigarrow \mathbf{M}_t(L) := (L(x^\alpha x^\beta))_{|\alpha|, |\beta| \leq t}$$

$M_t(L)$ is the *moment matrix* of L (of order t)

$$\textbf{Lemma: } L(f^2) \geq 0 \quad \forall f \in \mathbb{R}[x]_t \iff \mathbf{M}_t(L) \succeq 0$$

$$\textbf{Proof: } L(f^2) = \vec{f}^T M_t(L) \vec{f}$$

$$\textbf{Lemma: } L \geq 0 \text{ on } M(h)_{2t}$$

$$\text{i.e., } L(f^2 h_j) \geq 0 \quad \forall f \quad \forall j \quad \text{with } \deg(f^2 h_j) \leq 2t$$

$$\iff M_t(L) \succeq 0 \text{ and } \mathbf{M}_{t-d_j}(h_j L) \succeq 0 \quad \forall j$$

where $d_j := \lceil \deg(h_j)/2 \rceil$ and $h_j L \in \mathbb{R}[x]_{2t-\deg(h_j)}^*$ is defined by $(h_j L)(f) := L(h_j f)$.

Optimality criterion

Observation: Let L be an optimum solution to (MOMt).

If L comes from a probability measure μ on K ,

then (MOMt) is exact: $p_t^{\text{mom}} = p_{\min}$.

Question: How to recognize whether L has a representing measure on K ?

Next: Sufficient condition of Curto-Fialkow for the moment problem

A sufficient condition for the moment problem

Theorem [Curto-Fialkow 1996] Let $L \in \mathbb{R}[x]_{2t}^*$.

If $M_t(L) \succeq 0$ and (RC) $\text{rank} M_t(L) = \text{rank} M_{t-1}(L)$,
then L has a representing measure.

Corollary [Curto-Fialkow 2000] + [Henrion-Lasserre 2005]

Set $d := \max_j \lceil \deg(h_j)/2 \rceil$.

Let L be an optimum solution to (MOMt) satisfying

$\text{rank} M_t(L) = \text{rank} M_{t-d}(L)$. Then, $p_t^{\text{mom}} = p_{\min}$ and

$$V(\text{Ker} M_t(L)) \subseteq \{ \text{global minimizers of } p \text{ on } K \}$$

with **equality** if $\text{rank} M_t(L)$ is **maximum**.

Remarks

- Compute $V(\text{Ker} M_t(L))$ with the *eigenvalue method*.
- If the rank condition holds at a maximum rank solution, then (P) has *finitely many global minimizers*.

But the reverse is not true !

Example: $p_{\min} := \min p(x) \text{ s.t. } \sum_{i=1}^n x_i^2 \leq 1$, where p is a positive form, not SOS. Then, $p_t^{\text{mom}} = p_t^{\text{SOS}} < p_{\min}$, since $p \notin M(1 - \sum_i x_i^2)$.

- The rank condition *holds* in the **finite variety case**:

$$K = \{x \in \mathbb{R}^n \mid \underbrace{h_1(x) = 0, \dots, h_k(x) = 0}_{\text{ideal } I}, h_{k+1}(x) \geq 0, \dots\}$$

with $|V_{\mathbb{R}}(I)| < \infty$

Finite convergence in the finite variety case

$$K = \{x \in \mathbb{R}^n \mid \underbrace{h_1(x) = 0, \dots, h_k(x) = 0}_{\text{ideal } I}, h_{k+1}(x) \geq 0, \dots\}$$

Theorem: [La 2002] [Lasserre/La/Rostalski 2007]

(i) If $|V_{\mathbb{C}}(I)| < \infty$,

$$p_t^{\text{sos}} = p_t^{\text{mom}} = p_{\min} \quad \text{for some } t$$

(ii) If $|V_{\mathbb{R}}(I)| < \infty$,

$$p_t^{\text{mom}} = p_{\min} \quad \text{for some } t$$

The flat extension theorem

Theorem [Curto-Fialkow 1996] Let $L \in \mathbb{R}[x]_{2t}^*$.

If $\text{rank } M_t(L) = \text{rank } M_{t-1}(L)$,
then there is an extension $\tilde{L} \in \mathbb{R}[x]^*$ of L
for which $\text{rank } M(\tilde{L}) = \text{rank } M_t(L)$.

Main tool: $\text{Ker } M(\tilde{L})$ is an ideal.

[La-Mourrain 2009] The flat extension theorem can be generalized to matrices indexed by a set \mathcal{C} of monomials (*connected to 1*) and its *closure* $\mathcal{C}^+ = \mathcal{C} \cup x_1\mathcal{C} \cup \dots \cup x_n\mathcal{C}$, satisfying $\text{rank } M_{\mathcal{C}} = \text{rank } M_{\mathcal{C}^+}$.

The finite rank moment matrix theorem

Theorem: [Curto-Fialkow 1996] Let $L \in \mathbb{R}[x]^*$.

$$M(L) \succeq 0 \text{ and } \text{rank } M(L) =: r < \infty \\ \iff L \text{ has a (unique) } r\text{-atomic representation measure } \mu.$$

[La 2005] **Simple proof for \implies :**

- $I := \text{Ker } M(L)$ is a real radical ideal
- I is 0-dimensional, as $\dim \mathbb{R}[x]/I = r$

Hence: $V(I) = \{x_1, \dots, x_r\} \subseteq \mathbb{R}^n$

Verify: L is represented by $\mu = \sum_{i=1}^r L(p_i^2) \delta_{x_i}$, where the p_i 's are interpolation polynomials at the x_i 's

Implementations of the SOS/moment relaxation method

GloptiPoly by Henrion, Lasserre
(incorporates the optimality stopping criterion and the extraction procedure for global minimizers)

SOSTOOLS by Prajna, Papachristodoulou, Seiler, Parrilo

YALMIP by Löfberg

SparsePOP by Waki, Kim, Kojima, Muramatsu

Example 1

$$\begin{aligned}
 \min \quad & p = -25(x_1 - 2)^2 - (x_2 - 2)^2 - (x_3 - 1)^2 \\
 & \quad - (x_4 - 4)^2 - (x_5 - 1)^2 - (x_6 - 4)^2 \\
 \text{s.t.} \quad & (x_3 - 3)^2 + x_4 \geq 4, \quad (x_5 - 3)^2 + x_6 \geq 4 \\
 & x_1 - 3x_2 \leq 2, \quad -x_1 + x_2 \leq 2, \quad x_1 + x_2 \leq 6, \\
 & x_1 + x_2 \geq 2, \quad 1 \leq x_3 \leq 5, \quad 0 \leq x_4 \leq 6, \\
 & 1 \leq x_5 \leq 5, \quad 0 \leq x_6 \leq 10, \quad x_1, x_2 \geq 0
 \end{aligned}$$

order t	rank sequence	bound p_t^{mom}	solution extracted
1	1 7	unbounded	none
2	1 1 21	-310	(5, 1, 5, 0, 5, 10)

$$d = 1$$

The global minimum is found at the relaxation of order $t = 2$

Example 2

$$\begin{aligned} \min \quad & p = -x_1 - x_2 \\ \text{s.t.} \quad & x_2 \leq 2x_1^4 - 8x_1^3 + 8x_1^2 + 2 \\ & x_2 \leq 4x_1^4 - 32x_1^3 + 88x_1^2 - 96x_1 + 36 \\ & 0 \leq x_1 \leq 3, \quad 0 \leq x_2 \leq 4 \end{aligned}$$

order t	rank sequence	bound p_t^{mom}	solution extracted
2	1 1 4	-7	none
3	1 2 2 4	-6.6667	none
4	1 1 1 1 6	-5.5080	$(2.3295, 3.1785)$

$$d = 2$$

The global minimum is found at the relaxation of order $t = 4$

Example 3: Compute real roots of polynomial equations

$$\begin{aligned} \min p := 1 \quad \text{s.t.} \quad & 5x_1^9 - 6x_1^5x_2 + x_1x_2^4 + 2x_1x_3 = 0 \\ & -2x_1^6x_2 + 2x_1^2x_2^3 + 2x_2x_3 = 0, \quad x_1^2 + x_2^2 - 0.265625 = 0. \end{aligned}$$

order t	rank sequence	accuracy
5	1 4 8 16 25 34	—
6	1 3 9 15 22 26 32	—
7	1 3 8 10 12 16 20 24	—
8	1 4 8 8 8 12 16 20 24	4.6789e-5

$d = 5$, 20 complex roots, **8 real roots**:

$$\begin{aligned} v_1 &= (-0.515, -0.000153, -0.0124), & v_2 &= (-0.502, 0.119, 0.0124), \\ v_3 &= (0.502, 0.119, 0.0124), & v_4 &= (0.515, -0.000185, -0.0125), \\ v_5 &= (0.262, 0.444, -0.0132), & v_6 &= (-2.07\text{e-}5, 0.515, -1.27\text{e-}6), \\ v_7 &= (-0.262, 0.444, -0.0132), & v_8 &= (-1.05\text{e-}5, -0.515, -7.56\text{e-}7). \end{aligned}$$

An example where (RC) cannot hold

Perturbed Motzkin form:

$$p = x_1^2 x_2^2 (x_1^2 + x_2^2 - 3x_3^2) + x_3^6 + \epsilon(x_1^6 + x_2^6 + x_3^6)$$

$K = \{x \mid \sum_{i=1}^3 x_i^2 \leq 1\} \rightsquigarrow (0, 0)$ is the *unique minimizer*

But (RC) never holds

as $p \notin M(h)$ and $p_t^{\text{sos}} = p_t^{\text{mom}} < p_{\min} = 0$

order t	rank sequence	bound p_t^{mom}	val. moment vect.
3	1 4 9 13	$-2.11 \cdot 10^{-5}$	$1.67 \cdot 10^{-44}$
4	1 4 10 20 35	$-1.92 \cdot 10^{-9}$	$4.47 \cdot 10^{-60}$
5	1 4 10 20 35 56	$2.94 \cdot 10^{-12}$	$1.26 \cdot 10^{-44}$
6	1 4 10 20 35 56 84	$3.54 \cdot 10^{-12}$	$1.5 \cdot 10^{-44}$
7	1 4 10 20 35 56 84 120	$4.09 \cdot 10^{-12}$	$2.83 \cdot 10^{-43}$

$d = 3, \epsilon = 0.01$

Application to Unconstrained Polynomial Minimization

$$p_{\min} = \inf_{x \in \mathbb{R}^n} p(x)$$

where $\deg(p) = 2d$

As there is **no constraint**, the relaxation scheme just gives **one bound**:

$$p_t^{\text{sos}} = p_t^{\text{mom}} = p_d^{\text{sos}} = p_d^{\text{mom}} \leq p_{\min} \quad \text{for all } t \geq d$$

with equality iff $p(x) - p_{\min}$ is SOS

How to get better bounds ?

Idea: Transform the *Unconstrained* Problem into a *Constrained* Problem

If p has a minimum:

$$p_{\min} = p_{\text{grad}}^* := \inf_{x \in V_{\text{grad}}^{\mathbb{R}}} p(x)$$

where

$$V_{\text{grad}}^{\mathbb{R}} := \{x \in \mathbb{R}^n \mid \nabla p(x) = 0 \ (i = 1, \dots, n)\}$$

If, moreover, a bound R is known on the norm of a global minimizer:

$$p_{\min} = p_{\text{ball}}^* := \inf_{R^2 - \sum_i x_i^2 \geq 0} p(x)$$

When p attains its minimum

The ‘ball approach’:

- Convergence of the SOS/MOM bounds to $p_{\min} = p_{\text{ball}}^*$

The ‘gradient variety’ approach:

[Demmel, Nie, Sturmfels 2004]:

$$p > 0 \text{ on } V_{\text{grad}}^{\mathbb{R}} \implies p \in M_{\text{grad}}$$

$$p \geq 0 \text{ on } V_{\text{grad}}^{\mathbb{R}} \implies p \in M_{\text{grad}} \quad \text{if } I_{\text{grad}} \text{ radical}$$

$$M_{\text{grad}} := M(\pm \partial p / \partial x_i) = \Sigma_n + \underbrace{\sum_{i=1}^n \mathbb{R}[x] \partial p / \partial x_i}_{I_{\text{grad}}}$$

Convergence Result [Demmel, Nie, Sturmfels 2004]

Asymptotic convergence of the SOS/MOM bounds to p_{grad}^*

Finite Convergence to p_{grad}^* when the gradient ideal I_{grad} is radical

Hence: When p attains its minimum, we have a converging hierarchy of SDP bounds to p_{min}

Example: $p = x^2 + (xy - 1)^2$ does **not** attain its minimum

$$p_{\text{min}} = 0 < p_{\text{grad}}^* = 1$$

What if p is not known to have a minimum?

Strategy 1: Perturb the polynomial p

[Hanzon-Jibetean 2003] [Jibetean-Laurent 2004]

$$p_\epsilon(x) := p(x) + \epsilon \left(\sum_{i=1}^n x_i^{2d+2} \right) \quad \text{for small } \epsilon > 0$$

- p_ϵ has a minimum and $\lim_{\epsilon \rightarrow 0} (p_\epsilon)_{\min} = p_{\min}$
 - The global minimizers of p_ϵ converge to global minimizers of p as $\epsilon \rightarrow 0$
 - The gradient variety of p_ϵ is finite
- \rightsquigarrow finite convergence of $(p_\epsilon)_t^{\text{sos}}$, $(p_\epsilon)_t^{\text{mom}}$ to $(p_\epsilon)_{\min}$

Example: Perturb the polynomial $p = (xy - 1)^2 + x^2$

$$\inf p(x) \text{ s.t. } \partial p_\epsilon / \partial x_1 = 0, \partial p_\epsilon / \partial x_2 = 0$$

ϵ	order t	rank sequence	p_t^{mom}	extracted solutions
10^{-2}	3	2 6 8	0.00062169	
10^{-2}	4	2 2 2 7	0.33846	
10^{-2}	5	2 2 2 2 -	0.33846	$\pm(0.4729, 1.3981)$
10^{-3}	5	2 2 2 2 -	0.20824	$\pm(0.4060, 1.9499)$
10^{-4}	5	2 2 2 2 -	0.12323	$\pm(0.3287, 2.6674)$
10^{-5}	5	2 2 2 2 -	0.07132	$\pm(0.2574, 3.6085)$
10^{-6}	5	2 2 2 2 -	0.040761	$\pm(0.1977, 4.8511)$
10^{-7}	5	2 2 2 2 -	0.023131	$\pm(0.1503, 6.4986)$
10^{-8}	5	2 2 2 2 -	0.013074	$\pm(0.1136, 8.6882)$
10^{-9}	5	2 2 2 2 -	0.0073735	$\pm(0.0856, 11.6026)$
10^{-10}	5	2 2 2 2 -	0.0041551	$\pm(0.0643, 15.4849)$

When p does not have a minimum:

Algebraic/analytical approach of Schweighofer [2005]

Strategy 2: Minimize p over its ‘gradient tentacle’

If $p_{\min} > -\infty$, then

$$p_{\min} = \inf_{x \in K_{\text{grad}}} p(x)$$

where

$$K_{\text{grad}} := \{x \in \mathbb{R}^n \mid \|\nabla p(x)\|^2 \|x\|^2 \leq 1\} \supseteq V_{\text{grad}}^{\mathbb{R}}$$

$$\nabla p = (\partial p / \partial x_i)_{i=1}^n$$

Representation result on the gradient tentacle K_{grad}

[Schweighofer 2005]: Assume $p_{\min} > -\infty$ and p has **only isolated singularities at infinity (*)** (e.g. $n = 2$). Then,

$$\begin{aligned} p \geq 0 \text{ on } \mathbb{R}^n &\iff p \geq 0 \text{ on } K_{\text{grad}} \\ &\iff \forall \epsilon > 0 \quad p + \epsilon \in M(1 - \|\nabla p(x)\|^2 \|x\|^2) \end{aligned}$$

\rightsquigarrow Convergent SOS/moment bounds to $p_{\min} = \inf_{K_{\text{grad}}} p(x)$

(*): The system $\nabla p_d(x) = 0, p_{d-1}(x) = 0$ has **finitely many projective zeros**, where $p = p_d + p_{d-1} + \dots + p_0$ and p_i is the homogeneous component of degree i

Tools: Algebra (extension of Schmüdgen's theorem) + analysis (Parusinski's results on behaviour of polynomials at infinity)

When p does not have a minimum:

The ‘tangency variety’ approach of Vui-Son [2008]

Strategy 3: Minimize p over its ‘truncated tangency variety’:

$$\begin{aligned}\Gamma_p &:= \left\{ x \in \mathbb{R}^n \mid \text{rank} \begin{pmatrix} \nabla p(x) \\ x \end{pmatrix} \leq 1 \right\} \\ &= \{x \mid g_{ij} := x_i \partial p / \partial x_j - x_j \partial p / \partial x_i = 0 \ \forall i, j \leq n\}\end{aligned}$$

$$\Gamma_p^0 := \Gamma_p \cap \{x \mid p(x) \leq p(0)\}$$

[Vui-Son 2008]: For $p \in \mathbb{R}[x]$ such that $p_{\min} > -\infty$

$$\begin{aligned}p \geq 0 \text{ on } \mathbb{R}^n &\iff p \geq 0 \text{ on } \Gamma_p^0 \\ &\iff \forall \epsilon > 0 \quad p + \epsilon \in M(p(0) - p, \pm g_{ij})\end{aligned}$$

\rightsquigarrow Convergent SOS/moment bounds to $p_{\min} = \inf_{\Gamma_p^0} p(x)$

Further directions

- Exploit structure (equations, sparsity, symmetry, convexity, ...) to get smaller SDP programs
[Kojima, Grimm, Helton, Lasserre, Netzer, Nie, Riener, Schweighofer, Theobald, Parrilo, ...]
- Application to the generalized problem of moments, to approximating integrals over semi-algebraic sets, ...
[Henrion, Lasserre, Savorgnan, ...]
- Extension to NC variables
[Helton, Klep, McCullough, Schmüdgen, Schweighofer,...]