Semidefinite Programming in Polynomial Optimization

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Polynomial Optimization Problem (P)

$$p_{\min} := \inf_{x \in K} p(x)$$

where $$K := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0 \ldots g_m(x) \geq 0\}$$

and $p, g_1, \ldots, g_m$ are polynomials in n variables

Unconstrained Polynomial Minimization

$$p_{\min} := \inf_{x \in \mathbb{R}^n} p(x)$$

Example: An $n \times n$ matrix M is *copositive* if $p_{\min} \geq 0$ for $p(x) := \sum_{i,j=1}^{n} x_i^2 x_j^2 M_{ij}$, i.e., $p(x)$ is nonnegative on $\mathbb{R}^n$

Example: A sequence $a_1, \ldots, a_n \in \mathbb{N}$ can be *partitioned* if $p_{\min} = 0$ for $p(x) = (\sum_{i=1}^{n} a_i x_i)^2 + \sum_{i=1}^{n} (x_i^2 - 1)^2$

0/1 Linear Programming

$$\min\ c^T x\ \text{ s.t. } Ax \leq b,\ x_i^2 = x_i\ \forall i$$

Example: The *stability number* $\alpha(G)$ of $G = (V, E)$ can be computed via any of the programs:

$$\alpha(G) = \max \sum_{i \in V} x_i\ \text{ s.t. } x_i + x_j \leq 1\ (ij \in E),\ x_i^2 = x_i\ (i \in V)$$

$$\frac{1}{\alpha(G)} = \min\ x^T (I + A_G) x\ \text{ s.t. } \sum_{i \in V} x_i = 1,\ x_i \geq 0\ (i \in V)$$

Hence: (P) is an NP-hard problem

Strategy

Approximate (P) by a hierarchy of convex (semidefinite) relaxations

Shor (1987), Nesterov, Lasserre, Parrilo (2000-)

Such relaxations can be constructed using

representations of nonnegative polynomials as sums of squares of polynomials

and

the dual theory of moments

Underlying paradigm:

Testing whether a polynomial p is *nonnegative* is hard

but

one can test whether p is a *sum of squares of polynomials*

efficiently via *semidefinite programming*

Some notation about polynomials

- Multivariate polynomial:

$$p(x) = \sum_{\alpha \in \mathbb{Z}_+^n} p_\alpha \underbrace{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}_{x^\alpha} = \sum_{\alpha \in \mathbb{Z}_+^n} p_\alpha x^\alpha$$

- $\deg(x^\alpha) = |\alpha| := \sum_{i=1}^n \alpha_i$

- $\deg(p) = \max \deg(x^\alpha)$ s.t. $p_\alpha \neq 0$

- We identify a polynomial $p(x)$ with its sequence of coefficients $p = (p_\alpha)_{|\alpha| \leq d}$, if $\deg(p) = d$

- $p(x)$ is a *sum of squares of polynomials (SOS)* if

$$p(x) = \sum_{j=1}^{m} [u_j(x)]^2 \qquad \text{for some polynomials } u_1, \ldots, u_m$$

Then, $\deg(p)$ is even and $\deg(u_i) \leq \deg(p)/2$

Example: $x^2 + y^2 + 2xy + z^6 = (x+y)^2 + (z^3)^2$ is SOS

Recognizing SOS of pol's via SDP

$p(x) = \sum_{|\alpha| \le 2d} p_\alpha x^\alpha$ is a sum of squares of polynomials

i.e., $p(x) = \sum_j [u_j(x)]^2$, where $\deg(u_j) \le d$

$$\Updownarrow$$

$$p(x) = z^T \underbrace{(\sum_j u_j u_j^T)}_{X \succeq 0} z, \quad \text{setting } z := (x^\beta)_{|\beta| \le d}$$

$$\Updownarrow$$

The following semidefinite program is feasible:

$$\begin{cases} X \succeq 0 \\ \sum_{\substack{|\beta|,|\gamma| \le d \\ \beta+\gamma=\alpha}} X_{\beta,\gamma} = p_\alpha \quad (|\alpha| \le 2d) \end{cases}$$

$\rightsquigarrow$ system in matrix variable X of order $\binom{n+d}{d} \times \binom{n+d}{d}$, with $\binom{n+2d}{2d}$ equations

$\rightsquigarrow$ polynomial size *fixing either* n *or* d

When does a polynomial have a decomposition as a sum of squares?

Hilbert [1888] classified the pairs (n, d) for which *every nonnegative polynomial of degree d in n variables is a sum of squares of polynomials*:

- $n = 1$
- $d = 2$
- $n = 2,\ d = 4$

E.g., the **Motzkin polynomial:** $x^4y^2 + x^2y^4 - 3x^2y^2 + 1$ is nonnegative but *not* a SOS

Artin [1927] solved Hilbert's 17th problem [1900]

$$p \geq 0 \text{ on } \mathbb{R}^n \Longrightarrow p = \sum_i \left(\frac{p_i}{q_i}\right)^2, \text{ where } p_i, q_i \in \mathbb{R}[x_1, \ldots, x_n]$$

That is,

$$p \cdot q^2 \text{ is SOS for some } q \in \mathbb{R}[x_1, \ldots, x_n]$$

Sometimes, the common denominator is known:

Pólya [1928] + Reznick [1995]: For p *homogeneous*

$$p > 0 \text{ on } \mathbb{R}^n \setminus \{0\} \Longrightarrow p \cdot \left(\sum_{i=1}^n x_i^2\right)^r \text{ SOS for some } r \in \mathbb{N}$$

(Dual) SOS approximations for (P)

following [Lasserre 2001]

Recall: $K = \{x \in \mathbb{R}^n \mid g_\ell(x) \geq 0 \ (\ell = 1, \ldots, m)\}$

$$p_{\min} := \inf_{x \in K} \ p(x) = \sup \ \lambda \ \text{ s.t. } p(x) - \lambda \geq 0 \ \forall x \in K$$

Replacing the *nonnegativity* condition by a stronger *SOS* condition: formulate **(SOSt)**

$$\sigma_t := \sup \ \lambda \ \text{ s.t. } \quad p(x) - \lambda = s_0(x) + \sum_{\ell=1}^m s_\ell(x) g_\ell(x)$$
$$s_0, \ s_\ell \text{ are SOS}$$
$$\deg(s_0), \ \deg(s_\ell g_\ell) \leq 2t$$

for any $t \geq \lceil \deg(p)/2 \rceil, \ \lceil \deg(g_\ell)/2 \rceil$

$$\sigma_t \leq \sigma_{t+1} \leq p_{\min}$$

Example: $K = \mathbb{R}^n$ (unconstrained case), $\deg(p) = 2d$
$\sigma_t = \sigma_d \leq p_{\min}$, with equality iff $p(x) - p_{\min}$ is SOS

Equality in the univariate case (Shor [1987], Nesterov [2000])

(Primal) Moment Approximations

$$p_{\min} = \inf_{x \in K} p(x) = \inf_{\mu \in \mathcal{P}_K} \int p(x)\mu(dx) = \inf_{y \in \mathcal{M}_K} p^T y$$

$$\int p(x)\mu(dx) = \sum_\alpha p_\alpha \int x^\alpha \mu(dx)$$

$\mu \in \mathcal{P}_K$ if μ is a probability measure supported by K

$y \in \mathcal{M}_K$ if $y_\alpha = \underbrace{\int x^\alpha \mu(dx)}_{\substack{\text{moment of order } \alpha \\ \text{of the measure } \mu}} \ \forall \alpha$, for some $\mu \in \mathcal{P}_K$

Lemma:

If y is the sequence of moments of $\mu \in \mathcal{P}_K$ then

$$y_0 = 1$$

$$\boxed{M_t(y) := (y_{\alpha+\beta})_{|\alpha|,|\beta| \leq t} \succeq 0} \quad \text{for } t \geq 0$$

moment matrix of order t

$$\boxed{M_t(g_\ell y) := (\sum_\gamma (g_\ell)_\gamma y_{\alpha+\beta+\gamma})_{|\alpha|,|\beta| \leq t} \succeq 0} \quad \text{for } t \geq 0$$

localizing matrix

Moreover: $\boxed{\text{supp}(\mu) \subseteq \text{Zeros}(p)} \quad \forall p \in \text{Ker} M_t(y)$

Proof:

For $p = (p_\alpha)_{|\alpha| \leq t}$,

$$p^T M_t(y) p = \int p(x)^2 \mu(dx) \geq 0$$

$$p^T M_t(g_\ell y) p = \int g_\ell(x) p(x)^2 \mu(dx) \geq 0$$

If we relax the moment condition "$y \in \mathcal{M}_K$" by requiring positive semidefiniteness of its moment matrix, we obtain a hierarchy of tractable SDP relaxations:

(MOMt)

$$p_t := \inf \; p^T y \;\; \text{s.t.} \quad y_0 = 1,\; M_t(y) \succeq 0$$
$$M_{t-d_\ell}(g_\ell y) \succeq 0 \; (\ell = 1, \ldots, m)$$

for any $t \geq \lceil \deg(p)/2 \rceil,\; d_\ell := \lceil \deg(g_\ell)/2 \rceil$

$$p_t \leq p_{t+1} \leq p_{\min}$$

Lemma:

The semidefinite programs **(SOSt)** and **(MOMt)** are dual of each other.

Weak duality: $\boxed{\sigma_t \leq p_t}$

No duality gap, e.g., if K has a nonempty interior

Properties of the bounds

$$\sigma_t \leq p_t \leq p_{\min}$$

- For *fixed* t, one can compute σ_t, p_t in **polynomial time** (to any fixed precision)
- **Asymptotic (finite) convergence** to $p_{\min}$

via some representation theorems for positive polynomials from real algebraic geometry

- **Optimality certificate**

via some theorems of Curto and Fialkow about moment matrices

- **Extracting global minimizers**

via the eigenvalue method for solving systems of polynomial equations

Asymptotic convergence

Representation theorem: [Putinar 1993]
(more elementary proof by [Schweighofer 2003])

If $\exists u_\ell$ SOS : $\{x \mid \sum_\ell u_\ell(x) g_\ell(x) \geq 0\}$ is compact $\quad (*)$

then $p > 0$ on $K \Longrightarrow p = s_0 + \sum_\ell s_\ell g_\ell$ with s_0, s_ℓ SOS

Convergence theorem: [Lasserre 2001]

If $(*)$ then $\boxed{\lim_{t\to\infty} \sigma_t = \lim_{t\to\infty} p_t = p_{\min}}$

Note: $(*)$ holds, e.g.,

- if $\{x \mid g_\ell(x) \geq 0\}$ is compact for some ℓ
- if the equations $x_i^2 = x_i$ are present in the description of K **(0/1 case)**
- if the radius R of a ball containing K is known, then add the (redundant) constraint $R^2 - \sum_i x_i^2 \geq 0$ to the description of K

Note: A representation result valid for "$p \geq 0$ on K" would give a *finite* convergence result

Finite Convergence: Finite Variety

Assume the equations: $h_1(x) = 0 \ldots h_k(x) = 0$ are present in the description of K and

$$V := \{x \in \mathbb{C}^n \mid h_1(x) = 0 \ldots h_k(x) = 0\} \text{ is } \textit{finite}$$

Then: $p_t = p_{\min}$ for t large enough [Laurent 2002]

Moreover: $\sigma_t = p_t = p_{\min}$ for t large enough

if $h_1, \ldots, h_k$ form a *Groebner basis* [Laurent 2002]
or if they generate a *radical ideal* [Parrilo 2002]

Representation result in radical case: [Parrilo 2002]
$p \geq 0$ on $K \Longrightarrow p = s_0 + \sum_\ell s_\ell g_\ell$ (s_0, s_ℓ SOS)

In the 0/1 case

$$p_t = p_{\min} \quad \text{for } t \geq n + d$$

where $d := \max_\ell \lceil \deg(g_\ell)/2 \rceil$ [Lasserre 2001]

The hierarchy of bounds p_t refines other combinatorial hierarchies, e.g., by Lovász-Schrijver [1991]
(cf. [Laurent 2003])

Note: One can use the equations $h_1(x) = 0 \ldots h_k(x) = 0$ to *reduce the number of variables* in the moment relaxations (MOMt)

Optimality Certificate

Theorem: [Henrion-Lasserre 2005]
If y is an optimum solution to the program **(MOMt)** satisfying:

(RC) $\boxed{\text{rank}M_t(y) = \text{rank}M_{t-d}(y),}$

where $d = \max_\ell \lceil \deg(g_\ell)/2 \rceil$

then: $\boxed{p_t = p_{\min}}$

Proof

By a theorem of [Curto-Fialkow 1996] (cf. [Laurent 2004])

y is the sequence of moments of an r-atomic nonnegative measure μ

$$\mu = \sum_{i=1}^{r} \lambda_i \delta_{x_i}, \text{ where } \lambda_i > 0, \sum_i \lambda_i = 1, \; x_i \in K$$

$$\Longrightarrow \quad y = \sum_{i=1}^{r} \lambda_i (x_i^\alpha)_{|\alpha| \le 2t}$$

$$\Longrightarrow \quad p_{\min} \ge p_t = p^T y = \sum_{i=1}^{r} \lambda_i p(x_i) \ge p_{\min}$$

Moreover: $x_1, \ldots, x_r$ are global minimizers over K

Extracting global minimizers

Then, $\text{supp}(\mu) = \{x_1, \ldots, x_r\} \subseteq \{\text{global minimizers}\}$
Moreover,

$$\text{supp}(\mu) = \bigcap_{p \in \text{Ker} M_t(y)} \text{Zeros}(p)$$

Therefore: One can find the global minimizers $x_1, \ldots, x_r$ by *computing the common zeros to a system of polynomial equations*

$\rightsquigarrow$ Use the **'eigenvalue method'**

Based on computing the eigenvalues of the $r \times r$ multiplication matrices in the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]/I$, where I is the ideal generated by the kernel of $M_t(y)$

Implementations of the SOS/moment relaxation method

GloptiPoly by Henrion, Lasserre

SOSTOOLS by Prajna, Papachristodoulou, Parrilo

Example 1

$$
\begin{aligned}
\min \quad & p = -25(x_1-2)^2 - (x_2-2)^2 - (x_3-1)^2 \\
& \qquad -(x_4-4)^2 - (x_5-1)^2 - (x_6-4)^2 \\
\text{s.t.} \quad & (x_3-3)^2 + x_4 \geq 4, \;\; (x_5-3)^2 + x_6 \geq 4 \\
& x_1 - 3x_2 \leq 2, \;\; -x_1 + x_2 \leq 2, \; x_1 + x_2 \leq 6, \\
& x_1 + x_2 \geq 2, \; 1 \leq x_3 \leq 5, \; 0 \leq x_4 \leq 6, \\
& 1 \leq x_5 \leq 5, \; 0 \leq x_6 \leq 10, \; x_1, x_2 \geq 0
\end{aligned}
$$

order t	bound p_t	solution extracted
1	unbounded	none
2	-310	(5,1,5,0,5,10)

The global minimum is found at the relaxation of order $t = 2$

Example 2

$$
\begin{aligned}
\min \quad & p = -x_1 - x_2 \\
\text{s.t.} \quad & x_2 \leq 2x_1^4 - 8x_1^3 + 8x_1^2 + 2 \\
& x_2 \leq 4x_1^4 - 32x_1^3 + 88x_1^2 - 96x_1 + 36 \\
& 0 \leq x_1 \leq 3,\ 0 \leq x_2 \leq 4
\end{aligned}
$$

order t	bound p_t	solution extracted
2	-7	none
3	-6.6667	none
4	-5.5080	(2.3295,3.1785)

The global minimum is found at the relaxation of order $t = 4$

Unconstrained polynomial minimization

As $K = \mathbb{R}^n$, the relaxation scheme just gives *one* bound:
$\sigma_d = p_d \leq p_{\min}$, with equality iff $p(x) - p_{\min}$ is SOS

Idea:

Get better bounds by transforming the *unconstrained* problem into a *constrained* problem

If p has a minimum:

$$p_{\min} = p_{\text{grad}} := \inf_{x \in V_{grad}^{\mathbb{R}}} p(x)$$

where $$V_{\text{grad}}^{\mathbb{R}} := \{x \in \mathbb{R}^n \mid \frac{\partial p}{\partial x_i} = 0 \ (i = 1, \ldots, n)\}$$

If, moreover, a bound R is known on the norm of a global minimizer:

$$p_{\min} = p_{\text{ball}} := \inf_{R^2 - \sum_i x_i^2 \geq 0} p(x)$$

When p attains its minimum:

The 'ball approach':

• p_{ball} can be approximated via Lasserre's relaxation scheme (asymptotic convergence, as Putinar's assumption holds!)
• seems to work well only if the radius R of the ball is not too large ...

The 'gradient variety' approach:

Representation result: [Demmel, Nie, Sturmfels 2004]

$$p > 0 \text{ on } V_{\text{grad}}^{\mathbb{R}} \Longrightarrow p = s_0 + \sum_{i=1}^{n} u_i \frac{\partial p}{\partial x_i}$$

where s_0 SOS, $u_i \in \mathbb{R}[x_1, \ldots, x_n]$

Convergence result: [Demmel, Nie, Sturmfels 2004] There is asymptotic convergence of the SOS and moment bounds to p_{grad}

Moreover: When $\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n}$ generate a radical ideal, the representation result holds for $p \geq 0$, implying the *finite* convergence

What if p is not known to have a minimum?

Strategy: Perturb the polynomial p

[Hanzon-Jibetean 2003] [Jibetean-Laurent 2004]

$$p_\epsilon(x) := p(x) + \epsilon(\sum_{i=1}^{n} x_i^{2d+2})$$ for small $\epsilon > 0$

- p_ϵ has a minimum and $\lim_{\epsilon \to 0} (p_\epsilon)_{\min} = p_{\min}$
- the gradient variety of p_ϵ is *finite*

$\rightsquigarrow$ one can compute $(p_\epsilon)_{\min}$ via the 'gradient variety' approach

$\rightsquigarrow$ *finite convergence* of the moment bounds to $(p_\epsilon)_{\min}$ (in $2nd$ steps)

• one can use the gradient equations:

$$\frac{\partial p_\epsilon}{\partial x_i} = (2d+2)x_i^{2d+1} + \frac{\partial p}{\partial x_i} = 0 \;\; \forall i$$

to reduce # variables ($\rightsquigarrow$ only variables y_α with $\alpha_i \leq 2d \; \forall i$) and to express $(p_\epsilon)_{\min}$ via a SDP with a *single* LMI

• the limits of global minimizers of p_ϵ (as $\epsilon \to 0$) are global minimizers of p (if any)

Examples

Motzkin polynomial: $p = x^2y^2(x^2 + y^2 - 3) + 1$

Then, $p_{\min} = 0$, attained at $(\pm 1, \pm 1)$, and p is *not* SOS

ϵ	order t	(r_t, r_{t-1}, r_{t-2})	$(p_\epsilon)_t$	extracted solutions
	3	(10,6,3)	$-3.5070\ 10^4$	none
10^{-1}	4	(4,4,4)	0.0316	$(\pm 0.9453, \pm 0.9453)$
10^{-2}	4	(4,4,4)	$5.0177\ 10^{-4}$	$(\pm 0.9935, \pm 0.9935)$
10^{-3}	5	(4,4,4)	$5.2951\ 10^{-6}$	$(\pm 0.9993, \pm 0.9993)$
10^{-4}	5	(4,4,4)	$5.3170\ 10^{-8}$	$(\pm 0.9999, \pm 0.9999)$

Example:

$p = (xy - 1)^2 + y^2$ Then, $p_{\min} = 0$ is *not* attained

ϵ	order t	(r_t, r_{t-1}, r_{t-2})	$(p_\epsilon)_t$	extracted solutions
	2	(4,2,1)	$6.8791\ 10^{-5}$	none
10^{-2}	3	(2,2,2)	0.3385	$\pm(1.3981, 0.4729)$
10^{-3}	3	(2,2,2)	0.2082	$\pm(1.9499, 0.4060)$
10^{-4}	3	(2,2,2)	0.1232	$\pm(2.6674, 0.3287)$
10^{-5}	3	(2,2,2)	0.0713	$\pm(3.6085, 0.2574)$
10^{-6}	3	(2,2,2)	0.0408	$\pm(4.8511, 0.1977)$
10^{-7}	3	(3,2,2)	0.0231	$\pm(6.4986, 0.1503)$
10^{-8}	3	(3,2,2)	0.0131	$\pm(8.6882, 0.1136)$
10^{-9}	3	(8,4,2)	0.0074	none
10^{-10}	3	(7,4,2)	0.0041	none

Testing copositivity of a matrix M

M copositive iff $p_{\min} = 0$ for $p(x) := \sum_{i,j=1}^{n} x_i^2 x_j^2 M_{ij}$

Example 1:

$$M = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

M is copositive, e.g., since $(\sum_i x_i^2)p$ is SOS [Parrilo 2000]

ϵ	order t	(r_t, r_{t-1}, r_{t-2})	$(p_\epsilon)_t$	extracted solutions
	2	(5,15)	$-1.4955\ 10^6$	none
10^{-2}	3	(18,6,3)	-1.5407	none
10^{-2}	4	(4,4,4)	$1.3854\ 10^{-5}$	$(\pm 0.7058, 0, 0, \pm 0.7058)$
10^{-4}	4	(9,7,5)	$1.5544\ 10^{-7}$	none

Example 2:

$$M = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

ϵ	order t	(r_t, r_{t-1}, r_{t-2})	$(p_\epsilon)_t$	extracted solutions
1	3	(8,7,4)	-1.3333	none
1	4	(8,8,7)	-1.3333	$\pm(0.8165, 0.8165, 0.8165, 0, 0)$
10^{-1}	3	(8,7,4)	-133.3333	none
10^{-1}	4	(8,8,7)	-133.3333	$\pm(2.5820, 2.5820, 2.5820, 0, 0)$
10^{-2}	3	(8,7,4)	$-1.3333\ 10^4$	none

$\rightsquigarrow$ certificate that M is *not* copositive

Approximating the stability number via matrix copositivity and Pólya's representation result

[de Klerk-Pasechnik 2002] [Parrilo 2000]

$$M \textit{ strictly copositive}, \text{ i.e., } p_M := \sum_{i,j=1}^{n} x_i^2 x_j^2 M_{ij} > 0 \text{ on } \mathbb{R}^n \setminus \{0\}$$

$$\overset{\text{[Pólya]}}{\Longrightarrow} (\sum_i x_i^2)^r p_M \text{ is SOS for some } r \in \mathbb{N}$$

$$\Longrightarrow M \text{ copositive}$$

[Motzkin-Straus 1965]

$$\alpha(G) = \min\ t \text{ s.t. } \ t(I + A_G) - J \text{ is copositive}$$

Define: $\vartheta^{(r)}(G) := \min\ t \text{ s.t. } t(I + A_G) - J \in \mathcal{K}^{(r)}$

where $\mathcal{K}^{(r)} := \{M \mid (\sum_i x_i^2)^r p_M \text{ is SOS}\}$

Theorem: [dKP] $\lfloor \vartheta^{(r)}(G) \rfloor = \alpha(G)$ if $r \geq \alpha(G)^2$

Conjecture: [dKP] $\vartheta^{(r)}(G) = \alpha(G)$ if $r \geq \alpha(G) - 1$

Motivation: this rate of convergence holds for other hierarchies, e.g., by Lovász-Schrijver, Lasserre

Theorem: [Gvozdenović-L 2004] True for $\alpha(G) \leq 8$

Lasserre's hierarchy refines the hierarchy $\vartheta^{(r)}(G)$

Other results by [Peña-Vera-Zuluaga 2005]

Yet another representation result

Theorem: [Lasserre 2004]
$p \geq 0$ on $\mathbb{R}^n \implies \forall \epsilon > 0 \; \exists r \in \mathbb{N}$ for which

$$p_{r,\epsilon} := p + \epsilon \left(\sum_{k=0}^{r} \sum_{i=1}^{n} \frac{x_i^{2k}}{k!} \right) \text{ is SOS}$$

Note: $\|p_{r,\epsilon} - p\|_1 \to 0$ as $\epsilon \searrow 0$

Although there are *much more nonnegative polynomials than SOS polynomials* (at fixed degree, when $n \to \infty$) [Blekherman 2003]

$\rightsquigarrow$ new relaxation scheme for $p_{\min} = \inf_{x \in \mathbb{R}^n} p(x)$

$$\inf \; y^T p_{r,\epsilon} \;\; \text{s.t.} \; M_r(y) \succeq 0, \; y_0 = 1$$

$\rightsquigarrow$ extension to the constrained case

Many other interesting results unfortunately not covered here ....

- *SDP versus LP relaxations* [Sherali-Adams] [Lasserre]

- *Exploiting the symmetry to reduce the size of the SDP*

[Parrilo-Gaterman] [Schrijver] [Dukanovic-Rendl]
[de Klerk-Pasechnik-Schrijver] [Laurent] ...

$\rightsquigarrow$ invariant theory, block-diagonalization of C^*-algebras

- *Exploiting the sparsity of the polynomials to reduce the size of the SDP*

[Reznick 1978] [Kim-Kojima-Muramatsu-Waki] ...

- *Extension to matrix polynomial optimization*

[Kojima] [Hol-Scherer] [Henrion-Lasserre] ...