Quantum Computing Exercises # 1

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(to be handed in at or before the start of the lecture on Feb 8)

1. Is the controlled-NOT operation C Hermitian?
   Determine C^{-1}.

2. Compute the result of applying a Hadamard transform to both qubits of $|0\rangle \otimes |1\rangle$ in two ways (the first way using tensor product of vectors, the second using tensor product of matrices), and show that the two results are equal:
$$H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle).$$

3. Show that a bit-flip operation, preceded and followed by Hadamard transforms, equals a phase-flip operation: $HXH = Z$.

4. A matrix A is *inner product-preserving* if the inner product $\langle Av|Aw\rangle$ between Av and Aw equals the inner product $\langle v|w\rangle$, for all vectors v, w. A is *norm-preserving* if $\| Av \| = \| v \|$ for all vectors v, i.e., A preserves the Euclidean length of the vector. A is *unitary* if $A^*A = AA^* = I$.

   In the following, you may assume for simplicity that the entries of the vectors and matrices are real, not complex.

   (a) Prove that A is norm-preserving if, and only if, A is inner product-preserving.
   (b) Prove that A is inner product-preserving iff $A^*A = AA^* = I$.
   (c) Conclude that A is norm-preserving iff A is unitary.

   Bonus: prove the same for complex instead of real vector spaces.

5. Suppose Alice and Bob are not entangled. If Alice sends a qubit to Bob, then this can give Bob at most one bit of information about Alice (this is actually a deep statement, a special case of *Holevo's theorem*). However, if they share an EPR-pair, they can transmit *two* classical bits by sending one qubit over the channel. This exercise will show how.

   (a) They start with a shared EPR-pair, $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice has classical bits a and b. Suppose she does an X-operation on her half of the EPR-pair if $a = 1$, and then a Z-operation if $b = 1$ (she does both if $ab = 11$, and neither if $ab = 00$). Write the resulting 2-qubit state.
   (b) Suppose Alice sends her half of the state to Bob, who now has two qubits. Show that Bob can determine both a and b from his state. Write Bob's operation as a quantum circuit with Hadamard and CNOT gates.