Quantum Computing Exercises # 4

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(to be handed in before or at the start of the lecture on Mar 1)

1. For $\omega = e^{2\pi i/3}$ and $F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$, calculate $F_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $F_3 \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$

2. Prove that the Fourier coefficients of the convolution of vectors a and b are the product of the Fourier coefficients of a and b. In other words, prove that for every $a, b \in \mathbb{R}^N$ and every $\ell \in \{0, \ldots, N-1\}$ we have $\left(\widehat{a * b}\right)_\ell = \widehat{a}_\ell \cdot \widehat{b}_\ell$. Here the Fourier transform $\widehat{a}$ is defined as the vector $F_N a$, and the ℓ-entry of the convolution-vector $a * b$ is $(a * b)_\ell = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} a_j b_{\ell - j \bmod N}$.

3. The Euclidean distance between two states $|\phi\rangle = \sum_i \alpha_i |i\rangle$ and $|\psi\rangle = \sum_i \beta_i |i\rangle$ is defined as $\| \, |\phi\rangle - |\psi\rangle \, \| = \sqrt{\sum_i |\alpha_i - \beta_i|^2}$. Assume the states are unit vectors with (for simplicity) real amplitudes. Suppose the distance is small: $\| \, |\phi\rangle - |\psi\rangle \, \| = \epsilon$. Show that then the probabilities resulting from a measurement on the two states are also close: $\sum_i \left| \alpha_i^2 - \beta_i^2 \right| \leq 2\epsilon$.
Hint: use $|\alpha_i^2 - \beta_i^2| = |\alpha_i - \beta_i| \cdot |\alpha_i + \beta_i|$ and the Cauchy-Schwarz inequality.

4. The distance between two matrices A and B is defined as $\| A - B \| = \max_{v : \|v\| = 1} \| (A - B)v \|$
   (a) What is the distance between the 2×2 identity matrix and the phase-gate $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$?
   (b) Suppose we have a product of n-qubit unitaries $U = U_T U_{T-1} \cdots U_1$ (for instance, each U_i could be an elementary gate on a few qubits, tensored with identity on the other qubits). Suppose we drop the j-th gate from this sequence: $U' = U_T U_{T-1} \cdots U_{j+1} U_{j-1} \cdots U_1$. Show that $\| U' - U \| = \| I - U_j \|$.
   (c) Suppose we also drop the k-th unitary: $U'' = U_T U_{T-1} \cdots U_{j+1} U_{j-1} \cdots \cdots U_{k+1} U_{k-1} \cdots U_1$. Show that $\| U'' - U \| \leq \| I - U_j \| + \| I - U_k \|$. *Hint: use triangle inequality.*
   (d) Give a quantum circuit with $O(n \log n)$ elementary gates that has distance less than $1/n$ from the Fourier transform F_{2^n}. *Hint: drop all phase-gates with small angles $\phi < 1/n^3$ from the $O(n^2)$-gate circuit for F_{2^n} explained in the lecture. Calculate how many gates there are left in the circuit, and analyze the distance between the unitaries corresponding to the new circuit and the original circuit.*

5. Suppose $a \in \mathbb{R}^N$ is a vector which is r-periodic in the following sense: there exists an integer r such that $a_j = 1$ whenever j is an integer multiple of r, and $a_j = 0$ otherwise. Compute the Fourier transform $F_N a$ of this vector. Assuming $0 \ll r \ll N$, where (i.e., around which entries) are the entries with largest magnitude in the vector $F_N a$?