We next argue that Nash flows always exist and that their cost is unique. In order to do so, we use a powerful result from convex optimization. Consider the following program (CP):

\[
\begin{aligned}
\min & \sum_{a \in A} h_a(f_a) \\
\text{s.t.} & \sum_{P \in \mathcal{P}_i} f_P = r_i \quad \forall i \in [k] \\
& f_a = \sum_{P \in \mathcal{P}, a \in P} f_P \quad \forall a \in A \\
& f_P \geq 0 \quad \forall P \in \mathcal{P}.
\end{aligned}
\]

Note that the set of all feasible solutions for (CP) corresponds exactly to the set of all flows that are feasible for our selfish routing instance \((G, r, \ell)\). The above program is a linear program if the functions \((h_a)_{a \in A}\) are linear. (CP) is a convex program if the functions \((h_a)_{a \in A}\) are convex. A convex program can be solved efficiently by using, e.g., the ellipsoid method. The following is a fundamental theorem in convex (or, more generally, non-linear) optimization:

**Theorem 1.1** (Karush–Kuhn–Tucker optimality conditions). Consider the program (CP) with continuously differentiable and convex functions \((h_a)_{a \in A}\). A feasible flow \(f\) is an optimal solution for (CP) if and only if

\[
\forall i \in [k], \forall P, Q \in \mathcal{P}_i, f_P > 0: \quad h'_P(f) := \sum_{a \in P} h'_a(f_a) \leq \sum_{a \in Q} h'_a(f_a) =: h'_Q(f), \tag{4}
\]

where \(h'_a(x)\) refers to the first derivative of \(h_a(x)\).

Observe that (4) is very similar to the Wardrop equilibrium conditions (3). In fact, these two conditions coincide if we define for every \(a \in A\):

\[
h_a(f_a) := \int_0^{f_a} \ell_a(x) dx. \tag{5}
\]

**Corollary 1.2.** Let \((G, r, \ell)\) be a selfish routing instance with nondecreasing and continuous latency functions \((\ell_a)_{a \in A}\). A feasible flow \(f\) is a Nash flow if and only if it is an optimal solution to (CP) with functions \((h_a)_{a \in A}\) as defined in (5).

**Proof.** For every arc \(a \in A\), the function \(h_a\) is convex (since \(\ell_a\) is nondecreasing) and continuously differentiable (since \(\ell_a\) is continuous). The proof now follows from Theorem 1.1. \qed

**Corollary 1.3.** Let \((G, r, \ell)\) be a selfish routing instance with nondecreasing and continuous latency functions \((\ell_a)_{a \in A}\). Then a Nash flow \(f\) always exists. Moreover, its cost \(C(f)\) is unique.
Proof. The set of all feasible flows for (CP) is compact (closed and bounded). Moreover, the objective function of (CP) with (5) is continuous (since \( \ell_a \) is continuous for every \( a \in A \)). Thus, the minimum of (CP) must exist (by the extreme value theorem of Weierstraß). Since the objective function of (CP) is convex, the optimal value of (CP) is unique. It is not hard to conclude that the cost \( C(f) \) of a Nash flow is unique. \( \square \)

Note that, in particular, the above observations imply that we can compute a Nash flow for a given nonatomic selfish routing instance \((G, r, \ell)\) efficiently by solving the convex program (CP) with (5).

1.3 Optimal Flow

We define an optimal flow as follows:

**Definition 1.4.** A feasible flow \( f^* \) for the instance \((G, r, \ell)\) is an optimal flow if \( C(f^*) \leq C(x) \) for every feasible flow \( x \).

The set of optimal flows corresponds to the set of all optimal solutions to (CP) if we define for every arc \( a \in A \):

\[
h_a(f_a) := \ell_a(f_a)f_a.
\]

Since the cost function \( C \) is continuous (because \( \ell_a \) is continuous for every \( a \in A \)), we conclude that an optimal flow always exists (again using the extreme value theorem by Weierstraß). Moreover, we will assume that \( h_a \) is convex and continuously differentiable for each arc \( a \in A \); latency functions \((\ell_a)_{a \in A}\) that satisfy these conditions are called standard.

Using Theorem 1.1, we obtain the following characterization of optimal flows:

**Corollary 1.4.** Let the latency functions \((\ell_a)_{a \in A}\) be standard. A feasible flow \( f^* \) for the instance \((G, r, \ell)\) is an optimal flow if and only if:

\[
\forall i \in [k], \forall P, Q \in \mathcal{P}, f_P > 0 : \sum_{a \in P} \ell_a(f_a^*) + \ell'_a(f_a^*)f_a^* \leq \sum_{a \in Q} \ell_a(f_a^*) + \ell'_a(f_a^*)f_a^*.
\]

That is, an optimal flow is a Nash flow with respect to so-called marginal latency functions \((\ell'_a)_{a \in A}\), which are defined as

\[
\ell'_a(x) := \ell_a(x) + \ell'_a(x)x.
\]

1.4 Price of Anarchy

We study the inefficiency of Nash flows in comparison to an optimal flow. A common measure of the inefficiency of equilibrium outcomes is the price of anarchy.

**Definition 1.5.** Let \((G, r, \ell)\) be an instance of the selfish routing game and let \( f \) and \( f^* \) be a Nash flow and an optimal flow, respectively. The price of anarchy \( \rho(G, r, \ell) \) of the instance \((G, r, \ell)\) is defined as:

\[
\rho(G, r, \ell) = \frac{C(f)}{C(f^*)}.
\]
(Note that (7) is well-defined since the cost of Nash flows is unique.) The price of anarchy of a set of instances \( \mathcal{I} \) is defined as

\[
\rho(\mathcal{I}) = \sup_{(G, r, \ell) \in \mathcal{I}} \rho(G, r, \ell).
\]

### 1.5 Upper Bounds on the Price of Anarchy

Subsequently, we derive upper bounds on the price of anarchy for selfish routing games. The following variational inequality will turn out to be very useful.

**Lemma 1.1** (Variational inequality). A feasible flow \( f \) for the instance \((G, r, \ell)\) is a Nash flow if and only if it satisfies that for every feasible flow \( x \):

\[
\sum_{a \in A} \ell_a(f_a)(f_a - x_a) \leq 0. \tag{8}
\]

**Proof.** Given a flow \( f \) satisfying (8), we first show that condition (3) of Definition 1.3 holds. Let \( P, Q \in \mathcal{P}_i \) be two paths for some commodity \( i \in [k] \) such that \( \delta := \delta_P > 0 \). Define a flow \( x \) as follows:

\[
x_a := \begin{cases} 
    f_a & \text{if } a \in P \cap Q \text{ or } a \notin P \cup Q, \\
    f_a - \delta & \text{if } a \in P, \\
    f_a + \delta & \text{if } a \in Q.
\end{cases}
\]

By construction \( x \) is feasible. Hence, from (8) we obtain:

\[
\sum_{a \in A} \ell_a(f_a)(f_a - x_a) = \sum_{a \in P} \ell_a(f_a)(f_a - (f_a - \delta)) + \sum_{a \in Q} \ell_a(f_a)(f_a - (f_a + \delta)) \leq 0.
\]

We divide the inequality by \( \delta > 0 \), which yields the Wardrop conditions (3).

Now assume that \( f \) is a Nash flow. By Corollary 1.1, we have for every \( i \in [k] \) and \( P \in \mathcal{P}_i \) with \( f_P > 0 \): \( \ell_P(f) = c_i(f) \). Furthermore, for \( Q \in \mathcal{P}_i \) with \( f_Q = 0 \), we have \( \ell_Q(f) \geq c_i(f) \). It follows that for every feasible flow \( x \):

\[
\sum_{a \in A} \ell_a(f_a) f_a = \sum_{i \in [k]} \sum_{P \in \mathcal{P}_i} c_i(f) f_P = \sum_{i \in [k]} c_i(f) \left( \sum_{P \in \mathcal{P}_i} f_P \right) = \sum_{i \in [k]} c_i(f) \left( \sum_{P \in \mathcal{P}_i} x_P \right) \\
= \sum_{i \in [k]} \sum_{P \in \mathcal{P}_i} c_i(f) x_P \leq \sum_{i \in [k]} \sum_{P \in \mathcal{P}_i} \ell_P(f) x_P = \sum_{a \in A} \ell_a(f_a) x_a.
\]

\[\square\]