We derive an upper bound on the price of anarchy for affine linear latency functions with nonnegative coefficients:

\[ L_1 := \{ g : \mathbb{R}_+ \to \mathbb{R}_+ : g(x) = q_1 x + q_0 \text{ with } q_0, q_1 \in \mathbb{R}_+ \}. \]

**Theorem 1.2.** Let \((G, r, \ell)\) be an instance of a nonatomic routing game with affine linear latency functions \((\ell_a)_{a \in A} \in L_1^A\). The price of anarchy \(\rho(G, r, \ell)\) is at most \(\frac{4}{3}\).

**Proof.** Let \(f\) be a Nash flow and let \(x\) be an arbitrary feasible flow for \((G, r, \ell)\). Using the variational inequality (8), we obtain

\[
C(f) = \sum_{a \in A} \ell_a(f_a) f_a \leq \sum_{a \in A} \ell_a(f_a) x_a = \sum_{a \in A} \ell_a(f_a) x_a + \ell_a(x_a) x_a - \ell_a(x_a) x_a \\
= \sum_{a \in A} \ell_a(x_a) x_a + [\ell_a(f_a) - \ell_a(x_a)] x_a = \sum_{a \in A} \ell_a(x_a) x_a + \sum_{a \in A} W_a(f_a, x_a).
\]

We next bound the function \(W_a(f_a, x_a)\) in terms of \(\omega \cdot \ell_a(f_a) f_a\) for some \(0 \leq \omega < 1\), where

\[ \omega := \max_{f_a, x_a \geq 0} \frac{(\ell_a(f_a) - \ell_a(x_a)) x_a}{\ell_a(f_a) f_a} = \max_{f_a, x_a \geq 0} \frac{W_a(f_a, x_a)}{\ell_a(f_a) f_a}. \]

Note that for \(x_a \geq f_a\) we have \(\omega \leq 0\) (because latency functions are non-decreasing). Hence, we can assume \(x_a \leq f_a\). See Figure 3 for a geometric interpretation. Since latency functions are affine linear, \(\omega\) is upper bounded by \(\frac{1}{4}\). We obtain

\[ C(f) \leq C(x) + \frac{1}{4} \sum_{a \in A} \ell_a(f_a) f_a = C(x) + \frac{1}{4} C(f). \]

Rearranging terms and letting \(x\) be an optimal flow concludes the proof. \(\square\)

We can extend the above proof to more general classes of latency functions. For the latency function \(\ell_a\) of an arc \(a \in A\), define

\[ \omega(\ell_a) := \sup_{f_a, x_a \geq 0} \frac{(\ell_a(f_a) - \ell_a(x_a)) x_a}{\ell_a(f_a) f_a}. \]

We assume by convention \(0/0 = 0\). See Figure 4 for a graphical illustration of this value. For a given class \(L\) of non-decreasing latency functions, we define

\[ \omega(L) := \sup_{\ell_a \in L} \omega(\ell_a). \]
Theorem 1.3. Let \((G, r, \ell)\) be an instance of the nonatomic selfish routing game with latency functions \((\ell_a)_{a \in A} \in \mathcal{L}^A\). Let \(0 \leq \omega(\mathcal{L}) < 1\) be defined as above. The price of anarchy \(\rho(G, r, \ell)\) is at most \((1 - \omega(\mathcal{L}))^{-1}\).

Proof. Let \(f\) be a Nash flow and let \(x\) be an arbitrary feasible flow. We have

\[
C(f) = \sum_{a \in A} \ell_a(f_a) f_a \leq \sum_{a \in A} \ell_a(f_a) x_a = \sum_{a \in A} \ell_a(f_a) x_a + \ell_a(x_a) x_a - \ell_a(x_a) x_a = \sum_{a \in A} \ell_a(x_a) x_a + [\ell_a(f_a) - \ell_a(x_a)] x_a \leq C(x) + \omega(\mathcal{L}) C(f).
\]

Here, the first inequality follows from the variational inequality (8). The last inequality follows from the definition of \(\omega(\mathcal{L})\). Since \(\omega(\mathcal{L}) < 1\), the claim follows.

In general, we define \(\mathcal{L}_d\) as the set of latency functions \(g : \mathbb{R}_+ \to \mathbb{R}_+\) that satisfy

\[
g(\mu x) \geq \mu^d g(x) \quad \forall \mu \in [0, 1].
\]

Note that \(\mathcal{L}_d\) contains polynomial latency functions with nonnegative coefficients and degree at most \(d\).
\[
\frac{d}{\rho(G,r,\ell)} \approx 1.333 \approx 1.626 \approx 1.896 \ldots
\]

Table 1: The price of anarchy for polynomial latency functions of degree \(d\).

**Lemma 1.2.** Consider latency functions in \(L_d\). Then

\[
\omega(L_d) \leq \frac{d}{(d+1)^{(d+1)/d}}.
\]

**Proof.** Recall the definition of \(\omega(\ell_a)\):

\[
\omega(\ell_a) = \sup_{f_a, x_a \geq 0} \frac{(\ell_a(f_a) - \ell_a(x_a))x_a}{\ell_a(f_a)f_a}.
\]

We can assume that \(x_a \leq f_a\) since otherwise \(\omega(\ell_a) \leq 0\). Let \(\mu := \frac{x_a}{f_a} \in [0,1]\). Then

\[
\omega(\ell_a) = \max_{\mu \in [0,1], f_a \geq 0} \left( \frac{(\ell_a(f_a) - \ell_a(\mu f_a))\mu f_a}{\ell_a(f_a)f_a} \right) \leq \max_{\mu \in [0,1], f_a \geq 0} \left( \frac{(\ell_a(f_a) - \mu^d \ell_a(f_a))\mu f_a}{\ell_a(f_a)f_a} \right)
\]

\[
= \max_{\mu \in [0,1]} (1 - \mu^d)\mu. \tag{11}
\]

Here, the first inequality holds since \(\ell_a \in L_d\). Since this is a strictly convex program, the unique global optimum is given by

\[
\mu^* = \left( \frac{1}{d+1} \right)^{\frac{1}{d}}.
\]

Replacing \(\mu^*\) in (11) yields the claim. \(\square\)

**Theorem 1.4.** Let \((G, r, \ell)\) be an instance of a nonatomic routing game with latency functions \((\ell_a)_{a \in A} \in L_d^A\). The price of anarchy \(\rho(G, r, \ell)\) is at most

\[
\rho(G, r, \ell) \leq \left( \frac{1 - \frac{d}{(d+1)^{(d+1)/d}}} \right)^{-1}.
\]

**Proof.** The theorem follows immediately from Theorem 1.3 and Lemma 1.2. \(\square\)

The price of anarchy for polynomial latency functions with nonnegative coefficients and degree \(d\) is given in Table 1 for small values of \(d\).

### 1.6 Lower Bounds on the Price of Anarchy

We can show that the bound that we have derived in the previous section is actually tight.

**Theorem 1.5.** Consider nonatomic selfish routing games with latency functions in \(L_d\). There exist instances such that the price of anarchy is at least

\[
\left( 1 - \frac{d}{(d+1)^{(d+1)/d}} \right)^{-1}.
\]
Proof. See Exercise 1 of Assignment 1. □

1.7 Bicriteria Results

Theorem 1.6. Let \((G, r, \ell)\) be a nonatomic selfish routing instance. The cost of a Nash flow for \((G, r, \ell)\) is at most the cost of an optimal flow for the instance \((G, 2r, \ell)\).

Proof. Let \(f\) be a Nash flow for the instance \((G, r, \ell)\) and let \(x\) be an optimal flow for the instance \((G, 2r, \ell)\). Note that the flow \(\frac{1}{2}x\) is feasible for \((G, r, \ell)\). Using the variational inequality (8), we derive (similar as in the proof of Theorem 1.3)

\[
C(f) = \sum_{a \in A} \ell_a(f_a)f_a \leq \sum_{a \in A} \ell_a(f_a) \cdot \frac{1}{2}x_a \leq \frac{1}{2} \left( C(x) + \omega(L)C(f) \right).
\]

Observe that \(\omega(L) \leq 1\) which implies that \(C(f) \leq C(x)\). □