2 Potential Games

In this section, we consider so-called potential games which constitutes a large class of strategic games having some nice properties. We will address issues like the existence of pure Nash equilibria, price of stability, price of anarchy and computational aspects.

2.1 Connection Games

As a motivating example, we first consider the following connection game.

**Definition 2.1.** A connection game \( \Gamma = (G, (c_a)_{a \in A}, N, (s_i, t_i)_{i \in N}) \) is given by

- a directed graph \( G = (V, A) \);
- non-negative arc costs \( c : A \to \mathbb{R}_+ \);
- a set of players \( N := [n] \);
- for every player \( i \in N \) a terminal pair \( (s_i, t_i) \in V \times V \).

The goal of each player \( i \in N \) is to connect his terminal vertices \( s_i, t_i \) by buying a directed path \( P_i \) from \( s_i \) to \( t_i \) at smallest possible cost. Let \( S = (P_1, \ldots, P_n) \) be the paths chosen by all players. The cost of an arc \( a \in A \) is shared equally among the players that use this arc. That is, the total cost that player \( i \) experiences under strategy profile \( S \) is

\[
c_i(S) := \sum_{a \in P_i} \frac{c_a}{n_a(S)},
\]

where

\[
n_a(S) = |\{i \in N : a \in P_i \}|.
\]

Let \( A(S) \) be the set of arcs that are used with respect to \( S \), i.e., \( A(S) := \cup_{i \in N} P_i \). The social cost of a strategy profile \( S \) is given by the sum of all arc costs used by the players:

\[
C(S) := \sum_{a \in A(S)} c_a = \sum_{i \in N} c_i(S).
\]

**Example 2.1.** Consider the connection game in Figure 7 (a). There are two Nash equilibria: One in which all players choose the left arc and one in which all players choose the right arc. Certainly, the optimal solution is to assign all players to the left arc. The example shows that the price of anarchy can be as large as \( n \).
Figure 7: Examples of connection games showing that (a) Nash equilibria are not unique and (b) the price of stability is at least $H_n$.

**Example 2.2.** Consider the connection game in Figure 7 (b). Here the unique Nash equilibrium is that every player uses his direct arc to the target vertex. The resulting cost is

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

which is called the *n*-th harmonic number. ($H_n$ is about $\log(n)$ for large enough $n$.) An optimal solution allocates all players to the $1 + \varepsilon$ path. The example shows that the cost of a Nash equilibrium can be a factor $H_n$ away from the optimal cost.

Consider the following potential function $\Phi$ that maps every strategy profile $S = (P_1, \ldots, P_n)$ of a connection game to a real value:

$$\Phi(S) := \sum_{a \in A} c_a \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n_a(S)} \right) = \sum_{a \in A} c_a H_{n_a(S)}.$$

We derive some properties of $\Phi(S)$.

**Lemma 2.1.** Consider an instance $\Gamma = (G, (c_a)_{a \in A}, N, (s_i, t_i)_{i \in N})$ of the connection game. We have for every strategy profile $S = (P_1, \ldots, P_n)$:

$$C(S) \leq \Phi(S) \leq H_n C(S).$$

**Proof.** Recall that $A(S)$ refers to the set of arcs that are used in $S$. We first observe that $H_{n_a(S)} = 0$ for every arc $a \notin A(S)$ since $n_a(S) = 0$. Next observe that for every arc $a \in A(S)$ we have $c_a \leq c_a H_{n_a(S)} \leq c_a H_n$. Summing over all arcs concludes the proof.

For a given strategy profile $S = (P_1, \ldots, P_n)$ we use $(S_{-i}, P'_i)$ to refer to the strategy profile that we obtain from $S$ if player $i$ deviates to path $P'_i$, i.e.,

$$(S_{-i}, P'_i) = (P_1, \ldots, P_{i-1}, P'_i, P_{i+1}, \ldots, P_n).$$

The next lemma shows that the potential function reflects exactly the change in cost of a player if he deviates to an alternative strategy.
Lemma 2.2. Consider an instance $\Gamma = (G, (c_a)_{a \in A}, N, (s_i, t_i)_{i \in N})$ of the connection game and let $S = (P_1, \ldots, P_n)$ be a strategy profile. Fix a player $i \in N$ and let $P'_i \neq P_i$ be an alternative $s_i, t_i$-path. Consider the strategy profile $S' = (S_{-i}, P'_i)$ that we obtain if player $i$ deviates to $P'_i$. Then
\[ \Phi(S') - \Phi(S) = c_i(S') - c_i(S) \]

Proof. Note that for every $a \notin P_i \cup P'_i$ we have $n_a(S') = n_a(S)$. Moreover, for every $a \in P_i \cap P'_i$ we have $n_a(S') = n_a(S)$.

We thus have
\[ \Phi(S') - \Phi(S) = \sum_{a \in A} c_a H_{n_a(S')} - \sum_{a \in A} c_a H_{n_a(S)} \]
\[ = \sum_{a \in P_i \setminus P'_i} c_a (H_{n_a(S')} - H_{n_a(S)}) - \sum_{a \in P'_i \setminus P_i} c_a (H_{n_a(S')}) - \sum_{a \in P'_i \setminus P_i} c_a (H_{n_a(S')}) - \sum_{a \in P_i \setminus P'_i} c_a (H_{n_a(S)}) \]
\[ = \sum_{a \in P_i \setminus P'_i} c_a n_a(S) + 1 - \sum_{a \in P'_i \setminus P_i} c_a n_a(S) = c_i(S') - c_i(S). \]

We will see in the next section that the above two lemmas imply the following theorem.

Theorem 2.1. Let $\Gamma = (G, (c_a)_{a \in A}, N, (s_i, t_i)_{i \in N})$ be an instance of the connection game. Then $\Gamma$ has a pure Nash equilibrium and the price of stability is at most $H_n$, where $n$ is the number of players.

2.2 Potential games

The above connection game is a special case of the general class of potential games, which we formalize next.

Definition 2.2. A finite strategic game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is given by

- $N$ is the finite set of players;
- for every player $i \in N$ a finite set of strategies $X_i$;
- for every player $i \in N$ a utility function $u_i : X \to \mathbb{R}$ which maps every strategy profile $x \in X := \times_{i \in N} X_i$ to a real-valued utility $u_i(x)$ of player $i$.

The goal of every player is to choose a strategy $x_i \in X_i$ so as to maximize his own utility $u_i(x)$.

A strategy profile $x = (x_1, \ldots, x_n) \in X$ is a pure Nash equilibrium if for every player $i \in N$ and every strategy $y_i \in X_i$, we have
\[ u_i(x) \geq u_i(x_{-i}, y_i). \]

Here $x_{-i}$ denotes the strategy profile $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ excluding player $i$. Moreover, $(x_{-i}, y_i) = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ refers to the strategy profile that we obtain from $x$ if player $i$ deviates to strategy $y_i$.  

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Algorithmus 1 IMPROVING MOVES

Input: arbitrary strategy profile \( x \in X \)

Output: Nash equilibrium \( x^* \)

1: \( x^0 := x \)
2: \( k := 0 \)
3: while \( x^k \) is not a Nash equilibrium do
4: determine a player \( i \in N \) and \( y_i \in X_i \), such that \( u_i(x^k_{-i}, y_i) > u_i(x^k) \)
5: \( x^{k+1} := (x^k_{-i}, y_i) \)
6: \( k := k + 1 \)
7: end while
8: return \( x^* := x^k \)

In general, Nash equilibria are not guaranteed to exist in strategic games. Suppose \( x \) is not a Nash equilibrium. Then there is at least one player \( i \in N \) and a strategy \( y_i \in X_i \) such that
\[
u_i(x) < u_i(x_{-i}, y_i)\]

We call the change from strategy \( x_i \) to \( y_i \) of player \( i \) an improving move.

A natural approach to determine a Nash equilibrium is as follows: Start with an arbitrary strategy profile \( x^0 = x \). As long as there is an improving move, execute this move. The algorithm terminates if no improving move can be found. Let the resulting strategy profile be denoted by \( x^* \). A formal description of the algorithm is given in Algorithm 1. Clearly, the algorithm computes a pure Nash equilibrium if it terminates.

Definition 2.3. We associate a directed transition graph \( G(\Gamma) = (V,A) \) with a finite strategic game \( \Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N}) \) as follows:

- every strategy profile \( x \in X \) corresponds to a unique node of the transition graph \( G(\Gamma) \);
- there is a directed edge from strategy \( x \) to \( y = (x_{-i}, y_i) \) in \( G(\Gamma) \) iff the change from \( x_i \) to \( y_i \) corresponds to an improving move of player \( i \in N \).

Note that the transition graph is finite since the set of players \( N \) and the strategy set \( X_i \) of every player are finite. Every directed path \( P = (x^0, x^1, \ldots) \) in the transition graph corresponds to a sequence of improving moves. We therefore call \( P \) an improvement path. We call \( x^0 \) the starting configuration of \( P \). If \( P \) is finite its last node is called the terminal configuration.

Definition 2.4. A strategic game \( \Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N}) \) has the finite improvement property (FIP) if every improvement path in the transition graph \( G(\Gamma) \) is finite.

Consider the execution of IMPROVING MOVES. The algorithm computes an improving path \( P = (x^0, x^1, \ldots) \) with starting configuration \( x^0 \) and is guaranteed to terminate if \( \Gamma \) has the FIP. That is, \( \Gamma \) admits a pure Nash equilibrium if it has the FIP. In order to characterize games that have the FIP, we introduce potential games.