Definition 2.5. A finite strategic game \( \Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N}) \) is called exact potential game if there exists a function (also called potential function) \( \Phi : X \to \mathbb{R} \) such that for every player \( i \in N \) and for every \( x_{-i} \in X_{-i} \) and \( x_i, y_i \in X_i \):

\[
u_i(x_{-i}, y_i) - \nu_i(x_{-i}, x_i) = \Phi(x_{-i}, x_i) - \Phi(x_{-i}, y_i),
\]

\( \Gamma \) is an ordinal potential game if for every player \( i \in N \) and for every \( x_{-i} \in X_{-i} \) and \( x_i, y_i \in X_i \):

\[
u_i(x_{-i}, y_i) - \nu_i(x_{-i}, x_i) > 0 \iff \Phi(x_{-i}, x_i) - \Phi(x_{-i}, y_i) > 0.
\]

\( \Gamma \) is a generalized ordinal potential game if for every player \( i \in N \) and for every \( x_{-i} \in X_{-i} \) and \( x_i, y_i \in X_i \):

\[
u_i(x_{-i}, y_i) - \nu_i(x_{-i}, x_i) > 0 \Rightarrow \Phi(x_{-i}, x_i) - \Phi(x_{-i}, y_i) > 0.
\]

2.2.1 Existence of Nash Equilibria

Theorem 2.2. Let \( \Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N}) \) be an ordinal potential game. The set of pure Nash equilibria of \( \Gamma \) coincides with the set of local minima of \( \Phi \), i.e., \( x \) is a Nash equilibrium of \( \Gamma \) iff

\[
\forall i \in N, \: \forall y_i \in X_i : \quad \Phi(x) \leq \Phi(x_{-i}, y_i).
\]

Proof. The proof follows directly from the definition of ordinal potential games.

Theorem 2.3. Every generalized ordinal potential game \( \Gamma \) has the FIP. In particular, \( \Gamma \) admits a pure Nash equilibrium.

Proof. Consider an improvement path \( P = (x^0, x^1, \ldots) \) in the transition graph \( G(\Gamma) \). Since \( \Gamma \) is a generalized ordinal potential game, we have

\[
\Phi(x^0) > \Phi(x^1) > \ldots
\]

Because the transition graph has a finite number of nodes, the path \( P \) must be finite. Thus, \( \Gamma \) has the FIP. The existence follows now directly from the FIP and the IMPROVING MOVES algorithm.

One can show the following equivalence (we omit the proof here).

Theorem 2.4. Let \( \Gamma \) be a finite strategic game. \( \Gamma \) has the FIP if and only if \( \Gamma \) admits a generalized ordinal potential function.
2.2.2 Price of Stability

Consider an instance $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ of a potential game and suppose we are given a social cost function $c: X \rightarrow \mathbb{R}$ that maps every strategy profile $x \in X$ to some cost $c(x)$. We assume that the global objective is to minimize $c(x)$ over all $x \in X$. (The definitions are similar if we want to maximize $c(x)$.) Let $\text{opt}(\Gamma)$ refer to the minimum cost of a strategy profile $x \in X$ and let $\text{NE}(\Gamma)$ refer to the set of strategy profiles that are Nash equilibria of $\Gamma$.

The price of stability is defined as the worst case ratio over all instances of the game of the cost of a best Nash equilibrium over the optimal cost; more formally,

$$\text{POS} := \max \min_{\Gamma} \frac{c(x)}{\text{opt}(\Gamma)}.$$ 

In contrast, the price of anarchy is defined as the worst case ratio over all instances of the game of the cost of a worst Nash equilibrium over the optimal cost; more formally,

$$\text{POA} := \max \max_{\Gamma} \frac{c(x)}{\text{opt}(\Gamma)}.$$ 

**Theorem 2.5.** Consider a potential game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ with potential function $\Phi$. Let $c: X \rightarrow \mathbb{R}_{+}$ be a social cost function. If $\Phi$ satisfies for every $x \in X$:

$$\frac{1}{\alpha}c(x) \leq \Phi(x) \leq \beta c(x)$$

for some $\alpha, \beta > 0$, then the price of stability is at most $\alpha \beta$.

**Proof.** Let $x$ be a strategy profile that minimizes $\Phi$. Then $x$ is a Nash equilibrium by Theorem 2.2. Let $x^*$ be an optimal solution of cost $\text{opt}(\Gamma)$. Note that

$$\Phi(x) \leq \Phi(x^*) \leq \beta c(x^*) = \beta \text{opt}(\Gamma).$$

Moreover, we have $c(x) \leq \alpha \Phi(x)$, which concludes the proof. \qed

2.2.3 Characterization of Exact Potential Games

We show that every exact potential game can be decomposed into a coordination game and a coordination game.

**Definition 2.6.** A strategic game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is a

- *coordination game* if there exists a function $u: X \rightarrow \mathbb{R}$ such that $u_i = u$ for every $i \in N$ (all players have the same utility function);
- *dummy game* if for every $i \in N$, every $x_{-i} \in X_{-i}$ and every $x_i, y_i \in X_i$: $u_i(x_{-i}, x_i) = u_i(x_{-i}, y_i)$ (each player’s utility is independent of his own strategy choice).

**Theorem 2.6.** Let $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ be a finite strategic game. $\Gamma$ is an exact potential game if and only if there exist functions $(c_i)_{i \in N}$ and $(d_i)_{i \in N}$ such that

- $u_i = c_i + d_i$ for all $i \in N$;
- $(N, (X_i)_{i \in N}, (c_i)_{i \in N})$ is a coordination game;
• \((N, (X_i)_{i \in \mathbb{N}}, (d_i)_{i \in \mathbb{N}})\) is a dummy game.

**Proof.** Let \((c_i)_{i \in \mathbb{N}}\) and \((d_i)_{i \in \mathbb{N}}\) satisfy the statement of the theorem. We can then define a potential function

\[
\Phi(x) := -\sum_{i \in \mathbb{N}} c_i(x).
\]

Fix an arbitrary strategy profile \(x \in X\) and a player \(i \in N\). Then for every \(y_i \in X_i\), we have

\[
u_i(x_{-i}, y_i) - u_i(x) = c_i(x_{-i}, y_i) - c_i(x) + d_i(x_{-i}, y_i) = \Phi(x) - \Phi(x_{-i}, y_i),
\]

where the last equality holds because \((N, (X_i)_{i \in \mathbb{N}}, (d_i)_{i \in \mathbb{N}})\) is a dummy game. That is, \(\Gamma\) is an exact potential game.

Let \(\Phi\) be an exact potential function for \(\Gamma\). For every player \(i \in N\) we have \(u_i(x) = (u_i(x) + \Phi(x)) - \Phi(x)\). Clearly, \((N, (X_i)_{i \in \mathbb{N}}, (-\Phi)_{i \in \mathbb{N}})\) is a coordination game. Fix some player \(i \in N\) and \(x_{-i} \in X_{-i}\). Since \(\Gamma\) is an exact potential game, we have for every \(x_i, y_i \in X_i\)

\[
u_i(x_{-i}, y_i) - u_i(x_{-i}, x_i) = \Phi(x_{-i}, x_i) - \Phi(x_{-i}, y_i)
\]

\[\Leftrightarrow\]

\[
u_i(x_{-i}, y_i) + \Phi(x_{-i}, y_i) = u_i(x_{-i}, x_i) + \Phi(x_{-i}, y_i).
\]

Thus, \((N, (X_i)_{i \in \mathbb{N}}, (u_i + \Phi)_{i \in \mathbb{N}})\) is a dummy game. \(\square\)

### 2.2.4 Computing Nash Equilibria

We next consider the problem of computing a pure Nash equilibrium of an exact potential game \(\Gamma = (N, (X_i)_{i \in \mathbb{N}}, (u_i)_{i \in \mathbb{N}})\). The question of whether or not a Nash equilibrium can be computed in polynomial time is still open. We will relate the complexity of finding Nash equilibria for potential games to the complexity of computing local optima for local search problems. In particular, we will show that the problem of computing Nash equilibria is PLS-complete, where PLS stands for polynomial local search. PLS-complete problems constitute a large class of search problems for which (so far) no polynomial time algorithms are known.

We first define local search problems:

**Definition 2.7.** A local search problem \(\Pi\) is given by

- a set of instances \(\mathcal{I}\);
- for every instance \(I \in \mathcal{I}\):
  - a set \(F(I)\) of feasible solutions;
  - a cost function \(c : F(I) \rightarrow \mathbb{Z}\) that maps every feasible solution \(S \in F(I)\) to some value \(c(S)\);
  - for every feasible solution \(S \in F(I)\), a neighborhood \(N(S, I) \subseteq F(I)\) of \(S\).

The goal is to find a feasible solution \(S \in F(I)\) that is a local minimum, i.e., \(c(S) \leq c(S')\) for every \(S' \in N(S, I)\).

We associate a transition graph with an instance \(I \in \mathcal{I}\) of a local search problem \(\Pi\): Every solution \(S \in F(I)\) corresponds to a unique node \(v(S)\) and there is a directed arc from \(v(S_1)\) to \(v(S_2)\) if and only if \(S_2 \in N(S_1, I)\) and \(c(S_2) < c(S_1)\). The sinks of this graph are the local optima of \(\Pi\).
The problem of finding a Nash equilibrium of an exact potential game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ with potential function $\Phi$ can be formulated naturally as a local search problem: The set of feasible solutions corresponds to the set of possible strategy profiles $X$ and the objective function is the potential function $\Phi$. The neighborhood of a strategy profile $x \in X$ refers to the set of all possible strategy profiles that are obtainable from $x$ by single-player deviations. The local minima of the resulting local search problem corresponds exactly to the set of Nash equilibria of $\Gamma$ (see Theorem 2.2). Note that the transition graph $G(\Gamma)$ as defined in Definition 2.3 coincides with the transition graph of the resulting local search problem.

**Definition 2.8.** A local search problem $\Pi$ belongs to the complexity class $\text{PLS}$ (polynomial local search) if the following can be done in polynomial time for every given instance $I \in \mathcal{I}$:

- compute an initial feasible solution $S \in F(I)$;
- compute the objective value $c(S)$ for every solution $S \in F(I)$;
- determine for every feasible solution $S \in F(I)$ whether $S$ is locally optimal or not and, if not, find a better solution $S'$ in the neighborhood of $S$, i.e., some $S' \in N(S, I)$ with $c(S') < c(S)$.

It is not hard to see that the problem of computing a Nash equilibrium for potential games is in PLS.

We next define the concept of $\text{PLS-reducibility}$ (see also Figure 8):

**Definition 2.9.** Let $\Pi_1 = (\mathcal{I}_1, F_1, c_1, N_1)$ and $\Pi_2 = (\mathcal{I}_2, F_2, c_2, N_2)$ be two local search problems in PLS. $\Pi_1$ is $\text{PLS-reducible}$ to $\Pi_2$ if there are two polynomial time computable functions $f$ and $g$ such that

- $f$ maps every instance $I \in \mathcal{I}_1$ of $\Pi_1$ to an instance $f(I) \in \mathcal{I}_2$ of $\Pi_2$;
- $g$ maps every tuple $(S_2, I)$ with $S_2 \in F_2(f(I))$ to a solution $S_1 \in F_1(I)$;
- for all $I \in \mathcal{I}_1$: if $S_2$ is a local optimum of $f(I)$, then $g(S_2, I)$ is a local optimum of $I$. 

Figure 8: Illustration of PLS-reduction. A polynomial time algorithm for solving $\Pi_2$ gives rise to a polynomial time algorithm for solving $\Pi_1$. 

\[
\begin{align*}
\Pi_1: & \quad I \in \mathcal{I}_1 \\
\text{PLS-reducible} & \quad f \\
\Pi_2: & \quad f(I) \in \mathcal{I}_2 \\
\text{polynomial time} & \quad \text{algorithm} \\
S_1 = g(S_2, I) & \in F_1(I) \\
S_2 & \in F_2(f(I)) \\
g \text{ (local optima preserving)}
\end{align*}
\]