Kolmogorov Complexity

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Outline

- A. N. Kolmogorov.
- Kolmogorov Complexity
- Kolmogorov Complexity and Data Compression

Andrey Nikolaevich Kolmogorov

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The Notion of Information

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Example

$X = 111111111111111111111111111111111111111111$
$Y = 100101011101010011001000101010010111101010$

X carries as much information as "42 1s" while there is no immediate short description of Y.

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A description wraps all the information of a string. The minimal description length of a string is thus a measurement of its complexity, or the information it carries.

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$$K(x) := min_{p:\mathcal{U}(p)=x} l(p)$$

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Kolmogorov complexity is also referred to as "absolute information" in comparison with entropy.

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- Any initial segment of an algebraic number has low Kolmogorov complexity. e.g., first million digits of the greatest root of $x^2 = 2$.
- The Kolmogorov Complexity of a string that consists of n ones is at most $\log n + c$.
- Losslessly compressed data can be seen as a description.

Kolmogorov Complexity III

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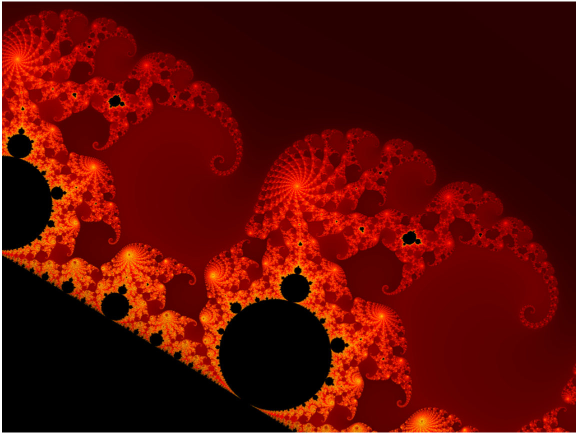

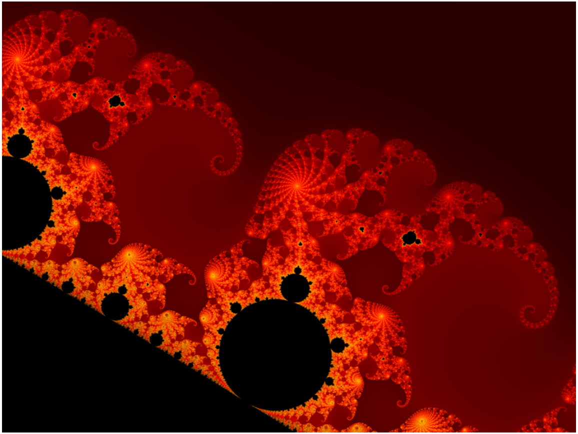

This 2.36MB fractal picture is generated by one complex polynomial. The description length of this picture is not much larger than that of the polynomial.

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$$K(x) \leq l(x) + c$$

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Proof.

Consider program "print string x". □

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Theorem (Lower Bound)

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Proof.

There are only $2^k - 1$ descriptions of length less than k. □

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For each n, at least one string of length n is of Kolmogorov Complexity at least n.

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A string is random, if it cannot be compressed. Intuitively, 111111111111111111 is not random, while 100101001011110110 is sort of random.

Randomness

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*For each n, at least one string of length n is of Kolmogorov Complexity at least n. This string is **not compressible**.*

A string is random, if it cannot be compressed. Intuitively, 1111111111111111111 is not random, while 10010100101111011010 is sort of random.

Definition (Random String)

x is **random** if $K(x) \geq l(x)$.

Kolmogorov Complexity and Data Compression II

The probability that a uniformly random binary string has low Kolmogorov complexity is fairly small:

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Let $X_i \sim$ *iid* $B(\frac{1}{2})$, *then*

$$P[K(X^n) < n - k] < 2^{-k}$$

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This is really a tiny fragment: for files of size $100kB$, only 2^{-8192} of them can be possibly compressed by $1kB$, and $2^{-409600}$ of them can be possibly compressed to half its original size. Compression softwares are, nevertheless, running happily, because the files we use them to compress, luckily, belongs to this tiny fragment.

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- By far the majority of the strings have their Kolmogorov complexity close to their length, thus cannot be compressed much. Compression is possible in practice because our files lie in the tiny bit that is compressible.

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- Philosophical thoughts. Occam's Razor.
- Worship Andrey N. Kolmogorov. http://www.kolmogorov.com/. http://theor.jinr.ru/~kuzemsky/ankolmogbio.html.

Thanks for your attention.