Stochastic Realization of σ-Algebras

Jan H. van Schuppen

CWI

P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

J.H.van.Schuppen@cwi.nl

Keywords Stochastic realization, σ-algebraic system, conditional independence relation.

Abstract

Stochastic realization problems are motivated by control and signal processing for stochastic systems. A stochastic system is defined by the property that at each time the current state makes the future and the past of the output and state processes conditionally independent. The stochastic realization problem for a σ-algebra family is then to construct the state σ-algebra family such that the families jointly satisfy the definition of a stochastic system. The first result is a characterization of minimal σ-algebra which make two σ-algebras minimally conditionally independent. The second result allows the construction of state σ-algebra families which together with a considered output σ-algebra family forms a stochastic system.

1 Introduction

Results are presented for the stochastic realization problem in the setting of σ-algebras.

Problems of control and signal processing require theoretical models in the form of deterministic or of stochastic systems. Stochastic systems are described in discrete time by mappings from the state and the input to the probability distributions of the next state and the current output and in continuous time by stochastic differential equations. For the filtering problem the Kalman filter or generalizations of it are applied. For stochastic control problems there are control laws for minimum variance cost functions and for cost functions of quadratic forms in the state and the input.

As has first been pointed out by R.E. Kalman, control and filtering theory is much advanced by the restriction for those problems to those stochastic systems for which finite-dimensional filters or finite-dimensional controllers exist. At Kalman's insistence, researchers have started research on the stochastic realization problem. The weak stochastic realization problem for Gaussian processes was solved by P. Faurre, the strong stochastic realization problem for Gaussian processes by A. Lindquist, G. Picci, and G. Ruckebusch, see [2] and the references provided in that paper. See also [3] for the importance of stochastic realization for system identification.

The author of this paper has formulated the stochastic realization problem for σ-algebras and for σ-algebraic families. For publications see, [6, 8, 9, 10, 11]. The realization problem for σ-algebras was partly solved in [10]. The characterization of minimal realizations is not described in that paper. Others have carried out related research, see [4]. The solution to a special case of the stochastic realization problem for σ-algebra families was solved by W.J. Stronegger, see [5]. The interest of the stochastic realization problem for σ-algebra families is in the development of general theory for stochastic control and filtering, not only for Gaussian processes but also for finite-valued processes. Stochastic realization theory for σ-algebras is inspired by that for Hilbert spaces yet more general.

The first result of this paper is a new characterization of minimal realizations for the σ-algebraic stochastic realization problem. The solution is inspired by the corresponding problem for Hilbert spaces. The second result is a characterization of stochastic realizations of σ-algebra families. Procedures for the construction of realizations have been partly developed.

The outline of the paper follows. Preliminaries on probability theory are mentioned in Section 2. Section 3 provides results on the stochastic realization problem for σ-algebras. Section 4 provides results on the stochastic realization for σ-algebra families.

2 Preliminaries

In this section notation is introduced.

Let (Ω, F, P) denote a complete probability space consisting of a set Ω, a σ-algebra F, and a probability measure P. Let

$$\mathbf{F} = \left\{ \begin{array}{l} G \subseteq F | G\ \sigma - \text{algebra containing} \\ \text{all null sets of } F \end{array} \right\}. \quad (1)$$

If $F_1, F_2 \in \mathbf{F}$ then $F_1 \vee F_2$ denotes the smallest σ-algebra in $\mathbf{F}$ that contains both F_1 and F_2. For $G \in \mathbf{F}$ let

$$L^+(G) = \{x : \Omega \to R_+ | x \text{ is } G \text{ measurable}\}.$$

If $x : \Omega \to R^n$ is a random variable then denote by F^x the σ-algebra generated by x. Denote by $(F_1, F_2, \ldots, F_n) \in I$ that the indicated σ-algebras are independent.

Definition 2.1 *The* conditional independence relation *for a triple of σ-algebras is defined by the condition that*

$$E[x_1 x_2 | G] = E[x_1|G] E[x_2|G], \quad (2)$$

for all $x_1 \in L^+(F_1)$, $x_2 \in L^+(F_2)$, and for $F_1, F_2, G \in \mathbf{F}$. Denote this condition by $(F_1, F_2|G) \in CI$ and one says that F_1, F_2 are conditional independent given G or that G splits F_1 and F_2.

It is an easily established fact that $(F_1, F_2|G) \in CI$ iff

$$E[x_1|F_2 \vee G] = E[x_1|G], \quad (3)$$

for all $x_1 \in L^+(F_1)$, see [1, II.45]. If $F_1 \subseteq G$ then $(F_1, F_2|G) \in CI$. Also, if $(F_1, F_2 \vee G) \in I$ then $(F_1, F_2|G) \in CI$. Consider on the set of σ-algebras $\mathbf{F}$ the partial order of the inclusion relation. One says that F_1, F_2, G form a minimal triple if $(F_1, F_2|G) \in CI$ and G is minimal; specifically, if $(F_1, F_2|G) \in CI$ and for any σ-algebra $G_1 \subseteq G$, $(F_1, F_2|G_1) \in CI$ implies that $G_1 = G$. Then one writes that $(F_1, F_2|G) \in CI_{min}$. For $F_1, F_2 \in \mathbf{F}$ define

$$\begin{aligned} &\sigma(F_2|F_1) \\ &= \sigma\left(\left\{ \begin{array}{l} E[x_2|F_1] : \Omega \to R_+ | \\ \forall x_2 \in L^+(F_2) \end{array} \right\}\right), \end{aligned} \quad (4)$$

where the right-hand side of this formula denotes the σ-algebra generated by all random variables in the indicated set.

3 Stochastic realization of a pair of σ-algebras

Problem 3.1 *Consider $F_1, F_2 \in \mathbf{F}$. Construct all σ-algebras $G \in \mathbf{F}$ such that:*

$$G \subseteq (F_1 \vee F_2),\ (F_1, F_2, |G) \in CI_{min}.$$

Define

$$\begin{aligned} &\mathbf{G}(F_1, F_2) \\ &= \left\{ \begin{array}{l} G \in \mathbf{F} | G \subseteq F_1 \vee F_2, \\ (F_1, F_2|G) \in CI \end{array} \right\}, \quad (5) \\ &\mathbf{G_{min}}(F_1, F_2) \\ &= \{G \in \mathbf{G}(F_1, F_2) | (F_1, F_2|G) \in CI_{min}\}. \quad (6) \end{aligned}$$

For $F_1, F_2 \in \mathbf{F}$ define

$$F_{21} = \sigma(F_2|F_1),\ F_{12} = \sigma(F_1|F_2). \quad (7)$$

Proposition 3.2 *[8, 10]. Consider $F_1, F_2, G \in \mathbf{F}$. Assume that $(F_1, F_2|G) \in CI$ and $G \subseteq F_1 \vee F_2$. Then $G \subseteq F_{21} \vee F_{12}$ if and only if $\sigma(G|F_1) = F_{21}$ and $\sigma(G|F_2) = F_{12}$.*

Proposition 3.3 *[8, 10]. Consider $F_1, F_2, G \in \mathbf{F}$. Assume that $G \subseteq F_{21} \vee F_{12}$. Then $(F_1, F_2|G) \in CI$ if and only if $(F_{21}, F_{12}|G) \in CI$. Moreover, $(F_1, F_2|G) \in CI_{min}$ if and only if $(F_{21}, F_{12}|G) \in CI_{min}$.*

There exist $F_1, F_2, G \in \mathbf{F}$ such that $(F_1, F_2|G) \in CI$ and $G \not\subseteq F_1 \vee F_2$. There exist $F_1, F_2, G \in \mathbf{F}$ such that $(F_1, F_2|G) \in CI_{min}$, $G \subseteq F_1 \vee F_2$, and $G \not\subseteq F_{21} \vee F_{12}$. Below attention is restricted to $G \subseteq F_{21} \vee F_{12}$ and hence to $\mathbf{G_{min}}(F_{21}, F_{12})$. The characterization of minimal σ-algebras which make two given σ-algebras conditional independent is expressed best in terms of a pair of σ-algebras. The following approach is inspired by the analogous problem for Hilbert spaces. One may call the σ-algebras H_1 and H_2 the forward and the backward σ-algebras respectively.

Definition 3.4 *Consider $F_1, F_2 \in \mathbf{F}$. Define the set*

$$\begin{aligned} \mathbf{H} &= \mathbf{H}(F_1, F_2) \quad (8) \\ &= \left\{ \begin{array}{l} (H_1, H_2) \in \mathbf{F} \times \mathbf{F} | \\ F_1 \subseteq H_1 \subseteq F_1 \vee F_2, \\ F_2 \subseteq H_2 \subseteq F_1 \vee F_2, \\ H_1 \cap H_2 = \sigma(H_1|H_2) = \sigma(H_2|H_1) \end{array} \right\}. \end{aligned}$$

Define on this set the partial order

$$\begin{aligned} &(H_{11}, H_{12}) \leq (H_{21}, H_{22}), \quad (9) \\ &\textit{if } H_{11} \subseteq H_{21} \textit{ and if } H_{12} \subseteq H_{22}. \end{aligned}$$

An element $(H_1, H_2) \in \mathbf{H}$ is said to be minimal if it is minimal with respect to this partial order. Define the set

$$\begin{aligned} & \mathbf{H_{min}}(F_1, F_2) && (10)\\ = & \ \{(H_1, H_2) \in \mathbf{H}(F_1, F_2) | (H_1, H_2) \textit{minimal}\}. \end{aligned}$$

Define the map

$$\begin{aligned} & f : \mathbf{H}(F_1, F_2) \to \mathbf{G}(F_1, F_2), \\ f(H_1, H_2) \ = \ & H_1 \cap H_2. && (11) \end{aligned}$$

Proposition 3.5 *Consider $F_1, F_2 \in \mathbf{F}$.*

(a) The function f is well defined.

(b) The function f is a bijection and

$$\begin{aligned} & f^{-1} : \mathbf{G} \to \mathbf{H}, \\ f^{-1}(G) \ = \ & (F_{21} \vee G, F_{12} \vee G). && (12) \end{aligned}$$

There exists an example for which $\mathbf{H_{min}}(F_1, F_2) \subsetneqq \mathbf{H}(F_1, F_2)$.

Theorem 3.6 *Consider $F_1, F_2 \in \mathbf{F}$. The restriction of the function $f : \mathbf{G} \to \mathbf{H}$ to $f : \mathbf{G_{min}} \to \mathbf{H_{min}}$ is a bijection.*

Problem 3.1 is with Theorem 3.6 replaced by a problem to characterize the set $\mathbf{H_{min}}(F_{21}, F_{12})$. The construction of the elements of this set is partly solved and will be included in the final version of the paper. The problem for the case in which the σ-algebras F_1 and F_2 are generated by finite-dimensional Gaussian random variables is solved in [7].

4 Stochastic realization problem for a σ-algebra family

Consider the discrete-time index set of the integers, $T = Z$ and the σ-algebra families $\{F_t, G_t, t \in T\}$. Define

$$\begin{aligned} F_t^+ &= \vee_{s \geq t} F_s, \ F_t^- = \vee_{s \leq t} F_s, \\ G_t^+ &= \vee_{s \geq t} G_s, \ G_t^- = \vee_{s \leq t} G_s. \end{aligned}$$

Definition 4.1 *The σ-algebra families $\{F_t, G_t, t \in T\}$ are said to form a σ-*algebraic system *if for all $t \in T$*

$$(F_t^+ \vee G_t^+, F_{t-1}^- \vee G_t^- | G_t) \in CI. \quad (13)$$

The reader may think of a σ-algebraic system as related to a discrete-time stochastic system with a state and output process. Thus, $F_t = F^{y_t}$ is the σ-algebra generated by the output at time $t \in T$ and $G_t = F^{x_t}$ is the σ-algebra generated by the state at time t. Condition (13) translates in words to: At any time $t \in T$ the current state σ-algebra G_t makes the future of the output and of the state conditional independent of the past of the output and of the state. The asymmetry of future and past with respect to the σ-algebra family $\{F_t, t \in T\}$ is necessary because of technical reasons.

Problem 4.2 *Consider the σ-algebra family $\{F_t, t \in T\}$.*

(a) Does there exist a σ-algebra family $\{G_t, t \in T\}$ on the same probability space of $\{F_t, t \in T\}$ such that $\{F_t, G_t, t \in T\}$ form a σ-algebraic system. Call $\{F_t, G_t, t \in T\}$ a realization *of the family $\{F_t, t \in T\}$.*

(b) Call $\{F_t, G_t, t \in T\}$ a minimal realization *of $\{F_t, t \in T\}$ if (1) it is a realization and if (2) for any other realization $\{F_t, \bar{G}_t, t \in T\}$, $\bar{G}_t \subseteq G_t$ for all $t \in T$ implies $\bar{G}_t = G_t$. Characterize a minimal realization.*

(c) Classify all σ-algebra families $\{G_t, t \in T\}$ such that (1) $G_t \subseteq F_\infty^-$ for all $t \in T$ and (2) $\{F_t, G_t, t \in T\}$ is a minimal realization of $\{F_t, t \in T\}$.

The solution procedure for the above problem makes use of the result of the realization problem of σ-algebras as discussed in Section 3. Consider the σ-algebra family $\{F_t, t \in T\}$ and for all $t \in T$ let $G_t \subseteq F_\infty^-$ be such that

$$(F_t^+, F_{t-1}^- | G_t) \in CI.$$

In general the σ-algebra family $\{F_t, G_t, t \in T\}$ does not form a σ-algebraic system. A condition is needed.

Definition 4.3 *A σ-algebra family $\{F_t, G_t, t \in T\}$ is called* transitive *if*

1. *$\{F_{t-1}^- \vee G_t, t \in T\}$ is increasing in T, or, equivalently, if*
$$s < t \Rightarrow (F_{s-1}^- \vee G_s) \subseteq (F_{t-1}^- \vee G_t);$$
2. *$\{F_t^+ \vee G_t, t \in T\}$ is decreasing in T, or, equivalently, if*
$$s < t \Rightarrow (F_s^+ \vee G_s) \supseteq (F_t^+ \vee G_t).$$

Proposition 4.4 *Consider the σ-algebra families $\{F_t, G_t, t \in T\}$. Assume that*

$$(F_t^+, F_{t-1}^- | G_t) \in CI, \; \forall t \in T.$$

Then

$$(F_t^+ \vee G_t^+, F_{t-1}^- \vee G_t^- | G_t) \in CI, \; \forall t \in T,$$

if and only if $\{F_t, G_t, t \in T\}$ is transitive.

Below σ-algebra families $\{H_t^+, H_t^-, t \in T\}$ are considered that may be interpreted as forward and backward families respectively. These concepts are inspired by the analogous Hilbert space case, see [2].

Theorem 4.5 *Consider σ-algebra families*

$$\{F_t, H_t^-, H_t^+, t \in T\}.$$

Define the conditions:

(1) $\{H_t^+, t \in T\}$ is an increasing family and $\{H_t^-, t \in T\}$ is a decreasing family.

(2) For all $t \in T$, $H_t^+ \vee H_t^- = F_t^+ \vee F_t^-$.

(3) For all $t \in T$ $F_t^- \subseteq H_t^+$ and $F_t^+ \subseteq H_t^-$.

(4) For all $t \in T$,

$$(H_t^+, H_t^- | H_t^+ \cap H_t^-) \in CI.$$

(a) Assume that the σ-algebra families

$$\{F_t, G_t, t \in T\}$$

form a σ-algebraic system such that for all $t \in T$ $G_t \subseteq F_\infty^-$. Define $H_t^+ = F_{t-1}^- \vee G_t$ and $H_t^- = F_t^+ \vee G_t$. Then the σ-algebra families $\{H_t^+, H_t^-, t \in T\}$ satisfy the conditions (1-4) above.

(b) Consider σ-algebra families

$$\{F_t, H_t^+, H_t^-, t \in T\}$$

that satisfy the conditions (1-4) above. Define for all $t \in T$ $G_t = H_t^+ \cap H_t^-$. Then $\{F_t, G_t, t \in T\}$ form a σ-algebraic system.

With this theorem one can prove that the filter σ-algebraic system studied in [5] is a σ-algebraic stochastic realization.

5 Conclusions

The stochastic realization problem for σ-algebra families has been considered. A characterization has been presented of those σ-algebras which make two considered σ-algebras minimally conditionally independent. A second result allows the construction of state σ-algebra family which together with the output σ-algebra family form a stochastic system. Further research is required on the algorithms for the set $\mathbf{G}_{min}(F_1, F_2)$ and for the properties of stochastic realizations.

References

[1] C. Dellacherie and P.A. Meyer. *Probabilités et potentiel, CH. I à IV.* Hermann, Paris, 1975.

[2] A. Lindquist and G. Picci. A geometric approach to modelling and estimation of linear stochastic systems. *J. Math. Systems, Estimation, and Control*, 1:241–333, 1991.

[3] A. Lindquist and G. Picci. Geometric methods for state space identification. In S. Bittanti and G. Picci, editors, *Identification, adaption, learning*, pages 1–69. Springer, London, 1996.

[4] M. Mouchart and J.-M. Rolin. On the σ-algebraic realization problem. Report 8604, Center for Operations Research and Econometrics, Université de Louvain, Louvain-la-Neuve, 1986.

[5] W.J. Stronegger. When is a sequence of sufficient σ-algebras a filter system. *Systems & Control Lett.*, 16:473–477, 1991.

[6] C. van Putten and J.H. van Schuppen. On stochastic dynamical systems. In *Proceedings Fourth International Symposium on the Mathematical Theory of Networks and Systems (MTNS79)*, pages Volume 3, 350–355, North Hollywood, CA, 1979. Western Periodical.

[7] C. van Putten and J.H. van Schuppen. The weak and strong gaussian probabilistic realization problem. *J. Multivariate Anal.*, 13:118–137, 1983.

[8] C. van Putten and J.H. van Schuppen. Invariance properties of the conditional independence relation. *Ann. Probab.*, 13:934–945, 1985.

[9] J.H. van Schuppen. Stochastic filtering theory: A discussion of concepts, methods and results. In W. Vogel M. Kohlmann, editor, *Stochastic control theory and stochastic differential systems*, volume 16 of *Lecture Notes in Control and Information Sciences*, pages 209–226, Berlin, 1979. Springer-Verlag.

[10] J.H. van Schuppen. The strong finite stochastic realization problem - Preliminary results. In A. Bensoussan and J.L. Lions, editors, *Analysis and optimization of systems*, volume 44 of *Lecture Notes in Control and Information Sciences*, pages 179–190, Berlin, 1982. Springer-Verlag.

[11] J.H. van Schuppen. Stochastic realization problems. In J.M. Schumacher H. Nijmeijer, editor, *Three decades of mathematical system theory*, volume 135 of *Lecture Notes in Control and Information Sciences*, pages 480–523. Springer-Verlag, Berlin, 1989.