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A control problem for affine dynamical systems on a full-dimensional polytope☆

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Abstract

Given an affine system on a full-dimensional polytope, the problem of reaching a particular facet of the polytope, using continuous piecewise-affine state feedback is studied. Necessary conditions and sufficient conditions for the existence of a solution are derived in terms of linear inequalities on the input vectors at the vertices of the polytope. Special attention is paid to affine systems on full-dimensional simplices. In this case, the necessary and sufficient conditions are equivalent and a constructive procedure yields an affine feedback control law, that solves the reachability problem under consideration.

Keywords: Affine systems; Piecewise-linear systems; Polytope; Facet; Reachability; Control law

1. Introduction

In this paper, a reachability problem for an affine dynamical system on a full-dimensional polytope is studied. The objective is to steer the state of the system to a particular facet of the polytope, using continuous (piecewise-) affine state feedback. The motivation for this study originates from the reachability problem for piecewise-linear hybrid systems. Therefore, we first explain the relationship between piecewise-linear hybrid systems on the one hand, and the control problem considered in this paper on the other.

In the modeling and control of engineering systems, one often has to deal with hybrid characterizations of a system. Whereas some phenomena may easily be described by ordinary differential equations, other aspects of a system are more suitably treated in a discrete-event framework. A hybrid description of a system combines these two types of dynamics, and specifies their mutual interaction. In the literature, several approaches have been proposed to model hybrid interactions. In this paper, we focus on one particular model, introduced in Sontag (1981, 1982, 1996): the class of *piecewise-linear* (or piecewise-affine) *hybrid systems*. Recently, this class of systems has received considerable interest, see e.g. Asarin, Bournez, Dang, and Maler (2000), Bemporad, Ferrari-Trecate, and Morari (1999) and Bemporad and Morari (1999). An overview of some of the current research in the area of general (i.e. not necessarily piecewise-linear) hybrid systems can be found in the conference proceedings (Di Benedetto & Sangiovanni-Vincentelli, 2001; Lynch & Krogh, 2000; Vaandrager & van Schuppen, 1999).

A piecewise-linear hybrid system consists of an automaton and, for each discrete mode, of an affine system on a polyhedral set. Every affine system is assumed to be controlled by a continuous input function. As soon as the continuous state reaches the boundary of the corresponding polyhedral set, a discrete event occurs, transferring the system to a new discrete mode. Also the continuous state is restarted and will continue to evolve according to the laws of the affine system corresponding to the new discrete mode. A discrete transition, and a resetting of the

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continuous state also occur if a so-called input event is applied to the system; in that case, it is not necessary that the continuous state has reached the boundary of the state set.

From now on, we consider piecewise-linear hybrid systems in which at each discrete mode the corresponding affine system is defined on a full-dimensional *polytope*. However, for every discrete mode, the corresponding affine system and polytope may be different. Furthermore, we assume that in every discrete mode the discrete event that occurs upon leaving the corresponding polytope, depends on the facet through which the polytope is left. In this setting, the problem studied in this paper, of steering the state of an affine system to a specific facet of a polytope, becomes relevant for the control of piecewise-linear hybrid systems: it describes the interaction between the continuous and discrete dynamics of a hybrid system. Therefore, the solution plays an important role in the reachability analysis of piecewise-linear hybrid systems (see e.g. van Schuppen, 1998; Habets & van Schuppen, 2001a, b).

At this point, we focus our attention on one discrete mode of a hybrid system, and consider an affine system on a full-dimensional polytope. Then the main question is to determine necessary conditions and sufficient conditions for the existence of a continuous piecewise-affine control law such that, independent of the initial state, all state-trajectories of the closed-loop system reach a particular facet of the polytope in finite time, without reaching other facets first. In the solution to this problem, convexity arguments play a major role. The necessary conditions are based on a continuity argument, and consist of a set of linear inequalities on the input vectors at the vertices of the polytope. For affine systems on a simplex, these conditions also turn out to be sufficient for the existence of an *affine* state feedback solution to our problem. For general polytopes, the sufficient conditions are somewhat more restrictive than the necessary conditions, but still they may be expressed as linear inequalities on the input vectors at the vertices of the polytope.

Once these linear inequalities have been obtained, existing algorithms may be used to check the existence of a solution. For this purpose, computer programs have been developed, for example in the research groups of Verimag (Jeannet, 1999) and of IRISA (Wilde, 1993). The final step is the computation of the continuous (piecewise-) affine control law. For systems on full-dimensional simplices, a simple procedure is provided for this, that may be extended to general polytopes using the concept of triangulation.

In the literature, there are several publications on the invariance of linear systems on polyhedral sets, see e.g. Castelan and Hennet (1993), Vassilaki and Bitsoris (1989) and Blanchini (1999) and on invariance of piecewise-linear hybrid systems, see Berardi, De Santis, and Di Benedetto (1999). The problem treated in this paper is different from that of those papers because the feedback law should not make the system invariant, but guarantee that a particular facet is reached in finite time. The ideas to solve this problem, by stating conditions on the input vectors at the vertices of the polytope, application of convexity arguments, and using triangulation for the construction of a piecewise affine state feedback, are similar to those used in Gutman and Cwikel (1986, 1987) for the solution of the invariance problem in the discrete-time case, though there far less explicit than in this paper.

The paper is organized as follows. The next section contains the problem formulation and terminology on polytopes and simplices. Necessary conditions for the existence of a continuous feedback law realizing the control objective are stated in Section 3. Sufficient conditions are derived in Section 4. We start with a rather general result on the existence of a continuous feedback solution. Subsequently, this result is specialized to systems on simplices and polytopes. Particular attention is paid to construction of a control law and to complexity issues. Concluding remarks are stated in Section 5. Appendix A is devoted to the detailed elaboration of the proof of one of the sufficient conditions.

2. Problem formulation

Let $N \in \mathbb{N}$, and consider the N-dimensional space $\mathbb{R}^N$. Throughout the paper, the Euclidean norm on real spaces is denoted by $\|\cdot\|$. Let $v_1, \dots, v_M$, with $M \geqslant N+1$, be M points in the space $\mathbb{R}^N$, such that there exists no hyperplane of $\mathbb{R}^N$, containing all these M points. The full-dimensional *polytope* P_N is defined as the convex hull of $v_1, \dots, v_M$. If a point v_i, $(i = 1, \dots, M)$, cannot be written as a convex combination of the points $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_M$ it is called a *vertex* of the polytope P_N. Since the convex hull of the vertices of a polytope is the polytope itself, a full-dimensional polytope is completely characterized by its set of vertices. A full-dimensional polytope with exactly $N+1$ vertices is called a full-dimensional *simplex*.

Alternatively, a polytope may be described as the intersection of a finite number of closed half spaces. I.e. there exist an integer $K \geqslant N+1$, non-zero vectors $n_1, \dots, n_K \in \mathbb{R}^N$, and scalars $\alpha_1, \dots, \alpha_K \in \mathbb{R}$, such that

$$P_N = \{x \in \mathbb{R}^N \mid \forall i = 1, \dots, K\colon n_i^{\mathrm{T}} x \leqslant \alpha_i\}. \tag{1}$$

Characterization (1) is called the *implicit description* of a polytope. The intersection of a full-dimensional polytope P_N with one of its supporting hyperplanes

$$F_i := \{x \in \mathbb{R}^N \mid n_i^{\mathrm{T}} x = \alpha_i\} \cap P_N$$

is called a *facet* of P_N, if the dimension of the intersection is equal to $N-1$. The vector n_i is the normal vector of the facet F_i, $(i = 1, \dots, K)$, and, by convention, n_i is of unit length and always points out of the polytope P_N. A full-dimensional simplex in $\mathbb{R}^N$ has exactly $N+1$ facets. Numerous results on polytopes can be found in the research monographs Grünbaum (1967) and Ziegler (1995). In the sequel the following result is needed.

Lemma 2.1. *Let P_N be a full-dimensional polytope in $\mathbb{R}^N$, and let v be a vertex of P_N. Let $F_1,\dots,F_L$ denote all facets of P_N that contain v. Then*

(1) *The normal vectors $n_1,\dots,n_L$ of $F_1,\dots,F_L$ generate $\mathbb{R}^N$.*
(2) *If P_N is a full-dimensional simplex, then $L=N$, and, assuming that the normal vector of every facet is pointing out of the polytope, the normal vector n_{N+1} of facet F_{N+1} is a negative linear combination of the vectors $n_1,\dots,n_N$. I.e. there exist $\lambda_1,\dots,\lambda_N<0$ such that*

$$n_{N+1}=\sum_{j=1}^{N}\lambda_j n_j.$$

Proof. (1) Let $F_{L+1},\dots,F_K$ denote the other facets of P_N, with normal vectors $n_{L+1},\dots,n_K$. Then there exist $\alpha_1,\dots,\alpha_K\in\mathbb{R}$ such that

$$P_N=\{x\in\mathbb{R}^N \mid \forall i=1,\dots,K\colon n_i^{\mathrm{T}}x\leqslant\alpha_i\}$$

is the implicit description of P_N. At the vertex v we know that $n_i^{\mathrm{T}}v=\alpha_i$ for $i=1,\dots,L$, and $n_i^{\mathrm{T}}v<\alpha_i$ for $i=L+1,\dots,K$. Assume that $n_1,\dots,n_L$ do not generate $\mathbb{R}^N$. Then there exists a vector $n\in\mathbb{R}^N\setminus\{0\}$ such that $n_i^{\mathrm{T}}n=0$ for $i=1,\dots,L$. Hence, there exists $\delta>0$ such that for all $\varepsilon\in(-\delta,\delta)$ the point $v+\varepsilon n$ still satisfies $n_i^{\mathrm{T}}(v+\varepsilon n)=\alpha_i$ for $i=1,\dots,L$ and $n_i^{\mathrm{T}}(v+\varepsilon n)<\alpha_i$ for $i=L+1,\dots,K$, which implies that $v+\varepsilon n\in P_N$. So v is a convex combination of two other points in P_N, contradicting the fact that v is a vertex of P_N (see e.g. Rockafellar, 1970, p. 162).

(2) Let $v_1,\dots,v_{N+1}$ denote the vertices of P_N, numbered in such a way that v_i does not belong to facet F_i. Then $v=v_{N+1}$. According to (1) there exist $\lambda_1,\dots,\lambda_N\in\mathbb{R}$ such that

$$n_{N+1}=\sum_{j=1}^{N}\lambda_j n_j.$$

Since v is the only vertex of P_N not belonging to facet F_{N+1} described by $n_{N+1}^{\mathrm{T}}x=\alpha_{N+1}$, it follows that for all vertices $v_1,\dots,v_N\neq v$, $n_{N+1}^{\mathrm{T}}(v_i-v)=\alpha_{N+1}-n_{N+1}^{\mathrm{T}}v>0$. Hence

$$0<n_{N+1}^{\mathrm{T}}(v_i-v)=\sum_{j=1}^{N}\lambda_j n_j^{\mathrm{T}}(v_i-v)$$

$$=\sum_{j=1}^{N}\lambda_j(n_j^{\mathrm{T}}v_i-\alpha_j)=\lambda_i(n_i^{\mathrm{T}}v_i-\alpha_i).$$

Since $n_i^{\mathrm{T}}v_i-\alpha_i<0$, we must have $\lambda_i<0$. □

On the full-dimensional polytope P_N, we consider an affine control system

$$\dot{x}=Ax+Bu+a,\quad x(0)=x_0 \tag{2}$$

with $A\in\mathbb{R}^{N\times N}$, $B\in\mathbb{R}^{N\times m}$ and $a\in\mathbb{R}^N$. So, on every time instant in a certain time interval, starting at time $t=0$, the state $x\in\mathbb{R}^N$ is assumed to be contained in the polytope P_N. Also the input u is assumed to take values in a polyhedral set $U\subset\mathbb{R}^m$ only. Note that the affine differential equation (2) only remains valid, as long as the state x is contained in the polytope P_N.

System (2) on the polytope P_N is assumed to describe the continuous dynamics at one specific discrete mode of a hybrid system. In this paper, we study the control problem of steering the state of system (2) in finite time to a particular facet of the polytope P_N. In a hybrid systems context, this describes the coupling between the discrete and continuous dynamics. In this respect, our results form an important building block for reachability analysis and control of hybrid systems.

Problem 2.2. Consider the affine system (2) on the full-dimensional polytope P_N, and let F_j be a facet of P_N, with normal vector n_j pointing out of the polytope P_N. For any initial state $x_0\in P_N$, we have to find a time-instant $T_0\geqslant 0$ and an input function $u:[0,T_0]\to U$, such that

(i) $\forall t\in[0,T_0]$: $x(t)\in P_N$,
(ii) $x(T_0)\in F_j$, and T_0 is the smallest time-instant in the interval $[0,\infty)$ for which the state reaches the exit facet F_j,
(iii) $n_j^{\mathrm{T}}\dot{x}(T_0)>0$, i.e. the velocity vector $\dot{x}(T_0)$ at the point $x(T_0)\in F_j$ has a positive component in the direction of n_j. This implies that in the point $x(T_0)$, the velocity vector $\dot{x}(T_0)$ points out of the polytope P_N.

Furthermore, this input function u should be realized by the application of a continuous feedback law

$$u(t)=f(x(t)) \tag{3}$$

with $f:P_N\to U$ a continuous function, that is independent of the initial state x_0.

Note that in Problem 2.2 the choice of the exit facet F_j is completely arbitrary. For notational convenience, we assume in the rest of the paper (without loss of generality) that the exit facet is chosen to be F_1.

For the solution of Problem 2.2, we are particularly interested in continuous feedback laws f that are (piecewise) affine. Note that if the feedback law f in (3) is affine, i.e. if $u=Fx+g$, with $F\in\mathbb{R}^{m\times N}$ and $g\in\mathbb{R}^m$, then the closed-loop system is also affine:

$$\dot{x}=(A+BF)x+(a+Bg),\quad x(0)=x_0. \tag{4}$$

Similarly, if the feedback law f is piecewise affine, then the closed-loop system is piecewise affine.

3. Necessary conditions for feedback control to a facet

In the next proposition, necessary conditions for the solution of Control Problem 2.2 are stated as linear inequalities on the input vectors at the vertices of the polytope P_N.

Proposition 3.1. *Let P_N be a full-dimensional polytope in $\mathbb{R}^N$ with vertices $v_1, \ldots, v_M$, $(M \geqslant N+1)$. Let $F_1, \ldots, F_K$ denote the facets of P_N, with normal vectors $n_1, \ldots, n_K$, respectively, pointing out of the polytope P_N. For $i \in \{1, \ldots, K\}$, let $V_i \subset \{1, \ldots, M\}$ be the index set such that $\{v_j \mid j \in V_i\}$ is the set of vertices of the facet F_i. Conversely, for every $j \in \{1, \ldots, M\}$, the set $W_j \subset \{1, \ldots, K\}$ contains the indices of all facets of which v_j is a vertex. Consider the affine system (2) on the polytope P_N. If Control Problem 2.2 with exit facet F_1 is solvable by a continuous state feedback f, then there exist inputs $u_1, \ldots, u_M \in U$ such that*

(1) $\forall j \in V_1$:
  (a) $n_1^{\mathrm{T}}(Av_j + Bu_j + a) > 0$,
  (b) $\forall i \in W_j \setminus \{1\}$: $n_i^{\mathrm{T}}(Av_j + Bu_j + a) \leqslant 0$.
(2) $\forall j \in \{1, \ldots, M\} \setminus V_1$:
  (a) $\forall i \in W_j$: $n_i^{\mathrm{T}}(Av_j + Bu_j + a) \leqslant 0$,
  (b) $\sum_{i \in W_j} n_i^{\mathrm{T}}(Av_j + Bu_j + a) < 0$.

Proof. Suppose that the continuous function $f : P_N \to U$ generates a feedback law $u(t) = f(x(t))$, that solves Control Problem 2.2. We show that the inputs $u_j = f(v_j) \in U$, $(j = 1, \ldots, M)$, obtained by applying feedback f to the vertices $v_1, \ldots, v_M$, satisfy (1) and (2).

(1a): For every $j \in V_1$, $v_j \in F_1$. So, as soon as the vertex v_j is reached, the state trajectory of the closed-loop system will leave the polytope P_N with a positive velocity in the n_1-direction. This implies that $n_1^{\mathrm{T}}(Av_j + Bu_j + a) = n_1^{\mathrm{T}}(Av_j + Bf(v_j) + a) > 0$.

(1b): If $N = 1$, then facets and vertices coincide, and condition (1b) is void. For $N > 1$, we use the continuity of the feedback f, to prove (1b) by contradiction. Let $j \in V_1$ and $i \in W_j \setminus \{1\}$ be such that $n_i^{\mathrm{T}}(Av_j + Bu_j + a) > 0$. Define the continuous function

$$h : P_N \to \mathbb{R} : \; h(x) = n_i^{\mathrm{T}}(Ax + Bf(x) + a).$$

Then $h(v_j) > 0$, and there exists $\delta > 0$, such that for all $x \in P_N$, with $\|x - v_j\| < \delta$: $h(x) > 0$. Let $L := \#V_i \geqslant 2$ denote the number of vertices of the facet F_i, and let

$$0 < \varepsilon < \min\left(\frac{\delta}{\sum_{k \in V_i} \|v_k\|}, 1\right).$$

We define

$$p := (1-\varepsilon)v_j + \frac{\varepsilon}{L-1} \sum_{k \in V_i \setminus \{j\}} v_k.$$

Because of the choice of $j \in V_1$ and $i \in W_j \setminus \{1\}$, we have $p \in F_i$, but $p \notin F_1$. Furthermore $h(p) > 0$ because

$$\|p - v_j\| = \left\| -\varepsilon v_j + \frac{\varepsilon}{L-1} \sum_{k \in V_i \setminus \{j\}} v_k \right\| \leqslant \varepsilon \sum_{k \in V_i} \|v_k\| < \delta.$$

Therefore, the trajectory of the closed-loop system $\dot{x} = Ax + Bf(x) + a$, with initial value $x(0) = p \in F_i$, will immediately leave the polytope P_N through the facet F_i because $n_i^{\mathrm{T}} \dot{x}(0) = h(p) > 0$. This contradicts the fact that the feedback law $u(t) = f(x(t))$ is a solution to Control Problem 2.2, irrespective of the initial state $x_0 \in P_N$.

(2a) and (2b): Let $j \in \{1, \ldots, M\} \setminus V_1$. Then $v_j \notin F_1$ and the velocity vector field at v_j of the closed-loop system does not point out of the polytope P_N. Since v_j is a vertex of all facets F_i with $i \in W_j$, this implies that for all $i \in W_j$:

$$n_i^{\mathrm{T}}(Av_j + Bu_j + a) = n_i^{\mathrm{T}}(Av_j + Bf(v_j) + a) \leqslant 0. \tag{5}$$

Moreover, there exists an $i \in W_j$ such that $n_i^{\mathrm{T}}(Av_j + Bu_j + a) < 0$. Otherwise, if $n_i^{\mathrm{T}}(Av_j + Bu_j + a) = 0$ for all $i \in W_j$, then $Av_j + Bu_j + a = 0$, because by Lemma 2.1(1), the set of normal vectors $\{n_i \mid i \in W_j\}$ generates $\mathbb{R}^N$. This would imply that the vertex $v_j \notin F_1$ is a fixed point of the closed-loop system, contradicting the assumption that the feedback f solves Control Problem 2.2. In combination with (5) we conclude that $\sum_{i \in W_j} n_i^{\mathrm{T}}(Av_j + Bu_j + a) < 0$. □

The necessary conditions for the solvability of Control Problem 2.2 stated in Proposition 3.1 only consist of a set of strict and non-strict linear inequalities on the inputs of the system at the vertices of the polytope P_N. Since also the input set U is assumed to be polyhedral, this has the advantage that the existence of a solution $u_1, \ldots, u_M \in U$ may be checked, using existing software for polyhedral sets, like e.g. Jeannet (1999); and Wilde (1993). This computation is further facilitated by the fact that the inequalities for each input are completely decoupled. Therefore, they may be checked for each input u_j separately.

Geometrically, inequalities (1) and (2) in Proposition 3.1 describe a polyhedral cone at each of the vertices of the polytope (see Fig. 1). The corresponding inputs have to be chosen in such a way that the velocity vector field at each vertex is pointing into the cone based at that vertex.

4. Sufficient conditions for feedback control to a facet

The goal of this section is to investigate to what extent the necessary conditions for the solvability of Control Problem 2.2 derived in the previous section, are also sufficient. First, we derive a set of sufficient conditions on the feedback function f, stated as linear inequalities that have to be satisfied on the whole polytope P_N or on its facets. Next we specialize to two particular cases. For systems on simplices we show that if the necessary conditions from Section 3 on the inputs at the vertices are satisfied, an affine

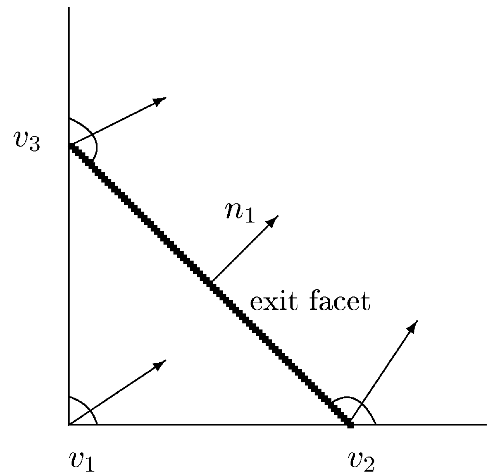

Fig. 1. Geometric interpretation of inequalities (1) and (2) of Proposition 3.1.

feedback control law can be constructed meeting the sufficient conditions, i.e. we obtain an affine feedback law that solves Control Problem 2.2. For arbitrary polytopes we combine the necessary conditions of Section 3 with an additional assumption on the direction of the velocity vector field in the vertices of the polytope P_N, in order to guarantee the existence of a continuous, piecewise-affine feedback law, that solves Control Problem 2.2. In both cases, sufficient conditions for solvability of Control Problem 2.2 are obtained in terms of linear inequalities on the inputs to the system at the vertices of the polytope. In the derivation of a feedback law, the convexity of the problem plays a major role.

We start with the formulation of necessary conditions in terms of the feedback law f. For this, we recall that a continuous function $f : P_N \to U$ satisfies the *Lipschitz condition* (or is called a *Lipschitz function*) if

$$\exists L \in \mathbb{R}\ \forall x, y \in P_N\colon \|f(x) - f(y)\| \leqslant L\|x - y\|.$$

Theorem 4.1. *Let P_N be a full-dimensional polytope in $\mathbb{R}^N$ with facets $F_1, \ldots, F_K$, and let $n_1, \ldots, n_K$ denote the normal vectors of $F_1, \ldots, F_K$, respectively, pointing out of the polytope P_N. Consider the affine system* (2) *on the polytope P_N. If there exists a Lipschitz function $f : P_N \to U$, such that*

(i) $\forall x \in P_N\colon n_1^{\mathrm{T}}(Ax + Bf(x) + a) > 0$,
(ii) $\forall i \in \{2, \ldots, K\}\ \forall x \in F_i\colon n_i^{\mathrm{T}}(Ax + Bf(x) + a) \leqslant 0$,

then the feedback law $u = f(x)$ solves Control Problem 2.2 *with exit facet F_1.*

Proof. If in condition (ii) the inequality is strict, then the proof is straightforward. Since the polytope P_N is compact, and the function $x \mapsto n_1^{\mathrm{T}}(Ax + Bf(x) + a)$ is continuous, condition (i) implies that there exists a $c > 0$, such that for all $x \in P_N\colon n_1^{\mathrm{T}}(Ax + Bf(x) + a) \geqslant c$. Therefore, the closed-loop system will move in the direction of F_1 with a strictly positive speed of at least c. Hence the polytope P_N is left in finite time. Furthermore, condition (ii) with strict inequality indicates that the state of the closed-loop system cannot leave P_N through any of the facets $F_2, \ldots, F_K$. So the state of the closed-loop system will leave P_N through F_1 in finite time.

The extension of the proof to the non-strict inequality in (ii) is quite involved. A detailed elaboration of this case is given in Appendix A. □

4.1. *Affine feedback for systems on simplices*

If the polytope under consideration is a full-dimensional simplex, then the necessary conditions of Proposition 3.1 and the sufficient conditions of Theorem 4.1 may be combined to obtain an affine feedback solution of Control Problem 2.2. The key observation is that every point in a simplex can be written in a *unique* way as a convex combination of its vertices. The input at this point is chosen to be the same convex combination of the inputs $u_1, \ldots, u_{N+1}$ at the vertices.

For the elaboration of this argument, we define for every $M \in \mathbb{N}$ the convex set Λ_M by

$$\Lambda_M := \left\{ (\lambda_1, \ldots, \lambda_M) \in [0,1]^M \;\middle|\; \sum_{j=1}^{M} \lambda_j = 1 \right\}. \tag{6}$$

Theorem 4.2. *Consider the affine system* (2) *on the full-dimensional simplex S_N with vertices $v_1, \ldots, v_{N+1}$ and facets $F_1, \ldots, F_{N+1}$. Assume that there exist inputs $u_1, \ldots, u_{N+1} \in U$, such that necessary conditions* (1) *and* (2) *of Proposition* 3.1 *are satisfied at all vertices. Define the affine mappings*

$$\varphi : \Lambda_{N+1} \to S_N\colon \varphi(\lambda_1, \ldots, \lambda_{N+1}) = \sum_{j=1}^{N+1} \lambda_j v_j,$$

$$\psi : \Lambda_{N+1} \to U\colon \psi(\lambda_1, \ldots, \lambda_{N+1}) = \sum_{j=1}^{N+1} \lambda_j u_j.$$

Then $f = \psi \circ \varphi^{-1}$ is an affine feedback solution to Control Problem 2.2.

Proof. To show that the feedback $f(x) = \psi(\varphi^{-1}(x))$ is well defined, we first note that the mapping $\varphi : \Lambda_{N+1} \to S_N$ is affine and bijective because S_N is a simplex. Therefore, also its inverse $\xi : S_N \to \Lambda_{N+1}$ exists and is affine. Furthermore, the affine mapping $\psi : \Lambda_{N+1} \to U$ is well defined because U is a polyhedral set, so every convex combination of $u_1, \ldots, u_{N+1} \in U$ belongs to U. Hence $f : S_N \to U$, being a composition of affine maps, is itself affine and therefore a Lipschitz function. Next we show that f satisfies sufficient conditions (i) and (ii) of Theorem 4.1.

Let v_1 denote the vertex not belonging to F_1. Since n_1 is a negative linear combination of $n_2, \ldots, n_{N+1}$ (see Lemma 2.1(2)), condition (2) of Proposition 3.1 guarantees that $n_1^{\mathrm{T}}(Av_1 + Bu_1 + a) > 0$. In combination with condition (1a) of Proposition 3.1, this yields

$$\forall j \in \{1,\dots,N+1\}\colon n_1^{\mathrm{T}}(Av_j + Bu_j + a) > 0.$$

Since every $x \in S_N$ may be written as $x = \sum_{j=1}^{N+1} \xi(x)_j v_j$ with $\xi(x) \in \Lambda_{N+1}$, we obtain for all $x \in S_N$:

$$\begin{aligned} &n_1^{\mathrm{T}}(Ax + Bf(x) + a) \\ &\quad = n_1^{\mathrm{T}}\left(A\sum_{j=1}^{N+1}\xi(x)_j v_j + B\sum_{j=1}^{N+1}\xi(x)_j u_j + \sum_{j=1}^{N+1}\xi(x)_j a\right) \\ &\quad = \sum_{j=1}^{N+1}\xi(x)_j \cdot n_1^{\mathrm{T}}(Av_j + Bu_j + a) > 0 \end{aligned}$$

and condition (i) of Theorem 4.1 is satisfied.

Let $i \in \{2,\dots,N+1\}$. Every $x \in F_i$ can be written as $x = \sum_{j\in V_i}\xi(x)_j v_j$, with $\sum_{j\in V_i}\xi(x)_j = 1$. Then conditions (1b) and (2a) of Proposition 3.1 indicate that

$$\begin{aligned} &n_i^{\mathrm{T}}(Ax + Bf(x) + a) \\ &\quad = n_i^{\mathrm{T}}\left(A\sum_{j\in V_i}\xi(x)_j v_j + B\sum_{j\in V_i}\xi(x)_j u_j + \sum_{j\in V_i}\xi(x)_j a\right) \\ &\quad = \sum_{j\in V_i}\xi(x)_j \cdot n_i^{\mathrm{T}}(Av_j + Bu_j + a) \leqslant 0, \end{aligned}$$

because $n_i^{\mathrm{T}}(Av_j + Bu_j + a) \leqslant 0$ for all $j \in V_i$ (note that $j \in V_i \Leftrightarrow i \in W_j$). We conclude that also condition (ii) of Theorem 4.1 is satisfied. Therefore, the feedback $u = f(x)$ solves Control Problem 2.2. □

In particular, Theorem 4.2 states that for affine systems on full-dimensional simplices the necessary conditions for solvability of Control Problem 2.2, that were obtained in Proposition 3.1, are also sufficient. Moreover, Theorem 4.2 gives a constructive way for finding an affine feedback solution.

For the actual computation of the affine feedback law f, the construction in Theorem 4.2 can be simplified considerably. Like any affine mapping, f is of the form $f(x) = Fx + g$ with fixed matrix $F \in \mathbb{R}^{m\times N}$ and fixed vector $g \in \mathbb{R}^m$. Additionally, f has the property that

$$\forall j \in \{1,\dots,N+1\}\colon u_j = f(v_j), \tag{7}$$

where the inputs u_j satisfy the conditions of Proposition 3.1. For a full-dimensional simplex S_N, property (7) determines F and g completely.

Algorithm 4.3. *Construction of an affine control law solving Control Problem* 2.2 *on a full-dimensional simplex*

Data: (i) vertices $v_1,\dots,v_{N+1}$ of a full-dimensional simplex S_N;

(ii) normal vectors $n_1,\dots,n_{N+1}$ to the facets of the simplex S_N;

(iii) an affine system $\dot{x} = Ax + Bu + a$ on S_N.

Step 1: Determine, if possible, $u_1,\dots,u_{N+1} \in U$ such that conditions (1) and (2) of Proposition 3.1 are satisfied.

If a solution exists, then go to Step 2.

If no solution exists, then Control Problem 2.2 is not solvable. Stop.

Step 2: Solve the following equation for F and g:

$$\begin{pmatrix} v_1^{\mathrm{T}} & 1 \\ \vdots & \vdots \\ v_{N+1}^{\mathrm{T}} & 1 \end{pmatrix}\begin{pmatrix} F^{\mathrm{T}} \\ \hline g^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} u_1^{\mathrm{T}} \\ \vdots \\ u_{N+1}^{\mathrm{T}} \end{pmatrix}. \tag{8}$$

Proof of correctness of algorithm. If a solution $u_1,\dots,u_{N+1} \in U$ for conditions (1) and (2) of Proposition 3.1 is obtained, then $F \in \mathbb{R}^{m\times N}$ and $g \in \mathbb{R}^m$ should satisfy $u_j = Fv_j + g$ for $j = 1,\dots,N+1$, or, after transposition, $v_j^{\mathrm{T}}F^{\mathrm{T}} + g^{\mathrm{T}} = u_j^{\mathrm{T}}$, $(j = 1,\dots,N+1)$. Collecting all equations in one matrix equation yields (8). For a full-dimensional simplex, Eq. (8) always has a *unique* solution for F and g, because the vectors $v_2 - v_1, v_3 - v_1, \dots, v_{N+1} - v_1$ are linearly independent and thus

$$\begin{aligned} \det\begin{pmatrix} v_1^{\mathrm{T}} & 1 \\ \vdots & \vdots \\ v_{N+1}^{\mathrm{T}} & 1 \end{pmatrix} &= \det\begin{pmatrix} 0 & 1 \\ v_2^{\mathrm{T}} - v_1^{\mathrm{T}} & 0 \\ \vdots & \vdots \\ v_{N+1}^{\mathrm{T}} - v_1^{\mathrm{T}} & 0 \end{pmatrix} \\ &= (-1)^N \det\begin{pmatrix} v_2^{\mathrm{T}} - v_1^{\mathrm{T}} \\ \vdots \\ v_{N+1}^{\mathrm{T}} - v_1^{\mathrm{T}} \end{pmatrix} \neq 0. \end{aligned}$$

So, the square $(N+1)\times(N+1)$ matrix on the left-hand side of (8) is invertible. □

The execution of Algorithm 4.3 involves the solution of N decoupled sets of N linear inequalities in m unknowns and the solution of one set of $N+1$ linear inequalities in m unknowns. Indeed, at every vertex of the exit facet, the corresponding input has to satisfy the N linear inequalities stated in condition (1) of Proposition 3.1; the input corresponding to the unique vertex not belonging to the exit facet, has to satisfy the $N+1$ linear inequalities in condition (2) of Proposition 3.1. If a solution exists, the computation of the corresponding feedback law just requires the unique solution of matrix equation (8) in $(N+1)\cdot m$ unknowns.

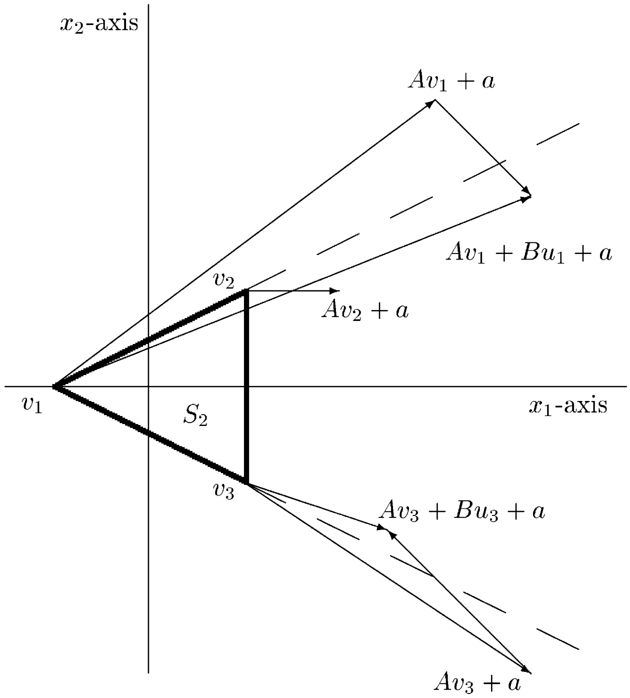

Fig. 2. Control of the vector field $\dot{x}$ at the vertices of S_2.

An illustration how Algorithm 4.3 is used, is given in the next example.

Example 4.4. Let $N = 2$, and let the simplex S_2 be the triangle in $\mathbb{R}^2$ with vertices $v_1 = (-1, 0)^{\mathrm{T}}$, $v_2 = (1, 1)^{\mathrm{T}}$, and $v_3 = (1, -1)^{\mathrm{T}}$ (see Fig. 2). The normal vectors on the three facets F_1, F_2, and F_3 of S_2 are $n_1 = (1, 0)^{\mathrm{T}}$, $n_2 = (1/\sqrt{5})\,(-1, -2)^{\mathrm{T}}$, and $n_3 = (1/\sqrt{5})\,(-1, 2)^{\mathrm{T}}$, respectively.

On the simplex S_2 we consider the system

$$\dot{x} = \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} x + \begin{pmatrix} 2 \\ -2 \end{pmatrix} u + \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

with state $x \in S_2$ and scalar input $-1 \leqslant u \leqslant 1$. We want to construct an affine feedback law $u = Fx + g$ such that the state of the closed-loop system can only leave the simplex S_2 through the facet F_1, the vertical line segment between the vertices v_2 and v_3.

For the existence of a solution it is necessary and sufficient that there exist an input u_1 at vertex v_1 satisfying condition (2) of Proposition 3.1, and inputs u_2, u_3 at the vertices v_2, v_3 satisfying condition (1). So, for u_1 the following inequalities should hold:

(a1) $n_2^{\mathrm{T}} B u_1 \leqslant -n_2^{\mathrm{T}}(Av_1 + a)$, so $u_1 \leqslant 5$,
(a2) $n_3^{\mathrm{T}} B u_1 \leqslant -n_3^{\mathrm{T}}(Av_1 + a)$, so $u_1 \geqslant \frac{1}{3}$,
(b) $n_2^{\mathrm{T}} B u_1 + n_3^{\mathrm{T}} B u_1 < -n_2^{\mathrm{T}}(Av_1 + a) - n_3^{\mathrm{T}}(Av_1 + a)$, so $u_1 \geqslant -2$

and, additionally, $-1 \leqslant u_1 \leqslant 1$. Therefore, all conditions are satisfied for $u_1 \in \left[\frac{1}{3}, 1\right]$. For u_2 we have

(a) $n_1^{\mathrm{T}} B u_2 > -n_1^{\mathrm{T}}(Av_2 + a)$, so $u_2 > -\frac{1}{2}$,
(b) $n_3^{\mathrm{T}} B u_2 \leqslant -n_3^{\mathrm{T}}(Av_2 + a)$, so $u_2 \geqslant -\frac{1}{6}$

with $-1 \leqslant u_2 \leqslant 1$. Hence $u_2 \in \left[-\frac{1}{6}, 1\right]$. Finally, u_3 has to satisfy

(a) $n_1^{\mathrm{T}} B u_3 > -n_1^{\mathrm{T}}(Av_3 + a)$, so $u_3 > -\frac{3}{2}$,
(b) $n_2^{\mathrm{T}} B u_3 \leqslant -n_2^{\mathrm{T}}(Av_3 + a)$, so $u_3 \leqslant -\frac{1}{2}$

and $-1 \leqslant u_3 \leqslant 1$. So every $u_3 \in \left[-1, -\frac{1}{2}\right]$ is a solution.

To obtain an affine feedback, we fix the inputs at the vertices by choosing $u_1 = \frac{1}{2}$, $u_2 = 0$, and $u_3 = -\frac{3}{4}$, and compute $F = (f_1 \;\, f_2)$ and g using formula (8)

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{3}{4} \end{pmatrix}.$$

This yields the following affine feedback solution for Control Problem 2.2 with exit facet F_1:

$$u = \begin{pmatrix} -\frac{7}{16} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{16}. \tag{9}$$

In Fig. 2, the vector fields of the open- and closed-loop system in the vertices of S_2 are depicted.

Remark 4.5. At first sight, it seems possible to use the freedom in the choice of the inputs at the vertices also for optimization purposes. Unfortunately, this idea does not work properly. Given an optimization criterion, the corresponding optimal affine feedback is dependent on the initial condition. So for every initial condition, an other optimization problem has to be solved. Therefore this procedure has a high complexity.

4.2. Piecewise-affine feedback for systems on polytopes

For affine systems on general full-dimensional polytopes, the idea of solving Control Problem 2.2 by applying a feedback that consists of a convex combination of suitable inputs at the vertices of the polytope still applies. However, in comparison with the simplex case, there are two additional complications.

(1) For general polytopes, the necessary conditions of Proposition 3.1 do not guarantee a positive speed in the direction of the exit facet. For this we have to impose an additional requirement. In the simplex case, this additional assumption is not needed because of the particular structure of a simplex; the result of Lemma 2.1(2) is only valid for full-dimensional simplices and cannot be extended to general polytopes. As a consequence, we obtain sufficient

conditions for the solvability of Problem 2.2 that are stronger than the necessary conditions of Proposition 3.1.

(2) Although any point in a polytope can be written as a convex combination of the vertices of the polytope, this choice is not necessarily unique. So, to construct a feedback based on the same philosophy as in the simplex case, we have to fix a specific choice in such a way that the corresponding feedback remains continuous. This is done by carrying out a *triangulation* of the polytope. In this way, we obtain a solution to Control Problem 2.2 in the form of a continuous piecewise-affine state feedback. Certainly, in a hybrid systems environment, it is no problem to use a feedback of this more general type.

Lemma 4.6. *Let P_N be a full-dimensional polytope in $\mathbb{R}^N$ with vertices $v_1,\dots,v_M$, $(M \geqslant N+1)$, and Λ_M the convex set defined in* (6). *Then there exists a continuous and piecewise-affine mapping $\xi : P_N \to \Lambda_M$ such that for all $x \in P_N$:*

$$x = \sum_{j=1}^{M} \xi(x)_j v_j. \tag{10}$$

Proof. First, we construct a triangulation of the polytope P_N: we make a subdivision of P_N into full-dimensional simplices $S_1,\dots,S_L$ such that

(i) $P_N = \bigcup_{i=1}^{L} S_i$,
(ii) For all $i,j \in \{1,\dots,L\}$ with $i \neq j$, the intersection $S_i \cap S_j$ is either empty or a common *face* of S_i and S_j. (A face of a polytope is the intersection (of arbitrary dimension) of a polytope with one of its supporting hyperplanes.)
(iii) The set of vertices of each simplex S_i, $(i = 1,\dots,L)$, is a subset of $\{v_1,\dots,v_M\}$.

According to Lee (1997), triangulations satisfying the properties (i)–(iii) exist; in the same reference several algorithms for the construction of triangulations are presented.

Given a triangulation, every $x \in P_N$ is uniquely represented as a convex combination of those vertices in $\{v_1,\dots,v_M\}$, which are vertices of all simplices of which x is an element. Let $\xi : P_N \to \Lambda_M$ be the corresponding function, that maps every point $x \in P_N$ to the M-tuple of coefficients corresponding to x. Clearly, ξ is affine in the interior of any of the simplices $S_1,\dots,S_L$. Furthermore, condition (ii) implies that ξ is continuous at the boundary of every simplex S_i $(i = 1,\dots,L)$. Hence, $\xi : P_N \to \Lambda_M$ is a continuous piecewise-affine mapping satisfying (10). □

Theorem 4.7. *Consider the affine system* (2) *on the full-dimensional polytope P_N with vertices $v_1,\dots,v_M$ and facets $F_1,\dots,F_K$. Assume that there exist inputs $u_1,\dots,u_M \in U$, such that at all vertices necessary conditions* (1) *and* (2) *of Proposition* 3.1 *are satisfied, and additionally*

(2c) $\forall j \in \{1,\dots,M\} \setminus V_1$: $n_1^{\mathrm{T}}(Av_j + Bu_j + a) > 0$,

where n_1 denotes the normal vector of F_1, pointing out of the polytope P_N. Let $\xi : P_N \to \Lambda_M$ be a continuous piecewise-affine mapping satisfying property (10), *and define the affine mapping*

$$\psi : \Lambda_M \to U\colon \psi(\lambda_1,\dots,\lambda_M) = \sum_{j=1}^{M} \lambda_j u_j.$$

Then $f = \psi \circ \xi$ is a continuous piecewise-affine feedback solution to Control Problem 2.2 *with exit facet F_1.*

Proof. Obviously, Lemma 4.6 indicates that $f : P_N \to U$ is well defined. Being a composition of continuous (piecewise-) affine maps, f is itself continuous and piecewise affine, and therefore a Lipschitz function. So it suffices to show that f satisfies conditions (i) and (ii) of Theorem 4.1.

(i): Let $x \in P_N$. Then x can be written as $x = \sum_{j=1}^{M} \xi(x)_j v_j$, with $\xi(x) \in \Lambda_M$. Hence, condition (1a) of Proposition 3.1 and condition (2c) imply that

$$\begin{aligned}
&n_1^{\mathrm{T}}(Ax + Bf(x) + a)\\
&\quad = n_1^{\mathrm{T}}\left(A\sum_{j=1}^{M} \xi(x)_j v_j + B\sum_{j=1}^{M} \xi(x)_j u_j + \sum_{j=1}^{M} \xi(x)_j a\right)\\
&\quad = \sum_{j=1}^{M} \xi(x)_j \cdot n_1^{\mathrm{T}}(Av_j + Bu_j + a) > 0.
\end{aligned}$$

(ii): Let $i \in \{2,\dots,K\}$, and $x \in F_i$. Then x can only be written as a convex combination of the vertices of F_i, i.e. $\forall j \in \{1,\dots,M\} \setminus V_i$: $\xi(x)_j = 0$. Hence, $x = \sum_{j \in V_i} \xi(x)_j v_j$, with $\sum_{j \in V_i} \xi(x)_j = 1$, and conditions (1b) and (2a) of Proposition 3.1 imply that

$$\begin{aligned}
&n_i^{\mathrm{T}}(Ax + Bf(x) + a)\\
&\quad = n_i^{\mathrm{T}}\left(A\sum_{j \in V_i} \xi(x)_j v_j + B\sum_{j \in V_i} \xi(x)_j u_j + \sum_{j \in V_i} \xi(x)_j a\right)\\
&\quad = \sum_{j \in V_i} \xi(x)_j \cdot n_i^{\mathrm{T}}(Av_j + Bu_j + a) \leqslant 0,
\end{aligned}$$

because $\xi(x)_j \geqslant 0$ and $n_i^{\mathrm{T}}(Av_j + Bu_j + a) \leqslant 0$ for all $j \in V_i$. □

Remark 4.8. If condition (2a) of Proposition 3.1 and (2c) of Theorem 4.7 hold, then condition (2b) of Proposition 3.1 is automatically satisfied. Indeed, if for some $j \in \{1,\dots,M\} \setminus V_1$, condition (2b) would not be satisfied, then $n_i^{\mathrm{T}}(Av_j + Bu_j + a) = 0$ for all $i \in W_j$, and Lemma 2.1(1) would imply that $Av_j + Bu_j + a = 0$. Hence, condition (2c) would be violated too.

Remark 4.9. Given a set of inputs $u_1, \ldots, u_M$, satisfying the necessary conditions of Theorem 4.7, an upper bound for the time T_0 at which the closed-loop system reaches the exit facet F_1 is easily obtained. Let $\alpha := \inf\{n_1^T x \mid x \in P_N\} = \min\{n_1^T v_j \mid j = 1, \ldots, M\}$ and $\beta := \sup\{n_1^T x \mid x \in P_N\} = \max\{n_1^T v_j \mid j = 1, \ldots, M\}$. Then the function $y(t) = n_1^T x(t)$ satisfies $\alpha \leqslant y(0) \leqslant \beta$ and $y(T_0) = \beta$, because $y(T_0) \in F_1$. Furthermore, the minimal rate of increase of y satisfies

$$\begin{aligned}
&\inf\{n_1^T \dot{x}(t) \mid t \in [0, T_0]\}\\
&\quad \geqslant \min\{n_1^T (Ax + Bf(x) + a) \mid x \in P_N\}\\
&\quad = \min\left\{\sum_{j=1}^{M} \xi(x)_j n_1^T (Av_j + Bu_j + a) \mid x \in P_N\right\}\\
&\quad = \min\{n_1^T (Av_j + Bu_j + a) \mid j = 1, \ldots, M\} =: c_1. \qquad (11)
\end{aligned}$$

This implies that

$$T_0 \leqslant \frac{\beta - \alpha}{c_1}, \qquad (12)$$

where α, β, and c_1 are easily determined with the previous formulae. This upper bound on T_0 is conservative, because T_0 depends both on the initial state x_0 and on the time-varying value of the decay rate $n_1^T(Ax + Bf(x) + a)$ along the solution trajectory $x(t) \in P_N$. If the solvability conditions of Theorem 4.7 admit some freedom in the choice of the inputs $u_1, \ldots, u_M \in U$, this may be used to decrease the upper bound for T_0 in (12), by increasing c_1. The upper bound may be optimized by solving a constrained maxmin problem for the inputs $u_1, \ldots, u_M$ at the vertices of the polytope P_N.

If the inputs u are unconstrained, i.e. if $U = \mathbb{R}^m$, and if the matrix B is right invertible, then the sufficient conditions of Theorem 4.7 for solving Control Problem 2.2 are always satisfied. Intuitively this is clear because in this situation we have full control, which makes it possible to prescribe the vector field of $\dot{x}$ at the vertices of P_N completely, due to the freedom in Bu_j $(j = 1, \ldots, M)$.

If B has not full rank, or if the inputs u are constrained, it is difficult to describe the parameter set for which the control-to-facet problem is solvable. In general, the solvability conditions will turn out to be restrictive, and are only satisfied for a rather limited number of exit facets. However, for the hybrid systems application we have in mind, this is not really a problem. In every discrete mode of a hybrid system it is to be expected that only a limited number of discrete transitions is possible. This corresponds to the situation of an affine system on a polytope with only a small number of possible exit facets.

For general full-dimensional polytopes, the sufficient conditions for solving Problem 2.2, formulated in Theorem 4.7, are stronger than the necessary conditions of Proposition 3.1. There exist even examples in which the conditions of Theorem 4.7 are too restrictive: although in these cases condition (2c) is not satisfied, Control Problem 2.2 is still solvable. For full-dimensional simplices however, it was shown in Section 4.1 that the necessary and sufficient conditions are equivalent. For systems on multi-dimensional rectangles, the necessary and sufficient conditions turn out to be almost equivalent.

Corollary 4.10. *Let R_N be the multi-dimensional rectangle defined by*

$$R_N := \{x \in \mathbb{R}^N \mid \forall i = 1, \ldots, N\colon a_i \leqslant x_i \leqslant b_i\}$$

with $a_i < b_i$, $(i = 1, \ldots, N)$, and consider the dynamical system (2), *with $x \in R_N$ and $u \in U$. Let $F_1 := R_N \cap \{x \in \mathbb{R}^N \mid x_1 = b_1\}$ be the exit facet of R_N with normal vector e_1. The normal vectors on the other facets are $-e_1$ and $\pm e_i$, $(i = 2, \ldots, N)$. Denote $M = 2^N$, and let $\xi : R_N \to \Lambda_M$ be a continuous and piecewise-affine mapping, satisfying* (10). *Assume that there exist inputs $u_1, \ldots, u_M \in U$, such that at the vertices $v_1, \ldots, v_M$ of R_N, conditions* (1) *and* (2) *of Proposition* 3.1 *are satisfied, and additionally*

(2c′) $\forall j \in \{1, \ldots, M\} \setminus V_1\colon e_1^T(Av_j + Bu_j + a) > 0.$

Define the affine mapping

$$\psi : \Lambda_M \to U\colon \psi(\lambda_1, \ldots, \lambda_M) = \sum_{j=1}^{M} \lambda_j u_j.$$

Then $f : R_N \to U$, defined by $f = \psi \circ \xi$, is a continuous and piecewise-affine feedback law, solving Control Problem 2.2.

In Corollary 4.10, the additional condition (2c′) is just slightly more restrictive than the necessary condition (2a) of Proposition 3.1. Indeed, if $j \in \{1, \ldots, M\} \setminus V_1$, then the vertex v_j belongs to the facet $R_N \cap \{x \in \mathbb{R}^N \mid x_1 = a_1\}$ of R_N, with normal vector $-e_1$. So condition (2a) of Proposition 3.1 states that $e_1^T(Av_j + Bu_j + a) \geqslant 0$. The only difference with condition (2c′) is a $\geqslant$ −sign instead of a $>$ −sign. Therefore the necessary conditions and the sufficient conditions are almost the same.

4.2.1. Complexity issues

We start by summarizing the solution method for Problem 2.2, proposed in this paper.

Algorithm 4.11. *Construction of a continuous piece-wise-affine control law solving Control Problem* 2.2 *on a full-dimensional polytope*:

Data: (i) vertices $v_1, \ldots, v_M$ of a full-dimensional polytope P_N in $\mathbb{R}^N$;

(ii) normal vectors $n_1,\dots,n_K$ to the facets of the polytope P_N;

(iii) an affine system $\dot{x} = Ax + Bu + a$ on P_N.

Step 1: Determine, if possible, $u_1,\dots,u_M \in U$ such that conditions (1) and (2) of Proposition 3.1 and condition (2c) of Theorem 4.7 are satisfied.

If a solution exists, go to Step 3.

If no solution exists, go to Step 2.

Step 2: Check whether there exist $u_1,\dots,u_M \in U$ such that necessary conditions (1) and (2) of Proposition 3.1 are satisfied.

If no solution exists, then Control Problem 2.2 is not solvable. Stop.

If a solution exists, then the method of this paper does not provide an answer whether Control Problem 2.2 is solvable. Stop.

Step 3: Carry out a triangulation of the polytope P_N, see below for details.

Step 4: Let S_ℓ be a simplex in the triangulation of P_N with vertices $v_{\ell_1},\dots,v_{\ell_{N+1}}$ from the set $\{v_1,\dots,v_M\}$. Then the affine feedback on S_ℓ is given by

$$u = F_\ell x + g_\ell, \tag{13}$$

where F_ℓ and g_ℓ are the unique solutions of the matrix equation

$$\begin{pmatrix} v_{\ell_1}^{\mathrm{T}} & 1 \\ \vdots & \vdots \\ v_{\ell_{N+1}}^{\mathrm{T}} & 1 \end{pmatrix} \begin{pmatrix} F_\ell^{\mathrm{T}} \\ \hline g_\ell^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} u_{\ell_1}^{\mathrm{T}} \\ \vdots \\ u_{\ell_{N+1}}^{\mathrm{T}} \end{pmatrix}, \tag{14}$$

where $u_{\ell_1},\dots,u_{\ell_{N+1}}$ are the inputs determined in Step 1 that correspond to the vertices $v_{\ell_1},\dots,v_{\ell_{N+1}}$.

Step 5: The collection of all affine feedback laws (13) on the simplices S_ℓ in the triangulation of P_N forms a continuous piecewise affine feedback solution of Control Problem 2.2.

The execution of Step 1 of Algorithm 4.11 requires the solution of M systems of linear inequalities in m unknowns: for every vertex $v_j \in \{v_1,\dots,v_M\}$, the corresponding input u_j has to satisfy

- $\#W_j$ linear inequalities if v_j is a vertex of exit facet F_1,
- $\#W_j + 1$ linear inequalities if v_j is not a vertex of the exit facet F_1.

Here $\#W_j$ denotes the number of facets of which v_j is a vertex. Since the total number of facets of P_N is K, $\#W_j < K$.

The complexity of Step 2 of the algorithm is comparable to Step 1. For each vertex v_j of the exit facet, a system of $\#W_j$ linear inequalities in m unknowns has to be solved. For all other vertices of the polytope P_N, the number of inequalities that has to be satisfied is $\#W_j + 1$.

There are several algorithms for carrying out a triangulation of a polytope, needed in Step 3. In Goodman and O'Rourke (1997), some chapters are devoted to this problem, see Chapter 14 (Lee, 1997), Chapter 20 (Fortune, 1997) and Chapter 22. In Chapter 20, a specific type of triangulation, the so-called *Delaunay triangulation*, is treated. In Chapter 14, special attention is paid to the triangulation of the N-dimensional unit cube.

In principle, the triangulation of a polytope can be carried out in two different ways. One possibility is to compute the triangulation at the same time that the polytope is constructed from a list of vertices. Alternatively, a triangulation may be computed afterwards; in this case triangulation is carried out by partitioning the polytope successively into smaller polytopes. For a reference and concepts on these algorithms see Lee (1997, Section 14.2). Recent algorithms for triangulation are used in the computation of the volume of a polytope, see Büeler, Enge and Fukuda (2000).

According to Fortune (1997, Table 20.2.1) and Fukuda (2000, Section 3.4), the time complexity for carrying out a Delaunay triangulation is of the order $O(n^{\lceil d/2 \rceil})$, where n denotes the number of vertices of the polytope, and d its dimension (so in the notation used in this paper: $n = M$ and $d = N$). In Fukuda (2000, Section 3.4) also the order of an upper bound for the number of simplices in the Delaunay triangulation is mentioned: $O(n^{\lfloor (d+1)/2 \rfloor})$. However, it should be noted that both mentioned orders for the complexity of the computation of a Delaunay triangulation are only valid in a worst case scenario. Usually, the algorithms will be considerably faster and produce less output.

An implementation of an algorithm for carrying out Delaunay triangulation is available in Matlab under the name `delaunayn`. Experiments on a modern workstation show that carrying out this triangulation method is still possible within a reasonable amount of time for polytopes up to a dimension of at least 10. For 20 randomly chosen points in $\mathbb{R}^8$, a Delaunay triangulation is carried out within a few seconds, generating a subdivision into 1000–1200 simplices. The case of 30 randomly chosen points in $\mathbb{R}^{10}$ is already more difficult: these examples required approximately 20 s of computing time, and generated a triangulation with about 20 000 simplices. These experiments seem to suggest that the large number of simplices in a triangulation is a more important bottleneck in Algorithm 4.11 than the actual computation of a triangulation in Step 3. Experiments with different triangulation techniques, using more specialized and sophisticated software, are reported in Büeler et al. (2000). There it is expected that triangulation remains tractable for polytopes with up to 10^2 hyperplanes and 10^4 vertices, or, 10^4 hyperplanes and 10^2 vertices, and with dimension up to 15.

For every simplex S_ℓ in the triangulation of the polytope, the computation of the corresponding feedback law $u = F_\ell x + g_\ell$ in Step 4 is relatively simple. It requires the solution of a matrix equation in $(N+1) \cdot m$ unknowns. Furthermore, it is known beforehand that this solution is unique. Whereas the computation of the feedback law in one simplex is relatively simple, a problem may arise if the total number of simplices in the triangulation becomes very large. In this situation, a careful implementation of the piecewise affine feedback as explained below may reduce the required computational effort.

In Step 5, all computed affine feedback laws are collected to be combined into one continuous piecewise affine state feedback. If the dimension of the problem is relatively low ($N = d = 4$ or 5), and the number of simplices in the triangulation of Step 3 is also small, then implementation of the total feedback law is not a real issue. For the application we have in mind, control of piecewise-affine hybrid systems, a number of 4 or 5 continuous states within every discrete mode seems reasonable. If in a discrete mode the number of continuous states is considerably higher, one will often be inclined to remodel the system in order to decrease the number of continuous states. However, also if the number of continuous states is higher (say $N = d = 10$), a careful implementation makes it possible to apply the continuous piecewise affine feedback law constructed in this paper. The main point of this implementation is that although the triangulation is carried out beforehand, the feedback law in a particular simplex is only computed at the time that it is really needed.

The first step in this implementation is the determination of the simplex in which the initial state of the system is located. To solve this problem, one first has to answer the question how to decide whether a given point is an element of a simplex. In case an N-dimensional simplex is characterized by its $N+1$ vertices, the answer requires the solution of a system of $N+1$ equations in $N+1$ unknowns, and the verification whether all unknowns are elements of the interval $[0,1]$. If a simplex is described as the intersection of $N+1$ halfspaces, one only has to check that the given point satisfies the $N+1$ linear inequalities that describe the simplex. However, if the number of simplices in the chosen triangulation is very large, the search for the right simplex in one of the ways described above may still be rather time consuming. Fortunately, the situation is different for the Delaunay triangulation: in Fukuda (2000, Section 3.7) it is described how in this situation the search for the right simplex may be carried out in an efficient way.

As soon as the initial simplex has been found, the corresponding feedback law is easily computed in the way described in Step 4 of Algorithm 4.11. During the application of this feedback, the state of the system should be monitored. At the moment that the state leaves the simplex, it is known through which facet the state has left. Since this facet should also be a facet of the simplex that has been entered by the state, this newly entered simplex is easily determined from a small list of all possible candidates. Next, the corresponding feedback law is determined and applied. In this way, one may continue until the state leaves the state polytope through the required exit facet. Note that in this implementation an affine state feedback on a simplex is only computed at the time that it is needed for feedback. Therefore, neither superfluous computations are carried out, nor superfluous memory space is occupied.

4.2.2. An example

Consider the system

$$\dot{x} = Ax + Bu + a$$

with

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\left[a = (\,8\ \ 1\ \ 1\ \ 1\,)^{\mathrm{T}}\right]$$

on the four-dimensional unit cube. We want to solve the problem of finding a continuous feedback law, guaranteeing that the state of the closed-loop system will leave the cube in finite time through the facet $x_1 = 1$, without leaving the cube through an other facet first.

The vertices of the unit cube are numbered according to the corresponding binary representation, so for $i_1, i_2, i_3, i_4 \in \{0, 1\}$, the corresponding vertex (i_1, i_2, i_3, i_4) is denoted by v_k, with $k = i_4 + 2i_3 + 2^2 i_2 + 2^3 i_1$. To test the existence of a feedback we have to check whether the conditions of Proposition 3.1 and Corollary 4.10 are satisfied. In this particular situation, we have to verify that for every vertex $v_k = (i_1, i_2, i_3, i_4)$ there exists an input $u_k \in \mathbb{R}^2$ such that

$$\begin{aligned} &e_1^{\mathrm{T}}(Av_k + Bu_k + a) > 0, \\ &(-1)^{i_2+1} e_2^{\mathrm{T}}(Av_k + Bu_k + a) \leqslant 0, \\ &(-1)^{i_3+1} e_3^{\mathrm{T}}(Av_k + Bu_k + a) \leqslant 0, \\ &(-1)^{i_4+1} e_4^{\mathrm{T}}(Av_k + Bu_k + a) \leqslant 0. \end{aligned}$$

So for each vertex we have to determine a corresponding input that satisfies four linear inequalities. It turns out that solutions to these inequalities do exist. One particular solution is

$$\begin{aligned} &u_0 = u_1 = u_2 = u_8 = u_9 = u_{10} = (0, 0)^{\mathrm{T}}, \\ &u_4 = u_5 = u_{12} = u_{13} = (0, 1)^{\mathrm{T}}, \quad u_3 = u_{11} = (1, 0)^{\mathrm{T}}, \\ &u_6 = u_{14} = (0, 2)^{\mathrm{T}}, \qquad u_7 = u_{15} = \left(1, \tfrac{3}{2}\right)^{\mathrm{T}}. \end{aligned}$$

For the determination of a feedback law, we have to carry out a triangulation of the four-dimensional unit cube. For this, we use the construction described in Mara (1976), yielding a triangulation into 16 simplices. According to Cottle (1982), this is the minimal number of simplices that is required for a triangulation of the four-dimensional unit cube. Below the simplices are characterized by their vertex numbers:

$T_1 = [0, 1, 2, 4, 8]$, $T_9 = [1, 2, 4, 8, 14]$,

$T_2 = [4, 8, 12, 13, 14]$, $T_{10} = [1, 4, 8, 13, 14]$,

$T_3 = [2, 8, 10, 11, 14]$, $T_{11} = [1, 2, 8, 11, 14]$,

$T_4 = [2, 4, 6, 7, 14]$, $T_{12} = [1, 2, 4, 7, 14]$,

$T_5 = [1, 8, 9, 11, 13]$, $T_{13} = [1, 8, 11, 13, 14]$,

$T_6 = [1, 4, 5, 7, 13]$, $T_{14} = [1, 4, 7, 13, 14]$,

$T_7 = [1, 2, 3, 7, 11]$, $T_{15} = [1, 2, 7, 11, 14]$,

$T_8 = [7, 11, 13, 14, 15]$, $T_{16} = [1, 7, 11, 13, 14]$.

Combining the computed inputs and the triangulation, a feedback law on each simplex is easily computed. As an example, we consider the simplex $T_8 = [7, 11, 13, 14, 15]$. In this situation, we find the affine state feedback by solving

$$\begin{pmatrix} v_7^{\mathrm{T}} & 1 \\ v_{11}^{\mathrm{T}} & 1 \\ v_{13}^{\mathrm{T}} & 1 \\ v_{14}^{\mathrm{T}} & 1 \\ v_{15}^{\mathrm{T}} & 1 \end{pmatrix} \begin{pmatrix} F_8^{\mathrm{T}} \\ g_8^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} u_7^{\mathrm{T}} \\ u_{11}^{\mathrm{T}} \\ u_{13}^{\mathrm{T}} \\ u_{14}^{\mathrm{T}} \\ u_{15}^{\mathrm{T}} \end{pmatrix}.$$

In this situation, this linear system of equations becomes

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \\ f_{13} & f_{23} \\ f_{14} & f_{24} \\ g_1 & g_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & \frac{3}{2} \end{pmatrix}.$$

So the affine feedback on simplex T_8 is given by $u = F_8x + g_8$ with

$$F_8 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad g_8 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Completely analogously the affine feedback for the other 15 simplices may be computed by solving the corresponding set of linear equations.

5. Concluding remarks

In this paper, a reachability problem for affine systems on full-dimensional polytopes was considered. First, necessary conditions were derived for the existence of a continuous feedback law, that realizes the control objective of steering the state in finite time to a particular facet of the polytope. These conditions consist of a number of linear inequalities on the inputs at the vertices of the polytope. For the same control problem also a different set of sufficient conditions was obtained. If the polytope is a simplex, the necessary and sufficient conditions were shown to be equivalent, and a solution was obtained using affine state feedback. Furthermore, a procedure has been described for the computation of the affine control law. For general full-dimensional polytopes, the sufficient conditions are somewhat stronger than the necessary conditions, but still may be described as linear inequalities on the inputs at the vertices of the polytope. If these conditions are satisfied, the problem is solvable by continuous piecewise-affine state feedback.

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Appendix A.

Proof of Theorem 4.1 (continuation). Let $f : P_N \to U$ be a Lipschitz function with Lipschitz constant L, satisfying conditions (i) and (ii) of Theorem 4.1. We start by extending f to the whole space $\mathbb{R}^N$, using a non-linear projection Π_{P_N} on the convex polytope P_N. This projection maps an arbitrary point $x \in \mathbb{R}^N$ to the point $w \in P_N$ that is closest to x:

$$\Pi_{P_N}(x) = \arg\min\{\|x - w\| \mid w \in P_N\}.$$

It may be verified that for all $x, y \in \mathbb{R}^N$

$$\|\Pi_V(x) - \Pi_V(y)\| \leqslant \|x - y\|.$$

Next, we define

$$\tilde{f} : \mathbb{R}^N \to U : \tilde{f}(x) = f(\Pi_{P_N}(x)).$$

Then $\tilde{f}$ is a Lipschitz function on $\mathbb{R}^N$ because for all $x, y \in \mathbb{R}^N$:

$$\begin{aligned} \|\tilde{f}(x) - \tilde{f}(y)\| &= \|f(\Pi_{P_N}(x)) - f(\Pi_{P_N}(y))\| \\ &\leqslant L\|\Pi_{P_N}(x) - \Pi_{P_N}(y)\| \leqslant L\|x - y\|. \end{aligned}$$

Now we consider the closed-loop system

$$\dot{x} = Ax + B\tilde{f}(x) + a \tag{A.1}$$

on the state space $\mathbb{R}^N$, and show that if $x(0) \in P_N$, then the state will leave the polytope P_N through the exit facet F_1 in finite time.

For this purpose, we introduce the vector field $g(x) = (p - x)/(1 + \|p - x\|)$, where p is a *fixed* vector in the interior of the full-dimensional polytope P_N. Clearly, $\|g(x)\| \leqslant 1$ for all $x \in \mathbb{R}^N$, and on all facets of P_N the vector field is pointing into the polytope P_N: if facet F_i is contained in the hyperplane $n_i^{\mathrm{T}} x = \alpha_i$, then $n_i^{\mathrm{T}} p < \alpha_i$ and for all $x \in F_i$, $n_i^{\mathrm{T}} g(x) = (1/(1 + \|p - x\|))(n_i^{\mathrm{T}} p - n_i^{\mathrm{T}} x) < 0$.

Let $\varepsilon > 0$, and consider the following perturbed closed-loop dynamics on $\mathbb{R}^N$:

$$\dot{x}_\varepsilon = Ax_\varepsilon + B\tilde{f}(x_\varepsilon) + a + \varepsilon g(x_\varepsilon). \tag{A.2}$$

We first show that a trajectory $x_\varepsilon(t)$, with $x_\varepsilon(0) \in P_N$ cannot leave the polytope P_N through one of the facets $F_2, \ldots, F_K$. Indeed, let $i \in \{2, \ldots, K\}$, and $w \in F_i$, then condition (ii) of Theorem 4.1 implies

$$\begin{aligned} n_i^{\mathrm{T}} \dot{x}_\varepsilon|_w &= n_i^{\mathrm{T}}(Aw + B\tilde{f}(w) + a + \varepsilon g(w)) \\ &= n_i^{\mathrm{T}}(Aw + Bf(w) + a) + \varepsilon n_i^{\mathrm{T}} g(w) < 0. \end{aligned}$$

So, unless $w \in F_1$, the velocity vector field of (A.2) is pointing strictly into the polytope P_N. This implies that any solution x_ε of the perturbed closed-loop system (A.2) with initial state in the polytope P_N, can only leave P_N through the facet F_1. By contradiction we will prove that the same is true for the unperturbed closed-loop system (A.1).

Suppose that there exists $x_0 \in P_N$ such that the solution $x(t)$ of system (A.1) with initial value $x(0) = x_0$ leaves the polytope P_N through facet F_i with $i \in \{2, \ldots, K\}$ before it has reached facet F_1. Let n_1 and n_i denote the normal vectors of the facets F_1 and F_i, respectively, and assume that they are of unit length and point out of the polytope P_N. Furthermore, the two hyperplanes containing the facets F_1 and F_i are described by the equations $n_1^{\mathrm{T}} x = \alpha_1$ and $n_i^{\mathrm{T}} x = \alpha_i$, respectively. Then there exists $t_0 > 0$ such that

(i) $\beta := n_i^{\mathrm{T}} x(t_0) > \alpha_i$,
(ii) $\forall t \in [0, t_0]$: $n_1^{\mathrm{T}} x(t) < \alpha_1$.

Define $\gamma := \max\{n_1^{\mathrm{T}} x(t) \mid t \in [0, t_0]\}$ and $H := \|A\| + \|B\| \cdot L$, where $\|\cdot\|$ denotes the operator norm of a matrix, and L is the Lipschitz constant of the function $\tilde{f}$. Let $0 < \varepsilon < (H/(\mathrm{e}^{Ht_0} - 1)) \cdot \min((\beta - \alpha_i), (\alpha_1 - \gamma))$. We will compare the solutions $x_\varepsilon(t)$ of the perturbed system (A.2) and $x(t)$ of the unperturbed system (A.1), both with initial value $x_0 \in P_N$, on the interval $[0, t_0]$. For all $t \in [0, t_0]$ we have

$$\begin{aligned} x(t) - x_\varepsilon(t) &= \int_0^t \dot{x}(s) - \dot{x}_\varepsilon(s)\,\mathrm{d}s \\ &= \int_0^t (Ax(s) + B\tilde{f}(x(s)) + a) \\ &\quad -(Ax_\varepsilon(s) + B\tilde{f}(x_\varepsilon(s)) \\ &\quad +a + \varepsilon g(x_\varepsilon(s)))\,\mathrm{d}s \\ &= \int_0^t A(x(s) - x_\varepsilon(s)) + B(\tilde{f}(x(s)) \\ &\quad -\tilde{f}(x_\varepsilon(s)))\,\mathrm{d}s - \int_0^t \varepsilon g(x_\varepsilon(s))\,\mathrm{d}s, \end{aligned}$$

hence

$$\begin{aligned} \|x(t) - x_\varepsilon(t)\| &\leqslant \int_0^t (\|A\| + \|B\| \cdot L)\|x(s) - x_\varepsilon(s)\|\,\mathrm{d}s \\ &\quad + \int_0^t \varepsilon \cdot \|g(x_\varepsilon(s))\|\,\mathrm{d}s \\ &\leqslant \varepsilon \cdot t + H \int_0^t \|x(s) - x_\varepsilon(s)\|\,\mathrm{d}s. \end{aligned}$$

Next, we apply Gronwall's Lemma (see e.g. Hille, 1969, p. 19) and find that for all $t \in [0, t_0]$

$$\begin{aligned} \|x(t) - x_\varepsilon(t)\| &\leqslant \varepsilon t + H \int_0^t \mathrm{e}^{H(t-s)} \varepsilon s\,\mathrm{d}s \\ &= \frac{\varepsilon}{H}(\mathrm{e}^{Ht} - 1). \end{aligned}$$

So, in particular

$$\|x(t) - x_\varepsilon(t)\| \leqslant \frac{\varepsilon}{H}(\mathrm{e}^{Ht_0} - 1) < \min(\beta - \alpha_i, \alpha_1 - \gamma)$$

for all $t \in [0, t_0]$. This implies that for every $t \in [0, t_0]$ the solution x_ε satisfies

$$\begin{aligned} n_1^{\mathrm{T}} x_\varepsilon(t) &= n_1^{\mathrm{T}}(x_\varepsilon(t) - x(t)) + n_1^{\mathrm{T}} x(t) \\ &\leqslant \|n_1\| \cdot \|x_\varepsilon(t) - x(t)\| + \gamma \\ &< 1 \cdot (\alpha_1 - \gamma) + \gamma = \alpha_1, \end{aligned}$$

i.e. on the interval $[0, t_0]$, the solution x_ε does not reach the facet F_1. Since x_ε can only leave the polytope P_N through this facet, this indicates that $x_\varepsilon(t) \in P_N$ for all $t \in [0, t_0]$. In combination with the observation that

$$\begin{aligned} n_i^{\mathrm{T}} x_\varepsilon(t_0) &= n_i^{\mathrm{T}}(x_\varepsilon(t_0) - x(t_0)) + n_i^{\mathrm{T}} x(t_0) \\ &= n_i^{\mathrm{T}}(x_\varepsilon(t_0) - x(t_0)) + \beta \\ &\geqslant \beta - \|n_i\| \cdot \|x_\varepsilon(t_0) - x(t_0)\| \\ &> \beta - 1 \cdot (\beta - \alpha_i) = \alpha_i, \end{aligned}$$

we obtain a contradiction. We conclude that every solution of the unperturbed closed-loop system (A.1), starting in a point $x_0 \in P_N$ can only leave the polytope P_N through the exit facet F_1.

Finally, since P_N is a compact set, condition (i) of Theorem 4.1 guarantees that in the polytope P_N the state of system (A.1) will move in the direction of F_1 with a speed of at least $c := \inf\{n_1^T(Ax + Bf(x) + a) \mid x \in P_N\} > 0$. Hence, the exit facet F_1 is reached in finite time. □

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