Observability of Piecewise-Affine Hybrid Systems

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Abstract. We consider observability for a class of piecewise-affine hybrid systems without inputs. The aim is to give verifiable conditions for observability in terms of linear equations and inequalities. We first discuss a number of important concepts, such as discrete-event detectability and trajectory observability. We give sufficient conditions for observability, observability in infinitesimal time, and observability after a single discrete event. The former conditions are used to construct an observer for the system, the latter are applied to deduce observability for an example system.

1 Introduction

Observability of a systems is a sufficient condition for the proper operation of an observer. Observability is also one of the conditions needed to characterise minimality of a realization of an input-output map. Of these two uses of the concept of observability, the first is the most practically useful while the second is of primary theoretical significance. For each new class of dynamic systems the concept of observability has to be explored and conditions for it derived.

A piecewise-affine hybrid system (PAHS) can be considered as a product of a finite state automaton and a family of finite-dimensional affine systems on polytopes. A formal definition is given in Sect. 2. Attention will be restricted to piecewise-affine hybrid systems without input function. Observability will be formulated as injectiveness of the map from the initial state to the future output trajectory, as is generally done in system theory, see for example, [1, 2]. The concept of final-state observability or reconstructability, in which one wishes to determine the final state from an output trajectory defined up to the current time, will not be considered. Observability of piecewise-affine hybrid systems is dual to reachability as developed in [3].

It was stated by E. Sontag [4] that observability is undecidable for a class of piecewise-linear hybrid systems, and this suggests that observability is also undecidable for the class of piecewise-affine systems considered here. We therefore concentrate on finding sufficient conditions for observability.

Observing a piecewise-affine hybrid system involves determining both the discrete state and the associated continuous state. Of critical importance is detecting the times of the discrete events. The interplay between the discrete and continuous dynamics means that the system may be observable even if the affine system associated with a particular discrete state is unobservable, since enough information may be available using a combination of outputs in different states to determine the initial state. Unlike linear systems, observability of piecewise-affine hybrid systems may be possible only after a finite time trajectory has been observed. As well as giving sufficient conditions for observability, we will also give a construction of an initial state observer.

An exponentially-stable observer for piecewise-affine hybrid systems was given by A. Balluchi et al in [5]. One of the conditions for the existence of the observer is that the affine system associated to each discrete state is observable. Due to this assumption, observability reduces to recovering the current discrete state, for which algorithms were provided in the paper. Sufficient conditions for final-state observability were given in [6]. Observability of piecewise-affine hybrid systems is also discussed in the paper [7]. During the Workshop Hybrid Systems—Computation and Control held in April 2003 in Prague [8], several papers on observability were presented. Of these papers, observability in the case of multiple discrete states was only explicitly discussed by R. Vidal et al [9], who gave necessary and sufficient conditions for observability of a restricted class of hybrid systems called *jump-linear systems*. This paper extends the results of [9] for jump-linear systems by considering affine systems, discontinuous jumps in the systems state, and switches induced by guard conditions.

An overview of this paper follows. Section 2 contains a definition of the class of piecewise-affine hybrid systems and of the concept of observability. The problem of characterising observability is formulated. In Sect. 3 a theorem is stated on how to recover the discrete state from the observed output, and results are formulated of how to recover the continuous state from the output functions. In Sect. 4 sufficient conditions are stated and proven for the observability of these systems. Procedures for the construction of the initial state are presented in Sect. 5. Examples of observable and of unobservable systems are presented in Sect. 6. The paper ends with concluding remarks.

2 Problem formulation

We now give a formal definition of piecewise-affine hybrid systems, and discuss various concepts related to the observability of such systems.

2.1 Piecewise-affine hybrid systems

Definition 1 (Piecewise-affine hybrid system).
A continuous-time piecewise-affine hybrid system *consists of*

- *A closed convex polyhedral* input set $U \subset \mathbb{R}^m$.
- *An* observation set $Y \subset \mathbb{R}^p$.
- *A finite* discrete state set Q.
- *A set* E *of* discrete events, *comprising a set* E_{in} *of* input events *and a set* E_{ct} *of* dynamically generated events.

– *A* discrete state transition function ρ*, which is a partial function* $Q \times E \to Q$*.*

– *For each discrete state* $q \in Q$*,*
  - *a convex polyhedral* continuous state space $X_q \subset \mathbb{R}^{n_q}$
  - *a convex polyhedral* initial state set $X_q^{\text{init}} \subset X_q$
  - *an* affine system $\mathcal{A}_q$ *given by*

$$\dot{x}(t) = a_q + A_q x(t) + B_q u(t)$$

  - *an affine* output map $\mathcal{C}_q : X_q \times U \to Y$ *given by*

$$y(t) = c_q + C_q x(t) + D_q u(t)$$

– *For each event* $e \in E$*, and each discrete state* $q \in Q$ *such that* $\rho(q, e)$ *is defined,*
  - *a closed convex polyhedral* guard set $X_{(q,e)}^{\text{guard}} \subset X_q$*.*
  - *an affine* continuous state transition function $\mathcal{F}_{(q,e)} : X_{(q,e)}^{\text{guard}} \to X_{\rho(q,e)}$ *given by*

$$\mathcal{F}_{(q,e)}(x) = f_{(q,e)} + F_{(q,e)} x.$$

The *state space* X of a PAHS is the set $\bigcup_{q \in Q} \{q\} \times X_q$. We can assume that each X_q has full dimension n_q, and so can be represented as the solution set of a system of linear inequalities. The initial sets X_q^{init} and the guard sets $X_{(q,e)}^{\text{guard}}$ can be represented by a combination of linear equations and linear inequalities. Since we shall be mostly interested in the linear equations for the initial and guard sets, we give explicit formulae:

- The linear equations for X_q^{init} are

$$\mathcal{J}_q(x) = j_q + J_q x = 0.$$

- The linear equations for $X_{(q,e)}^{\text{guard}}$ are

$$\mathcal{G}_{(q,e)}(x) = g_{(q,e)} + G_{(q,e)} x = 0.$$

However, the results in this paper extend easily to the case of mixed linear equations and inequalities defining polyhedral sets for X_q, X_q^{init} and $X_{(q,e)}^{\text{guard}}$.

Definition 2. *An* trajectory *of the PAHS system* $\mathcal{H}$ *on the time index set* $[t_0, t_1) \subset \mathbb{R}$ *with continuous input* $u : [t_0, t_1) \to U$ *is a right-continuous function* $(q, x) : [t_0, t_1) \to X$ *such that*

1. $x(t_0) \in X_{q(t_0)}^{\text{init}}$.
2. *If an event* $e \in E$ *occurs at time* t*, then*

$$x^-(t) \in X_{(q^-(t),e)}^{\text{guard}}, \qquad q(t) = \rho(q^-(t), e) \quad \text{and} \quad x(t) = \mathcal{F}_{(q^-(t),e)}(x^-(t)).$$

3. *If no event occurs at time* t*, then* x *is continuous at* t*, and*

$$\frac{d}{dt} x(t) = a_{q(t)} + A_{q(t)} x(t) + B_{q(t)} u(t).$$

Here we use $q^{-}(t)$ and $x^{-}(t)$ to denote, respectively, $\lim_{\tau \nearrow t} q(\tau)$ and $\lim_{\tau \nearrow t} x(\tau)$. Events in E_{ct} model events driven by the continuous dynamics, while events in E_{in} model user input events. If a event in E_{ct} is enabled at time t, then an event in E_{ct} must occur.

We further assume that every trajectory can be continued for infinite time (non-blocking), and only finitely many events occur on any finite time interval (non-Zenoness). Note that the non-blocking condition requires that each point of the boundary of X_q at which the continuous evolution leaves X_q must be contained in some guard set $X_{(q,e)}^{\text{guard}}$. For simplicity of exposition, we assume (unless otherwise stated) that no event occurs at the initial time t_0, and that at most one event occurs at any other subsequent time. The case of more than one event at a given time is a fairly straightforward extension.

For most of this paper we shall restrict to the class of PAHS *without inputs*. For such systems, there are no input events (i.e. $E = E_{\text{ct}}$) and the continuous dynamics in discrete state q reduces to

$$\dot{x}(t) = a_q + A_q x(t), \qquad y(t) = c_q + C_q x(t).$$

To simplify much of the notation, for each discrete state q and each time t we construct a *time-evolution map* $\mathcal{S}_{(q,t)}$ such that for any $x \in X_q$,

$$\frac{d}{dt}\mathcal{S}_{(q,t)}(x) = a_q + A_q\,\mathcal{S}_{(q,t)}(x).$$

By solving the continuous evolution equations, we find $\mathcal{S}_{(q,t)}$ is the affine map

$$\mathcal{S}_{(q,t)}(x) = s_{(q,t)} + S_{(q,t)}x = \left(\exp\left(A_q t\right) - I\right) A_q^{-1} a_q + \exp\left(A_q t\right)x. \qquad (1)$$

Note that $(\exp(A_q t) - I)A_q^{-1}$ is well-defined by its power series even if A_q is not invertible.

In examples, we will usually label states by integers i, and events e_{ij}, with $\rho(i, e_{ij}) = j$

2.2 Observability and observers

The concept of observability is best formulated in terms of the *state-output* map of a system. The state-output map of a deterministic system on the time interval $[t_0, t_1)$ is the functional $\lambda : X \times U^{[t_0,t_1)} \to Y^{[t_0,t_1)}$ assigning to each initial state $x_0 \in X$ and each admissible input function $u(t)$ the output function $y(t)$ for the trajectory $x(t)$ giving the response of the system to the input function $u(t)$ with $x(t_0) = x_0$.

As proposed by Sontag [1, 2], there are many different notions of observability, each relating to the degree to which the state can be determined from the state-output map. For systems without inputs, these observability concepts reduce to determining either the initial state $x(t_0)$ or the final state $x(t_1)$. A system is *(initial-state) observable* if the initial state can be determined from the output function $y(t) \in Y^{[t_0,t_1)}$, and *final-state observable* if the final state can be determined from the output function. Final-state observability is sometimes referred to as *current-state observability* or *reconstructability* in the literature [10, 6]. Some observability concepts for discrete event systems are given in [11]

We can further distinguish observability concepts by considering the dependence on the time domain.

Definition 3. *A system is* observable in time T *if the initial state can be determined from the output function* η *restricted to* $[0, T)$.

- *If the system is observable in time* ϵ *for all* $\epsilon > 0$*, then the system is* observable in infinitesimal time.
- *If the system is observable in time* T *for some finite* T*, it is* observable in finite time.
- *If the system is observable in time* ∞*, it is* observable in infinite time.

Unlike linear systems, which are either unobservable, or are observable in infinitesimal time, observable PAHS may be observable in infinitesimal, finite or infinite time.

By an *observer* for a system, we mean a dynamic system driven by the output $y(t)$ which produces an estimate of the state of the plant system. Just as for notions of observability we can consider *initial-state observers*, which estimate the initial state of the plant, $x(t_0)$, and *current-state observers*, which estimate the current state $x(t)$. A number of different classes of observer can be considered, including

- *point estimates* $\widehat{x}(t)$ satisfying $\lim_{t\to\infty} d(\widehat{x}(t), x(t)) = 0$,
- *set estimates* $\widehat{X}(t)$ such that $x(t) \in \widehat{X}(t)$, and
- *probabilistic estimates* which give a probability distribution for $x(t)$.

For linear systems, observers are usually constructed as point estimates evolving under a differential equation, and observability implies convergence of the estimate. For discrete-event systems, observers are usually constructed as set estimates [12]. For stochastic systems, probabilistic estimates such as Kalman filters are used.

For the theory of observability, the most natural concept of observer is an initial-state set estimator. We construct such an observer for PAHS in Sect. 5. A current-state point estimator for PAHS has been considered by Balluchi et al [5]

2.3 Observability of PAHS

The main problem we shall consider is to determine necessary and sufficient conditions for observability of a PAHS without inputs, and the construction of observers. We shall see that the determination of the times of the discrete events is of critical importance. We say an event is *detectable* at a point x if it produces a measurable change in output, otherwise it is *undetectable at* x. An event may be detectable at certain points and not at others, even in the same discrete state. An event is *detectable* in a state q if it is detectable at all points in the guard set

$X^{\mathrm{guard}}_{(q,e)}$, and a system is *event detectable* if all events are detectable in all states. We shall see that conditions for event detectability can be expressed in affine form, and so is possible to determine whether an event is detectable.

Notice that by detectability we only require that the *time* that an event occurs can be determined; we do not require that actual event is known. Indeed, the determination of the event depends upon knowledge of the continuous state, and so may be possible at some points in the guard set but not others. The situation for hybrid systems is therefore more complicated than that of discrete-event systems, in which an event is either observable or unobservable.

The *event-time sequence* of a trajectory is the sequence (t_i) of event times. It is possible to compute the event-time sequence for an event-detectable hybrid system. The *timed event sequence* of a trajectory is the sequence of pairs (e_i, t_i) of events and event times. Since there are only finitely many events, for any event-time sequence of finite length, there are only a finite number of possible timed event sequences.

Under certain conditions, we can actually determine the discrete state completely from the continuous-time dynamics. We say discrete states q and q' are *distinguishable* if for any $x \in X_q$ and $x' \in X_{q'}$ the observations $y(t)$ for the trajectory through x and $y'(t)$ for the trajectory through x' are different on any interval $[0, \epsilon)$.

However, it is possible to obtain more information by considering events which have *not occurred* on some time interval (t_i, t_{i+1}) between consecutive events. These conditions take the form that $x(t)$ cannot satisfy the guard conditions for any $e \in E_{\mathrm{ct}}$. Unfortunately, since $x(t)$ does not depend in an affine way on t, these conditions cannot be expressed as linear equations and inequalities. This means that while it is possible to give sufficient conditions for observability, the formulation of necessary conditions requires non-linear conditions. Some theoretical progress can be made by considering o-minimal systems [13], but these conditions do not seem particularly amenable to analysis. We shall therefore restrict attention to finding sufficient conditions for observability which can be expressed in terms of affine equations and inequalities.

3 Observability equations

Rather than immediately give necessary or sufficient conditions for observability of PAHS, we consider important sub-problems for which we can formulate conditions in terms of linear equations. We give conditions for deducing the continuous state given the discrete state, detecting the occurrence of discrete events and determining the discrete state of the system. We then formulate equations which must be satisfied if the initial discrete state and the timed event sequence are known. Most of these equations follow in a straightforward way from the definitions, and are similar to those of [9], which are formulated in terms of *joint observability matrices* and *switching observability matrices*.

If it is not possible to detect an event, it is possible to formulate linear equations which hold between the continuous state after the previous detected event and before the next detected event. We shall not discuss the construction here.

3.1 Observability equations for affine systems

Consider the affine system with state $x \in \mathbb{R}^n$ evolving by $\dot{x} = a + Ax$, and observations $y \in \mathbb{R}^p$ given by $y = c + Cx$. Computing derivatives of y, we obtain the general formula

$$\frac{d^k y}{dt^k} = CA^{k-1}a + CA^k x \quad \text{for } k \geqslant 1.$$

The linear terms are the same as for linear systems, and give the *observability matrix* O of linear systems theory. The constant expressions give rise to an *observability vector* o, and the derivatives give the *output derivative vector* $\mathcal{Y}(t)$. We have

$$O = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix}, \qquad o = \begin{pmatrix} c \\ Ca \\ CAa \\ \vdots \end{pmatrix} \quad \text{and} \quad \mathcal{Y}(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \\ \vdots \end{pmatrix}. \tag{2}$$

If we further define the *observability map* $\mathcal{O}$ by $\mathcal{O}\left(x(t)\right) = o + Ox(t)$ we obtain the following *observability equation*

$$\mathcal{Y}(t) = \mathcal{O}\left(x(t)\right) = o + Ox(t) \tag{3}$$

which is satisfied at all points on the trajectory. Further, if $\mathcal{Y}(t_0) = \mathcal{O}\left(x(t_0)\right)$ for some t_0, then $\mathcal{Y}(t) = \mathcal{O}\left(x(t)\right)$ for all t.

Recall that a linear system is observable if and only if $\operatorname{rank}(O) = n$, or equivalently, if $\operatorname{nullity}(O) = 0$. (The *nullity* of a matrix is the dimension of the null space.) An observable linear system is necessarily observable in infinitesimal time.

Remark 1. As defined here, the observability matrices have infinitely many rows. However, as is standard in systems theory for linear systems, we can restrict to matrices with pn rows, since CA^n can be expressed as a linear combination of the CA^k for $k < n$. In examples, we shall always consider enough rows to obtain the strongest available conditions.

3.2 Determining the discrete dynamics

The main difficulty in the observability analysis of hybrid systems is in determining the discrete state and events. We first aim to find conditions under which we can deduce that a discrete event has occurred. Since the system output is analytic, we can deduce the presence of a discrete event by the non-smoothness of the output function.

The following result gives necessary and sufficient conditions for event detectability, as discussed in Subsection 2.3.

Proposition 1 (Event detection). *The event e is detectable in $q \in Q$ if and only if the linear equations*

$$\mathcal{G}_{(q,e)}(x) = 0 \quad \textit{and} \quad \mathcal{O}_q(x) = \mathcal{O}_{\rho(q,e)}\left(\mathcal{F}_{(q,e)}(x)\right) \tag{4}$$

have no solutions with $x \in X_q$.

The condition $\mathcal{O}_q(x) = \mathcal{O}_{\rho(q,e)}\left(\mathcal{F}_{(q,e)}(x)\right)$ is obtained by equating the vectors $\mathcal{Y}^-(t)$ and $\mathcal{Y}(t)$, and the condition $\mathcal{G}_{(q,e)}(x) = 0$ is simply the guard condition in X_q.

The following result gives a criterion for discrete states q and q' to be distinguishable, and follows immediately by considering $\mathcal{Y}(t)$.

Proposition 2 (Discrete state distinguishability). *If the linear equations*

$$\mathcal{O}_q(x) = \mathcal{O}_{q'}(x') \tag{5}$$

have no solution with $(x, x') \in X_q \times X_{q'}$, then the discrete states q and q' are distinguishable.

Clearly, if $c_q + C_q x \neq c_{q'} + C_{q'} x'$ for all $(x, x') \in X_q \times X_{q'}$, then q and q' are distinguishable; indeed, they can be distinguished by a single observation value without computing derivatives. Distinguishability of discrete states is a very strong condition.

Event detection is of critical importance in finding affine equations for observability, since the set of all initial points x_0 with a first discrete transition at time t_1 is an affine space, since $x^-(t_1) = \mathcal{S}_{(q_0, t_1 - t_0)}\left(x(t_0)\right)$ is an affine function of $x(t_0)$, but the set of all points in X_q with a discrete transition e at *some* time $t \geqslant 0$ is *not* an affine space, as $x(t) = \mathcal{S}_{(q, t - t_0)}\left(x(t_0)\right)$ does not depend in an affine way on t.

3.3 Determining the continuous state

If the timed event sequence of the system is known, it is easy to write down equations for the continuous state. The system evolution can be written as

$$x(t_{n+1}) = \mathcal{F}_{(q_n, e_{n+1})} \mathcal{S}_{(q_n, t_{n+1} - t_n)} x(t_n) \tag{6}$$

Combining this with the guard equations and observability equations we obtain the following result.

Proposition 3. *Let $\mathcal{H}$ be a PAHS system, and suppose the initial time t_0, the initial state q_0, the timed event sequence (t_i, e_i) and the output function $y(t)$ are known. Let q_i be the state immediately after the ith event. Then the initial state $x = x(t_0)$ satisfies the following linear equations, the* trajectory equations

$$\begin{array}{rl}
\mathcal{J}_{q_0} x(t_0) & = \quad 0 \\
\mathcal{O}_{q_0} x(t_0) & = \mathcal{Y}(t_0) \\
\mathcal{G}_{(q_0,e_1)} \mathcal{S}_{(q_0, t_1 - t_0)} x(t_0) & = \quad 0 \\
\mathcal{O}_{q_1} \mathcal{F}_{(q_0,e_1)} \mathcal{S}_{(q_0, t_1 - t_0)} x(t_0) & = \mathcal{Y}(t_1) \\
\mathcal{G}_{(q_1,e_2)} \mathcal{S}_{(q_1, t_2 - t_1)} \mathcal{F}_{(q_0,e_1)} \mathcal{S}_{(q_0, t_1 - t_0)} x(t_0) & = \quad 0 \\
\mathcal{O}_{q_2} \mathcal{F}_{(q_1,e_2)} \mathcal{S}_{(q_1, t_2 - t_1)} \mathcal{F}_{(q_0,e_1)} \mathcal{S}_{(q_0, t_1 - t_0)} x(t_0) & = \mathcal{Y}(t_2) \\
 & \quad \vdots
\end{array} \tag{7}$$

Remark 2. If $t_{i+1} = t_i$ for some i, then we remove the equations containing $\mathcal{O}_{q_i}$, since no output is observed in state q_i.

4 Conditions for observability of PAHS

We now present several sufficient conditions for observability of a PAHS. The conditions given in Theorem 1 are the sharpest, but cannot be verified since they contain nonlinear dependencies on the event times. However, these conditions can be used to construct an observer for the system. The conditions in Theorems 2 and 3 are expressed purely in terms of linear equations, and hence can be solved for a given system.

4.1 Sufficient conditions for observability

Unfortunately, it is not possible to give necessary and sufficient conditions for the observability of a PAHS purely in terms of linear equations involving the initial state and event times. The following conditions are sufficient for observability, and are expressed as equations which are linear in the initial state, but depend in a nonlinear way on the possible event times.

Theorem 1. *A PAHS $\mathcal{H}$ is observable if:*

1. *All events e of $\mathcal{H}$ are detectable.*
2. *For all possible event-time sequences (t_i), there exists at most one value (q_0, x_0) which is a solution of the trajectory equations (7) for any possible event sequence.*

Proof. Since all events are detectable, the event time sequence can be uniquely determined. Any possible initial state (q_0, x_0) must satisfy the trajectory equations (7) for some event sequence. Hence if there is only one such initial state which satisfies these equations for any event sequence, this must be the initial state of the system given the output, hence $\mathcal{H}$ is observable.

Remark 3. If for some initial state q_0 and timed event sequence (t_i, e_i) there are two initial points x_0 and x_0' with the same output, we can eliminate the constant terms in the trajectory equations (7). Hence the conditions under which this occur reduce to a nullity (or rank) condition on the matrix giving the linear part of equations (7). Hence a necessary condition for the system to be observable is

$$\text{nullity} \begin{pmatrix} J_{q_0} \\ O_{q_0} \\ G_{(q_0,e_1)} S_{(q_0,t_1-t_0)} \\ O_{q_1} F_{(q_0,e_1)} S_{(q_0,t_1-t_0)} \\ \vdots \end{pmatrix} = 0. \tag{8}$$

4.2 Observability in infinitesimal time

Unfortunately, since the sufficient conditions for observability given in Theorem 1 include nonlinear dependencies on the event times t_i, which are not known a priori, they do not give checkable conditions. A simple checkable condition, which is akin to the observability conditions for linear systems, are the following necessary and sufficient conditions for observability in infinitesimal time.

Theorem 2 (Conditions for observability in infinitesimal time). *A PAHS $\mathcal{H}$ without inputs is observable in infinitesimal time if, and only if, for initial conditions in X^{init}:*

1. *All discrete states are distinguishable, and*
2. *For all discrete states, the corresponding affine system is observable.*

Proof. If all discrete states are distinguishable, then the initial discrete state q_0 can be determined from the initial trajectory. The initial condition x_0 can then be determined since the continuous dynamics in the discrete state discrete state is observable.

Conversely, if the system $\mathcal{H}$ is observable in infinitesimal time, then it must be possible to determine the initial condition without seeing any discrete events. Hence the discrete states must be distinguishable, since it must be possible to determine the initial discrete state, and the affine systems must be observable, or else it is impossible to determine the initial continuous state.

In terms of linear equations, the conditions for observability in infinitesimal time become $\mathcal{J}_q(x) = \mathcal{J}_{q'}(x')$ and $\mathcal{O}_q(x) = \mathcal{O}_{q'}(x')$ has no solutions in $X_q \times X_{q'}$ if $q \neq q'$, and $J_q x = 0$ and $O_q x = 0$ has a single solution for all q.

4.3 Single-event observability

A second condition under which a hybrid system is observable using only linear equations is the following *single-event observability.*

Definition 4 (Single-event observability). *A hybrid system is* single-event observable *if any two trajectories are distinguishable for any time after their first discrete event.*

Conditions for single-event observability can be formulated in terms of linear equations if we ignore the dependence on the initial condition, since then the time-evolution map $\mathcal{S}_{(q_0,t_1-t_0)}$ which has nonlinear dependence on t_1 does not enter into the equations.

Theorem 3 (Conditions for single-event observability). *A PAHS $\mathcal{H}$ is single-event observable if:*

1. *All events e of $\mathcal{H}$ are detectable.*
2. *If q and q' are discrete states for which there exist initial conditions for which no discrete events occur, then q and q' are distinguishable and the corresponding affine systems are observable.*

3. For every pair of events e, e′, the only solutions of the linear equations

$$\begin{gathered}\mathcal{O}_q(x) = \mathcal{O}_{q'}(x') \qquad \mathcal{G}_{(q,e)}(x) = \mathcal{G}_{(q',e')}(x') = 0 \\ \mathcal{O}_{\rho(q,e)}\big(\mathcal{F}_{(q,e)}(x)\big) = \mathcal{O}_{\rho(q',e')}\big(\mathcal{F}_{(q',e')}(x')\big)\end{gathered} \tag{9}$$

are $(q, x) = (q', x')$.

If the initial conditions for $\mathcal{H}$ *are* $X_q^{\text{init}} = \mathbb{R}^{n_q}$ *for all* q*, and every trajectory undergoes a discrete event, then these conditions are also necessary.*

Proof. Since all events are detectable, it is possible to distinguish trajectories which have a discrete event from those which do not, and it is possible to distinguish trajectories which do not contain a discrete event by Theorem 2.

If the first event e_1 takes place at time t_1, let $x = x^-(t_1) \in X_{q_0}$. Then $x(t_0) = \mathcal{S}^{-1}_{(q_0.t_1-t_0)}(x)$, so $x(t_0)$ can be reconstructed from x. The guard conditions give $\mathcal{G}_{(q_0,e_1)}(x) = 0$, and the observations give $\mathcal{O}_{q_0}(x) = \mathcal{Y}^-(t_1)$ and $\mathcal{O}_{(q_1,e_1)}\big(\mathcal{F}_{(q_0,e_1)}(x)\big) = \mathcal{Y}(t_1)$.

If $\mathcal{H}$ is single-event observable, then the initial discrete state must be uniquely determined, so there can be no solutions of (9) for $q \neq q'$ and $(x, x') \in X_q \times X_{q'}$. Hence the discrete state q can be determined from $y(t)$. Then there can be no solutions of (9) with $q = q'$ and $x \neq x'$.

The results of this section give sufficient conditions for observability of a PAHS. The main thrust of these results is that events should be detectable, and for all possible timed event sequences, there should be a single solution to the trajectory equation. The main difficulty with applying this result is determining the possible timed-event sequences. We therefore have stronger conditions for instantaneous observability and single-event observability.

5 Observers for PAHS

We now consider the construction of observers for an event-detectable PAHS. From the previous discussions of observability it is clear that an observer must contain a discrete-state estimate and some estimate of the (initial or current) continuous state. Since the initial conditions X_q^{init} and guard sets $X_{(q,e)}^{\text{guard}}$ polyhedra, it is natural to consider polyhedral estimates for the initial state, which can be given in terms of mixed linear equations and inequalities, and to update the estimate whenever an event is detected. This does not result in much loss of information; the only information that is lost is whether a state estimate is impossible due to some guard condition being met.

It is also possible to consider only the linear equations for the initial conditions and guard sets. This has the advantage that instead of considering systems of equations and inequalities defining polyhedra, which may contain arbitrarily many equations, an estimate of the initial state in X_q in terms of linear equations can be reduced to at most n_q equations. However, the loss of information may result in more discrete states than necessary being considered.

We remark that while our observers will be constructed based on the computation of derivatives of the output function, similar observers can be constructed by sampling $y(t)$ at discrete times; for generic sampling times and error-free sampling, the same information is obtained from n samples as from n derivatives of $y(t)$. See [14] for sampling of continuous-time linear systems.

5.1 Affine observers

We now construct an affine observer for a PAHS.

Definition 5. *An* affine observer *is a pair* $(\tau, \mathbf{O})$ *where* τ *is the last event time, and* $\mathbf{O}$ *is a set of tuples* $(q^{\text{init}}, q, \Lambda, \Sigma)$ *where*

- $q^{\text{init}} \in Q$ *is the* initial state
- $q \in Q$ *is the* current discrete state
- Λ *is a system of linear equations and inequalities with domain* $\mathbb{R}^{n_{q_{\text{init}}}}$
- Σ *is an affine map from* $X_{q_{\text{init}}}$ *to* X_q.

The observer dynamics are given by the procedure below.

Procedure 4.

0. Set Obs $= (\tau, \mathbf{O})$ where τ is the initial time t_0 and $\mathbf{O} = \{O^{\text{init}}(q) : q \in Q\}$ where $O^{\text{init}}(q) = (q, q, \mathcal{J}_q, I)$. Here, $\mathcal{J}_q$ is a set of linear equations defining X_q^{init} and I is the identity on $\mathbb{R}^{n_q}$. Go to step 3.
1. Set $\delta t = t - \tau$, the time since the last event, and set $\tau = t$.
2. Replace each $O \in \mathbf{O}$ with a copy of itself for each $e \in E$ for which $\rho(q, e)$ is defined, and modify these copies as follows:
   (a) Set $\Sigma = \mathcal{S}_{(q,\delta t)} \circ \Sigma$.
   (b) Append the equations $(\mathcal{G}_{(q,e)} \circ \Sigma)(x) = 0$ to Λ.
   (c) Set $\Sigma = \mathcal{F}_{(q,e)} \circ \Sigma$.
   (d) Set $q = \rho(q, e)$.
3. After time ϵ update each $O \in \mathbf{O}$ by appending the equations $(\mathcal{O}_q \circ \Sigma)(x) = 0$ to Λ.
4. For each $O \in \mathbf{O}$, reduce the system of equations Λ.
5. If for any $O \in \mathbf{O}$, the system Λ is inconsistent, discard O.
6. If there exists $q \in Q$ and $x \in X_q$ such that for all $O \in \mathbf{O}$, the state q_{init} of O is q, and x is the only solution of $\Lambda x = 0$, the algorithm terminates. The initial state is (q, x).
7. Wait for a discrete event to be detected, then go to step 1.

This observer is a generalisation of an observer for a discrete-event system, which keeps a set of possible current states for each initial state. For PAHS, we also need to store the equations satisfied by the continuous state, and the time evolution map from the initial continuous state to the current continuous state given the discrete-event sequence the observer models. Note that not all the information obtained about the system needs to be stored. In particular, it is not necessary to store the complete time sequence, nor the discrete-event sequence modelled by each element of the observer array.

The observer given above only considers the linear equations for X^{init} and X^{guard}, but it is straightforward to include linear inequalities to Λ. At every stage, Λ therefore defines a closed convex polyhedral set of possible initial points.

6 Examples

Example 1. Let $\mathcal{H}$ be the PAHS with two states $Q = \{1, 2\}$ with $X_1 = X_2 = \mathbb{R}^2$, and events $E = \{e_{12}, e_{21}\}$ for which

$$X_1^{\mathrm{init}} = \{(x_1, x_2) : x_1 = 5\}, \quad X_2^{\mathrm{init}} = \{(x_1, x_2) : x_1 = 1\},$$

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 5 & 6 \end{pmatrix},$$

$$X_{12}^{\mathrm{guard}} = \{(x_1, x_2) : x_1 \leqslant 1\}, \quad X_{21}^{\mathrm{guard}} = \{(x_1, x_2) : x_1 \geqslant 5\},$$

$$F_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The observability matrices are

$$O_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}, \quad O_2 = \begin{pmatrix} 5 & 6 \\ 15 & 24 \\ 45 & 96 \end{pmatrix}.$$

The transition observability matrix and joint observability matrix are

$$O_{12} = -O_{21} = \begin{pmatrix} 4 & 6 \\ 16 & 24 \\ 44 & 96 \end{pmatrix} \quad \text{and} \quad O_{(1,2)} = \begin{pmatrix} -1 & 0 & 5 & 6 \\ 1 & 0 & 15 & 24 \\ -1 & 0 & 45 & 96 \end{pmatrix},$$

Both events e_{12} and e_{21} are detectable since the transition observability matrices are nonsingular. However, the only solution to $O_1 x = O_2 x'$ has $x' = 0$, which is not a possible state of X_2, hence the discrete states are distinguishable. The system is therefore observable since the system is observable in discrete state 2, and every initial state is either in 2 or enters 2 in a detectable event. The trajectory observability matrix for any trajectory entering 2 is the full-rank matrix

$$\begin{pmatrix} O_1 \\ O_2 S_{1,t_1 - t_0} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 6 \\ 15 & 24 \end{pmatrix}.$$

Example 2. Consider the jump-linear system with $Q = \{1, 2, 3\}$, $X_1 = X_2 = X_3 = \mathbb{R}^2$,

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

The observability matrices are given by

$$O_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad O_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and the difference between any two observability matrices has full rank 2.

Suppose the output trajectory is given by

$$\eta(t) = \begin{cases} 0 \text{ if } t < t_1, \\ e^{t-t_1} \text{ if } t \geqslant t_1. \end{cases}$$

Then from the output on $[t_1, \infty)$ we deduce that for $t \geqslant t_1$ the system is in state 3, so for $t = t_1$ we have $x_1 + x_2 = 1$. This is consistent both with $q_0 = 1$, $x_0 = (0, 1)$ and $q_0 = 2$ and $x_0 = (1, 0)$, so the system not observable.

This is a counterexample to a claim of Vidal et al given in Item 4 in Sect. 3 of [9], which states that if a *jump linear system* with n-dimensional state space satisfies $\text{rank}(O_q - O_{q'}) = n$ for all states q, q', then the initial state is reconstructible from the output after the first even time t_1 if the output on $[t_0, t_1)$ is zero. For this system, although all switching times are observable, and the observability subspaces for each discrete state have empty intersection, the discrete states themselves are not observable, since (32) of [9] is satisfied by both states q_1 and q_2.

7 Concluding remarks

The results of the paper are concepts for the observability of piecewise-affine hybrid systems. The case where one or more continuous-space systems at different discrete states are unobservable is explicitly treated. Sufficient conditions for observability of piecewise-affine hybrid systems are stated and proven. Open issues remain concerning the decidability of observability.

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References

1. Sontag, E.: Realization theory of discrete-time nonlinear systems: I. the bounded case. IEEE Trans. Circuits & Systems **26** (1979) 342–356
2. Sontag, E.D.: Mathematical Control Theory. Springer-Verlag, Berlin (1998)
3. van Schuppen, J.: A sufficient condition for controllability of a class of hybrid systems. In Henzinger, T., Sastry, S., eds.: Hybrid systems: Computation and control. Number 1386 in Lecture Notes in Computer Science, Berlin, Springer (1998) 374–383
4. Sontag, E.D.: From linear to nonlinear: Some complexity questions. In: Proceedings of the 34th IEEE Conference on Decision and Control, New York, IEEE Press (1995) 2916–2920
5. Balluchi, A., Benvenuti, L., Di Benedetto, M.D., Sangiovanni-Vincentelli, A.L.: Design of observers for hybrid systems. In Tomlin, C.J., Greenstreet, M.R., eds.: Hybrid Systems: Computation and Control. Volume 2289 of Lecture Notes in Computer Science. Springer-Verlag, Berlin Heidelberg New York (2002) 76–89

6. Balluchi, A., Benvenuti, L., Di Benedetto, M.D., Sangiovanni-Vincentelli, A.L.: Observability for hybrid systems. In: Proc. 42nd IEEE Conference on Decision and Control, Maui, Hawaii, USA (2003)
7. Bemporad, A., Ferrari-Trecate, G., Morari, M.: Observability and controllability of piecewise affine and hybrid systems. IEEE Trans. Automatic Control **45** (2000) 1864–1876
8. Maler, O., Pnueli, A., eds.: Hybrid Systems: Computation and Control. Number 2623 in Lecture Notes in Computer Science. Springer, Berlin (2003) jhvsb.
9. Vidal, R., Chiuso, A., Soatto, S., Sastry, S.: Observability of linear hybrid systems. In Maler, O., Pnueli, A., eds.: Hybrid Systems: Computation and Control (Prague). Number 2623 in Lecture Notes in Computer Science, Springer (2003) 527–539
10. Kalman, R.E., Falb, P.L., Arbib, M.A.: Topics in mathematical systems theory. McGraw-Hill, New York (1969)
11. Özveren, C., Willsky, A.: Observability of discrete event systems. IEEE Trans. Automatic Control **35** (1990) 797–806
12. Cassandras, C., Lafortune, S.: Introduction to discrete event systems. Kluwer Academic Publishers, Boston (1999)
13. Lafferiere, G., Pappas, G.J., Sastry, S.: O-minimal hybrid systems. Math. Control Signals Systems **13** (2000) 1–21
14. Åström, K.J., Wittenmark, B.: Computer-Controlled Systems. Prentice Hall (1997)