Control to facet problems for affine systems on simplices and polytopes – With applications to control of hybrid systems

L.C.G.J.M. Habets and J.H. van Schuppen

Abstract— **In this paper, a general control-to-facet problem for affine systems on polytopes is studied: find an affine feedback law such that all trajectories of the closed-loop system leave the state polytope through an a priori specified (possibly empty) set of facets. Solutions are presented in terms of (bi)linear inequalities in the coefficients of the affine feedback. The result is applied to control synthesis for piecewise-affine hybrid systems. Using a backward recursion algorithm, a sufficient condition for reachability of hybrid systems is obtained, and a piecewise-affine controller is computed that realizes the required reachability property.**

I. INTRODUCTION

In the last decade, modeling and control of hybrid systems has attracted considerable attention. Nowadays large numbers of engineering systems are controlled by computers, generating an interaction between the continuous dynamics of a physical system, and the discrete dynamics of a computer. This leads to a closed-loop system that is in fact a hybrid system. Also the modeling of complex engineering systems themselves is often facilitated by the use of hybrid system models. Examples of hybrid control systems are manifold, and range from the control of car engines to the description of the evolution of biomolecular networks.

In the literature, a specific subclass of hybrid systems, called *piecewise-affine hybrid systems*, introduced by Sontag in [12], [13], has been studied quite extensively (see e.g. [2], and numerous papers in [3], [10], [1]). A piecewise-affine hybrid system consists of an automaton, with at each discrete mode of the automaton an affine system on a polyhedral set, evolving in continuous time. As soon as the continuous state crosses the boundary of the polyhedral set, a discrete event occurs, and the automaton switches to a new discrete mode. There the continuous state is restarted and will evolve according to the system dynamics of the affine system corresponding to the new discrete mode. In every discrete mode, the dynamics of the corresponding continuous-time affine system, and the polyhedral set on which this system is defined, may be different. In this paper we assume that at each discrete mode the corresponding affine dynamics are defined on a polytope, i.e. on a *bounded* polyhedral set, and that the discrete event that occurs upon reaching the boundary of a polytope, depends only on the facet through which the polytope is left.

In this setting, we study the following problem on the interaction between the continuous and discrete dynamics of a piecewise-affine hybrid system: *is it possible to use affine state feedback to guarantee that in a given discrete mode, the continuous state trajectory will leave the state polytope through an element of an a priori specified set of possible exit facets?* A solution to this problem can be used to influence the discrete behavior of a hybrid system in the following way. In each discrete location, affine state feedback is applied to enable or disable certain sets of events in the discrete automaton, by steering the continuous state to a suitable set of exit facets. By combining the control laws for the continuous state in all discrete locations, an abstraction of the hybrid system is obtained in terms of a finite, possibly non-deterministic, automaton. The goal is to find an affine control law in each discrete location such that the overall closed-loop system meets the a priori given control objectives, like certain reachability properties or guaranteeing safety.

To study the problem described above, we focus our attention on one discrete mode of a hybrid system, and consider an affine system on a full-dimensional polytope P. Given a subset $\mathcal{E}$ of the set of all facets of P, the question is to find necessary and sufficient conditions for the existence of an affine state feedback, such that all trajectories of the closed-loop system can only leave P through one of the facets in $\mathcal{E}$, and do so in finite time. In several respects, this problem may be considered as an extension of the control problem that was solved in [6]. Although the problem formulation in [6] looks similar, there are major differences. First of all in [6] only one exit facet is allowed, i.e. $\mathcal{E}$ contains one single element. In this paper, $\mathcal{E}$ may contain an arbitrary number of elements. Secondly, in [6] it is assumed that every state trajectory starting on the exit facet must leave the state polytope instantaneously. In [6] this requirement is needed for technical reasons, although it has no particular meaning in a hybrid systems context. In the present paper, this additional assumption is not needed any longer.

The approach to reachability analysis and control synthesis for piecewise-affine hybrid systems described in this

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L. Habets is with the Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, P.O. Box 513, NL - 5600 MB Eindhoven, The Netherlands, and is part-time affiliated with CWI - Centrum voor Wiskunde en Informatica, P.O. Box 94079, NL - 1090 GB Amsterdam, The Netherlands. E-mail: l.c.g.j.m.habets@tue.nl

J.H. van Schuppen is with CWI - Centrum voor Wiskunde en Informatica, P.O. Box 94079, NL - 1090 GB Amsterdam, The Netherlands. E-mail: J.H.van.Schuppen@cwi.nl

paper is based on a decomposition of the continuous and discrete dynamics as proposed in [11]. In [5], the relationship of these questions with control-to-facet problems was investigated. Note however that in general reachability of piecewise-affine hybrid systems is undecidable (see [14], and [8] for some related results). One problem is that an exact analysis would require integration of the continuous dynamics in combination with an iterative refinement technique, which makes the problem intractable. In this paper, we use a more pragmatic approach, that can be regarded as an extension of the results in [7]. By avoiding integration of the continuous dynamics, one may obtain sufficient conditions for reachability of hybrid systems, that can be verified algorithmically in a finite number of steps. Note however, that the method is conservative in that it may fail to find a controller, even if one exists. Nevertheless, this is the best one can expect for a problem that is undecidable. In comparison with [7], this paper is more general because it also covers the situation of general full-dimensional polytopes. At the same time the derivation of the result has been simplified.

The paper is organized as follows. The next section contains an exact formulation of the control-to-facet problems considered in this paper, and introduces some terminology needed in the sequel. In Section III an important intermediate result is presented, relating the existence of fixed points with the existence of state trajectories never leaving the state polytope. In Section IV the control-to-facet problems formulated in Section II are studied. If the state polytope is a simplex, a complete solution of all problems is possible in terms of linear inequalities on the inputs at the vertices of the simplex. For general polytopes, a solution is presented, mainly in terms of linear inequalities on the coefficients of the affine state feedback. In Section V it is explained how solutions to the control-to-facet problems can be applied in the control of hybrid systems. Conclusions and final remarks are stated in Section VI.

II. PROBLEM DESCRIPTION

Let $N \in \mathbb{N}$, and let $v_1, \ldots, v_M$ be M points in $\mathbb{R}^N$ ($M \geq N+1$), such that there is no hyperplane in $\mathbb{R}^N$ containing all M points. Then the convex hull $P := \mathrm{Conv}(\{v_1, \ldots, v_M\})$ is called a full-dimensional *polytope* in $\mathbb{R}^N$. The points in $\{v_1, \ldots, v_M\}$ that cannot be written as the convex combination of two other points in P are called the *vertices* of P; the set of all vertices of P is denoted by $\mathcal{V}(P)$. A full-dimensional polytope in $\mathbb{R}^N$ with exactly $N+1$ vertices is called a *simplex*.

Alternatively, a polytope is characterized as the intersection of a finite number of halfspaces. A *face* of a polytope P is the intersection of P with one of its supporting hyperplanes. The faces of dimension $N-1$ are called *facets*. Let $\mathcal{F}(P) = \{H_1, \ldots, H_K\}$ denote the set of all facets of the polytope P. Every facet H_i, ($i = 1, \ldots, K$), is contained in a hyperplane $n_i^T x = \alpha_i$. Here $n_i \in \mathbb{R}^N$ is a normal vector of H_i, and $\alpha_i \in \mathbb{R}$. Throughout the paper we use the convention that n_i is of unit length, and points out of the polytope P, implying that $P = \{x \in \mathbb{R}^N \mid n_i^T x \leq \alpha_i, i = 1, \ldots, K\}$.

On the full-dimensional polytope P we consider an affine system

$$\dot{x} = Ax + Bu + a, \qquad x(0) = x_0, \tag{1}$$

with state $x \in P \subset \mathbb{R}^N$, and input $u \in U$, where U denotes a closed polytope in $\mathbb{R}^m$, implicitly described by

$$U = \{u \in \mathbb{R}^m \mid \forall j = 1, \ldots, L : m_j^T u \leq \beta_j\}, \tag{2}$$

with $m_j \in \mathbb{R}^m$ and $\beta_j \in \mathbb{R}$, ($j = 1, \ldots, L$). We assume that differential equation (1), fixed by the matrices $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times m}$, and vector $a \in \mathbb{R}^N$, remains valid as long as the state x is contained in the state polytope P. Then (1) describes the continuous dynamics of a piecewise-affine hybrid system at one discrete location. In this hybrid setting, departure of the continuous state through a specific facet of P enforces the occurrence of a particular event, that transfers the hybrid system to a new discrete location. In order to understand and influence this interaction between the continuous and discrete dynamics of a hybrid system, we study the following control problems for affine systems on polytopes.

Problem 2.1: Consider affine system (1) on polytope P, and with input $u \in U$. Let $\mathcal{E}$ be a (non-empty) subset of the set $\mathcal{F}(P)$ of facets of P. Find $F \in \mathbb{R}^{m\times N}$ and $g \in \mathbb{R}^m$ such that the affine feedback law $u = Fx + g$ is admissible, i.e. $Fx + g \in U$ for all $x \in P$, and all solutions of the corresponding closed-loop system

$$\dot{x} = (A + BF)x + (a + Bg), \qquad x(0) = x_0, \tag{3}$$

have the following property: for every $x_0 \in P$ there exist $T \geq 0$ and $\varepsilon > 0$ such that solution $x(t, x_0)$ of (3) satisfies

(i) $x(t, x_0) \in P$ for all $t \in [0, T]$,
(ii) $x(t, x_0) \notin P$ for all $t \in (T, T+\varepsilon)$,
(iii) $\exists H \in \mathcal{E} : x(T, x_0) \in H$.

Conditions (i) and (ii) in Problem 2.1 state that at time T trajectory $x(t, x_0)$ leaves the state polytope P for the first time. In particular, it is guaranteed that all solutions of (3) leave P in finite time. According to condition (iii), solutions always leave P through a facet in $\mathcal{E}$. Therefore, the facets in $\mathcal{E}$ are called *admissible exit facets*.

In a hybrid systems context, it is also useful to search for control laws that guarantee that no further events take place. The corresponding continuous-time control problem is stated next.

Problem 2.2: Consider affine system (1) on polytope P, and with input $u \in U$. Find $F \in \mathbb{R}^{m\times N}$ and $g \in \mathbb{R}^m$ such that the affine feedback law $u = Fx + g$ is admissible, i.e. $Fx + g \in U$ for all $x \in P$, and all solutions of the corresponding closed-loop system

$$\dot{x} = (A + BF)x + (a + Bg), \qquad x(0) = x_0,$$

satisfy

$$\forall x_0 \in P\ \forall t \geq 0: \ x(t, x_0) \in P. \qquad (4)$$

Condition (4) in Problem 2.2 states that all solution trajectories of the closed-loop system remain in P forever, guaranteeing that none of the facets of P is ever crossed.

III. GUARANTEEING DEPARTURE FROM A POLYTOPE IN FINITE TIME

In this section we consider an autonomous affine system on a polytope, and focus on a sub-problem of Problem 2.1: we derive necessary and sufficient conditions that guarantee that all solution trajectories of an autonomous affine system leave a polytope in finite time. In a hybrid system context, this result is needed to assure that in the corresponding discrete location, eventually a discrete event will occur. In this way it is possible to avoid undesirable deadlocks in the automaton, underlying the hybrid system.

Theorem 3.1: Let P be a closed full-dimensional polytope in $\mathbb{R}^N$, and let $A \in \mathbb{R}^{N\times N}$ and $a \in \mathbb{R}^N$. Consider the affine autonomous system

$$\dot{x} = Ax + a, \qquad x(0) = x_0, \qquad (5)$$

and let $x(t, x_0)$ denote the solution trajectory of (5) with initial state x_0. Then the following properties are equivalent

(i) For all $x_0 \in P$, trajectory $x(t, x_0)$ leaves P in finite time.
(ii) $\forall x \in P$: $\quad Ax + a \neq 0$.
(iii) $\exists n \in \mathbb{R}^N$: $\quad \forall x \in P$: $\quad n^T(Ax + a) > 0$.
(iv) $\exists n \in \mathbb{R}^N$: $\quad \forall v \in \mathcal{V}(P)$: $\quad n^T(Av + a) > 0$.

Theorem 3.1 states that the required property that all state trajectories leave the polytope P in finite time is equivalent with the absence of a fixed point in P. Alternatively, the property is equivalent with the existence of a direction n, such that in all vertices of P the vector field $Ax + a$ has a positive component in the direction of n.

Proof of Theorem 3.1: (i) $\Longrightarrow$ (ii): If $x_0 \in P$ is such that $Ax_0 + a = 0$, then $x(t, x_0) \equiv x_0$ is a solution that remains in P forever, which contradicts (i).

(ii) $\Longrightarrow$ (iii): If $Ax + a \neq 0$ for all $x \in P$, then $Q := \{Ax + a \mid x \in P\}$ is a closed polytope, not containing 0. So there exists a hyperplane $n^T x = \alpha$, with $\alpha > 0$, separating Q and $\{0\}$. It follows that $\forall y \in Q : \ n^T y > \alpha$. In particular, $n^T(Ax + a) > \alpha > 0$ for all $x \in P$.

(iii) $\Longrightarrow$ (iv): Trivial.

(iv) $\Longrightarrow$ (i): Let $n \in \mathbb{R}^N$ be such that $n^T(Av + a) > 0$ for all vertices $v \in \mathcal{V}(P)$. Choose $\beta \in \mathbb{R}$ such that $\forall x \in P : \ n^T x < \beta$, and $c > 0$ such that in every vertex $v \in \mathcal{V}(P)$: $n^T(Av + a) > c$. Then $n^T(Ax + a) > c$ for all $x \in P$. Indeed, if $x \in P$, there exist $v_1, \dots, v_k \in \mathcal{V}(P)$ and $\lambda_1, \dots, \lambda_k \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i v_i = x$. Hence

$$\begin{aligned} n^T(Ax + a) &= n^T(A\sum_{i=1}^k \lambda_i v_i + \sum_{i=1}^k \lambda_i a) \\ &= \sum_{i=1}^k \lambda_i n^T(Av_i + a) > \sum_{i=1}^k \lambda_i c = c. \end{aligned}$$

Let $x(t, x_0)$ be a solution of (5), and suppose that $x(t, x_0) \in P$ for all $t \geq 0$. Define $y(t) = n^T x(t, x_0)$. Then on the one hand $y(t) < \beta$ for all $t \geq 0$, because $x(t, x_0)$ remains in P. On the other hand, $\dot{y}(t) = n^T \dot{x}(t, x_0) = n^T(Ax(t, x_0) + a) > c$ for all $t \geq 0$, i.e. y grows with a speed of at least c. So $y(t) \geq n^T x_0 + ct$ for all $t \geq 0$, which leads to a contradiction because $y(t) < \beta$ for all $t \geq 0$. We conclude that $x(t, x_0)$ must leave P in finite time. □

IV. SOLVING CONTROL TO FACET PROBLEMS

In this section we present solutions to the control problems stated in Section II, mainly in terms of linear inequalities on the coefficients of the pair $(F, g) \in \mathbb{R}^{m\times(N+1)}$, that describes the affine feedback law $u = Fx + g$. In most results, convexity plays a major role, because it can be used to extend certain properties valid at all vertices of a facet to all points of this facet.

In both control problems of Section II it is required to design a control law that guarantees that no solution of the closed-loop system can leave the state polytope P through a facet that does not belong to the set $\mathcal{E}$ of admissible exit facets. Therefore, we first consider the problem of avoiding non-admissible exit facets.

Lemma 4.1: Consider an affine system $\dot{x} = Ax + Bu + a$ on a full-dimensional polytope P, and with $u \in U$, where U is a closed polytope, implicitly described by (2). Let $\mathcal{E}$ be the (possibly empty) set of admissible exit facets. For every facet H of P, we denote by $\mathcal{V}(H)$ the set of vertices of H, and by n_H the normal vector of H of unit length and pointing out of P. Let $F \in \mathbb{R}^{m\times N}$ and $g \in \mathbb{R}^m$. Then the following two statements are equivalent:

(A) The affine state feedback $u = Fx + g$ is admissible, and for all $x_0 \in P$ solution $x(t, x_0)$ of the closed-loop system satisfies the following property

$$\begin{aligned} \exists T \geq 0\ \exists \varepsilon > 0 : \ (\forall t \in [0, T] : x(t, x_0) \in P \wedge \\ \forall t \in (T, T + \varepsilon) : x(t, x_0) \notin P) \end{aligned}$$

$$\Longrightarrow$$

$$\exists H \in \mathcal{E} : \ x(T, x_0) \in H.$$

(B) The following set of linear inequalities is satisfied:
  (i) $\forall v \in \mathcal{V}(P)\ \forall j \in \{1, \dots, L\} : m_j^T(Fv + g) \leq \beta_j$,
  (ii) $\forall H \in \mathcal{F}(P)\backslash\mathcal{E}\ \forall v \in \mathcal{V}(H) : n_H^T((A + BF)v + (a + Bg)) \leq 0$.

Proof: (A) $\Longrightarrow$ (B): If feedback $u = Fx + g$ is admissible, then $Fv + g \in U$ for all $v \in \mathcal{V}(P)$, and formula (2) for U implies that (i) holds. To prove (ii), let $H \in \mathcal{F}(P)\backslash\mathcal{E}$, and suppose that there is a vertex $v \in \mathcal{V}(H)$, such that

$n_H^T((A+BF)v+(a+Bg)) > 0$. Then, due to continuity, there also exists a point x_0 in the relative interior of H, such that $n_H^T((A+BF)x_0+(a+Bg)) > 0$. Hence $x(t, x_0)$ leaves P at time 0, despite the fact that H is the only facet to which x_0 belongs. This yields a contradiction, because trajectories of the closed-loop system can only leave P through an admissible exit facet.

(B) $\Longrightarrow$ (A): Condition (i) states that feedback $u = Fx + g$ is admissible: $u = Fx + g \in U$ for all $x \in P$. Condition (ii) of (B) expresses that the state of the closed-loop system can only leave P through an admissible exit facet $H \in \mathcal{E}$. If the inequalities in (ii) are strict, this is obvious, and otherwise, in case of non-strict inequalities, one may use the same perturbation argument as in [6, Appendix A] to prove this claim. □

Next, solutions to Control Problems 2.2 and 2.1 are obtained as special cases of Lemma 4.1.

Theorem 4.2: Consider an affine system (1) on a full-dimensional polytope P, and with inputs u from an input polytope U, implicitly described by (2). Let $F \in \mathbb{R}^{m\times N}$ and $g \in \mathbb{R}^m$. Then $u = Fx + g$ is an admissible affine state feedback that solves Problem 2.2 if and only if

(i) $\forall v \in \mathcal{V}(P)\ \forall j \in \{1, \ldots, L\} : m_j^T(Fv + g) \le \beta_j$,
(ii) $\forall H \in \mathcal{F}(P)\ \forall v \in \mathcal{V}(H) : n_H^T((A + BF)v + (a + Bg)) \le 0$.

Proof: Apply Lemma 4.1 with $\mathcal{E} = \varnothing$. □

Theorem 4.3: Consider an affine system (1) on a full-dimensional polytope P, and with inputs u from an input polytope U, implicitly described by (2). Let $\mathcal{E} \subset \mathcal{F}(P)$ be a non-empty set of admissible exit facets. Let $F \in \mathbb{R}^{m\times N}$ and $g \in \mathbb{R}^m$. Then $u = Fx + g$ is an admissible affine state feedback that solves Problem 2.1 if and only if

(i) $\forall v \in \mathcal{V}(P)\ \forall j \in \{1, \ldots, L\} : m_j^T(Fv + g) \le \beta_j$,
(ii) $\forall H \in \mathcal{F}(P)\backslash\mathcal{E}\ \forall v \in \mathcal{V}(H) : n_H^T((A + BF)v + (a + Bg)) \le 0$.
(iii) $\exists n \in \mathbb{R}^N :\ \forall v \in \mathcal{V}(P) :\ n^T((A+BF)v+(a+Bg)) > 0$.

Proof: Condition (i) is equivalent with the fact that feedback $u = Fx + g$ is admissible. According to Theorem 3.1, all trajectories of the closed-loop system leave P in finite time if and only if condition (iii) is satisfied. Finally, Lemma 4.1 states that condition (ii) guarantees that trajectories can only leave through an admissible exit facet. □

Note that conditions (i) and (ii) of Theorem 4.2 just consist of a system of linear inequalities in the coefficients of F and g. Therefore, the existence of a solution (F, g) to this system of linear inequalities can be tested using existing software on polyhedral sets. Furthermore, every solution to the linear inequalities (i) and (ii) corresponds to an affine feedback law $u = Fx + g$ that solves Control Problem 2.2. In Theorem 4.3 the situation is slightly different. Still, conditions (i) and (ii) of Theorem 4.3 consist of linear inequalities in the coefficients of F and g. Condition (iii) however, is not just a linear inequality in several unknowns. In (iii), both the pair (F, g) of feedback coefficients and the normal vector n are unknowns, and since also products of these terms occur, the inequalities in (iii) seem to be bilinear. The algorithmic solution of this type of equations, in combination with the linear inequalities in (i) and (ii), is subject of current research of the authors, and is not presented in this paper. Instead, we consider the special case in which the state polytope of the affine system is a *simplex*. In this case, explicit solution of the inequalities in (iii) can be avoided, using an alternative approach purely based on the solution of linear inequalities. The rest of this section is devoted to the elaboration of this result.

First we observe that every affine function f is uniquely determined by its values at the vertices of a full-dimensional simplex. So, for a system on a simplex S, we may determine suitable inputs at the vertices first, and compute the corresponding affine feedback law realizing these inputs afterward, instead of working with the closed-loop system from the very beginning. To solve Problem 2.1, the inputs at the vertices have to be chosen in such a way that the vector field of the corresponding closed-loop system has no fixed points inside S and does not point outward on non-admissible exit facets. Furthermore, instead of collecting all inequalities in which the normal vector n_H of one particular facet is involved, all inequalities valid at one vertex of a simplex are collected.

Theorem 4.4: Consider an affine system (1) on a full-dimensional simplex S in $\mathbb{R}^N$, and with inputs u from an input polytope $U \subset \mathbb{R}^m$. Let $\mathcal{E} \subset \mathcal{F}(S)$ be a non-empty set of admissible exit facets. Let $\mathcal{V}(S) = \{v_1, \ldots, v_{N+1}\}$ be the set of vertices of S. For $i = 1, \ldots, N+1$, let $\mathcal{F}_i = \{H \in \mathcal{F}(S) \mid v_i \in \mathcal{V}(H)\}$ denote the set of all facets of S of which v_i is a vertex. For every $i \in \{1, \ldots, N+1\}$ we define the (possibly empty) polytope

$$U_i := \{u \in U \mid n_H^T(Av_i + Bu + a) \le 0 \text{ for all } H \in \mathcal{F}_i\backslash\mathcal{E}\}.$$

Then Control Problem 2.1 is solvable if and only if the following conditions are satisfied:

(i) $U_i \ne \varnothing$ for all $i = 1, \ldots, N+1$,
(ii) For all $i \in \{1, \ldots, N+1\}$ there is a vertex $w_i \in \mathcal{V}(U_i)$ such that

$$0 \notin \mathrm{Conv}(\{Av_i + Bw_i + a \mid i = 1, \ldots, N+1\}).$$

In particular, if inputs $u_i \in U_i$, $(i = 1, \ldots, N+1)$ are chosen in such a way that $0 \notin \mathrm{Conv}(\{Av_i + Bu_i + a \mid i = 1, \ldots, N+1\})$, then an admissible affine state feedback solving Problem 2.1 is given by $u = Fx + g$, with $F \in \mathbb{R}^{m\times N}$ and $g \in \mathbb{R}^m$ the unique solution of

$$\begin{pmatrix} v_1^T & 1 \\ \vdots & \vdots \\ v_{N+1}^T & 1 \end{pmatrix} \begin{pmatrix} F^T \\ \hline g^T \end{pmatrix} = \begin{pmatrix} u_1^T \\ \vdots \\ u_{N+1}^T \end{pmatrix}. \quad (6)$$

Proof: (Necessity) Assume that $u = Fx + g$ is an admissible affine state feedback solving Control Problem 2.1. For $i = 1, \dots, N+1$ we define $u_i := Fv_i + g$. Then $u_i \in U_i$, because $u = Fx + g$ is admissible, and, by interchanging the universal quantifiers in condition (ii) of Theorem 4.3, $n_H^T(Av_i + Bu_i + a) = n_H^T((A + BF)v_i + (a + Bg)) \le 0$ for all $H \in \mathcal{F}_i \backslash \mathcal{E}$. Furthermore, condition (iii) of Theorem 4.3 implies that there exists an $n \in \mathbb{R}^N$ such that $n^T(Av_i + Bu_i + a) > 0$ for all $i = 1, \dots, N+1$. So, every polytope U_i contains a point u_i such that $Av_i + Bu_i + a$ belongs to the open half space $n^T x > 0$. Then also one of the vertices w_i of U_i satisfies $n^T(Av_i + Bw_i + a) > 0$. Hence, $\mathrm{Conv}(\{Av_i + Bw_i + a \mid i = 1, \dots, N+1\})$ is contained in the half space $n^T x > 0$, and condition (ii) is satisfied.

(Sufficiency) Assume that conditions (i) and (ii) are satisfied. Then there exist inputs $u_i \in U_i$, $(i = 1, \dots, N+1)$ such that $0 \notin \mathrm{Conv}(\{Av_i + Bu_i + a \mid i = 1, \dots, N+1\})$. Let $u = Fx + g$ be the corresponding affine state feedback, obtained by solving the set of linear equalities (6). Then condition (i) of Theorem 4.3 is satisfied, because $Fv_i + g = u_i \in U$ for all $i = 1, \dots, N+1$. Let H be a facet in $\mathcal{F}(S) \backslash \mathcal{E}$, and v_i be a vertex of H. Then $H \in \mathcal{F}_i \backslash \mathcal{E}$, and $n_H^T((A + BF)v_i + (a + Bg)) = n_H^T(Av_i + Bu_i + a) \le 0$, because $u_i \in U_i$. Hence, also condition (ii) of Theorem 4.3 is satisfied. Finally, since $0 \notin \mathrm{Conv}(\{Av_i + Bu_i + a \mid i = 1, \dots, N+1\}) =: R$, there exists a hyperplane $n^T x = \alpha$ with $\alpha > 0$, that separates R and $\{0\}$. So, for all $i = 1, \dots, N+1$ we have $n^T((A + BF)v_i + (a + Bg)) = n^T(Av_i + Bu_i + a) > \alpha > 0$, and condition (iii) of Theorem 4.3 is valid. So $u = Fx + g$ solves Control Problem 2.1. □

Remark 4.5: The necessary and sufficient conditions stated in Theorem 4.4 can be checked in a finite number of steps, using existing software on polyhedral sets (see e.g. [4], [9], [15]). Furthermore, the result provides a constructive method to compute an admissible affine feedback solution to Control Problem 2.1.

V. APPLICATION TO CONTROL OF HYBRID SYSTEMS

In this section we show how solutions to the control-to-facet problems that were derived in this paper, can be applied to control synthesis for hybrid systems. For this, we have to introduce the class of piecewise-affine hybrid systems, mentioned in Section I, in a more formal way.

Definition 5.1: A *piecewise-affine hybrid system* consists of a discrete event system (Q, E, f), (where Q denotes a finite state set, E a set of events, and $f :\subset (Q \times E) \to Q$ the partial transition function), interacting with $|Q|$ affine systems on polytopes. For every $q \in Q$, the continuous dynamics at state q is described by the affine system

$$\dot{x}_q(t) = A_q x_q(t) + B_q u(t) + a_q, \quad x_q(t_0) = x_q^0, \qquad (7)$$

on a full-dimensional polytope $P_q \subset \mathbb{R}^N$, and with input $u \in U$, where U is a polytope in $\mathbb{R}^m$. Differential equation (7) remains valid until $x_q(t)$ crosses a guard $G_q(e) \subset \partial P_q$. At that time instant $t_1 \ge t_0$, discrete event e occurs and

1) the discrete state transfers to $q^+ = f(q, e)$,
2) the continuous state is reset by an affine reset map, $x_{q^+}^0 = R(q, e, q^+) x_q(t_1) + r(q, e, q^+)$, and continues to evolve according to the new differential equation

$$\dot{x}_{q^+}(t) = A_{q^+} x_{q^+}(t) + B_{q^+} u(t) + a_{q^+}, \quad x_{q^+}(t_1) = x_{q^+}^0.$$

In this paper, it is additionally assumed that every guard set $G_q(e)$ consists of facets of the polytope P_q, i.e. $\forall (q, e) \in \mathrm{dom}(f) :\ G_q(e) \subset \mathcal{F}(P_q)$.

Clearly, the behavior of a piecewise-affine hybrid system consists of a discrete and a continuous component. The continuous component may be influenced directly by application of an admissible affine feedback law

$$u = k_q(x_q) = F_q x_q + g_q, \qquad (8)$$

at each discrete state $q \in Q$. A piecewise-affine state feedback $\{k_q \mid q \in Q\}$ also influences indirectly the discrete component of the behavior of a hybrid system. After application of a piecewise-affine state feedback, an autonomous hybrid system is obtained. The discrete component of the behavior of this system may be regarded as the trajectories of a discrete event system. In this section we will present a method to compute an admissible affine state feedback (8) in each discrete state $q \in Q$, such that the discrete component of the behavior of the closed-loop hybrid system satisfies one of the following reachability properties.

Problem 5.2 (Reach-avoid problem): Given a piecewise-affine hybrid system, with initial discrete state q_0 and required final discrete state q_f. Let $Q_u \subset Q$ be a set of unsafe discrete states that have to be avoided. Determine a piecewise-affine state feedback $\{k_q \mid q \in Q\}$, such that any hybrid trajectory of the closed-loop system, starting in discrete state q_0 (and with continuous initial state $x_{q_0}(0)$ an arbitrary element of P_{q_0}), reaches discrete state q_f in finite time, without visiting unsafe states $q \in Q_u$. In the *reach-avoid-stay problem* there is the additional requirement that every discrete state trajectory that reaches q_f, terminates because the affine feedback $u = F_{q_f} x_{q_f} + g_{q_f}$ at discrete state q_f is such that the continuous state x_{q_f} never leaves the state polytope P_{q_f} after arriving there.

Sufficient conditions for solving Problem 5.2 can be obtained using the solutions to the control-to-facet problems in Section IV. Let $q \in Q$, with corresponding state polytope P_q, and let $Q_c \subset Q$. We define

$$E(q, Q_c) = \{e \in E \mid (q, e) \in \mathrm{dom}(f) \text{ and } f(q, e) \in Q_c\},$$

$$\mathcal{H}(q, Q_c) = \{H \in \mathcal{F}(P_q) \mid \exists e \in E(q, Q_c) :\ H \subset G_q(e)\}.$$

I.e. $E(q, Q_c)$ consists of all events that lead to a transition from discrete state q to a discrete state belonging to the set Q_c. $\mathcal{H}(q, Q_c)$ is the corresponding set of facets; an event $e \in E(q, Q_c)$ is triggered if and only the continuous state leaves polytope P_q through a facet in $\mathcal{H}(q, Q_c)$. So, to guarantee that in finite time a discrete transition takes place from location q to a discrete state in Q_c, by using an affine feedback control law on the continuous state, Problem 2.1 has to be solved for the set $\mathcal{H}(q, Q_c)$ of possible exit facets. Since necessary and sufficient conditions for this problem have been formulated in Theorem 4.3, we obtain the following backward recursion algorithm for solving Problem 5.2

Algorithm 5.3: Given: a piecewise-affine hybrid system, with initial discrete state q_0, required final discrete state q_f, and unsafe state set Q_u. Assume that $q_f \notin Q_u$.

(1) $j := 0$; $Q_{-1} := \varnothing$; $Q_0 := \{q_f\}$;
(2) While $q_0 \notin Q_j$ and $Q_j \neq Q_{j-1}$ do

$$\begin{aligned} Q_{j+1} \;:=\; & Q_j \cup \{q \in Q \backslash Q_u \mid \text{Problem 2.1 with} \\ & \mathcal{E} = \mathcal{H}(q, Q_j) \text{ is solvable at location } q\}; \end{aligned}$$

$j := j + 1$;
end

The output of the algorithm is an increasing sequence of sets of discrete locations Q_j, $j = -1, 0, 1, \ldots, \ell$ and a set of affine feedback laws $\{k_q \mid q \in Q_\ell \backslash \{q_f\}\}$.

Theorem 5.4: Consider a piecewise-affine hybrid system with initial location q_0, required final location q_f, and set Q_u of unsafe states. Let the sequence Q_j, $j = -1, 0, 1, \ldots, \ell$ and the piecewise affine feedback law $\{k_q \mid q \in Q_\ell \backslash \{q_f\}\}$ be the outputs of Algorithm 5.3. If $q_0 \in Q_\ell$, then the reach-avoid problem 5.2 is solved by application of the piecewise-affine feedback $\{k_q \mid q \in Q_\ell \backslash \{q_f\}\}$. If, additionally, there exists an admissible affine feedback k_{q_f} that satisfies the conditions of Theorem 4.2 at discrete location q_f, then the reach-avoid-stay problem 5.2 is solved by the piecewise-affine feedback law $\{k_q \mid q \in Q_\ell\}$.

Remark 5.5: Theorem 5.4 only provides a *sufficient* condition for the solution of Problem 5.2. Therefore the condition is conservative in the sense that Problem 5.2 may be solvable, although Algorithm 5.3 fails to find a solution. This conservatism is mainly due to the fact that the reset maps and the corresponding initial states of the continuous dynamics are not used in the algorithm. Nevertheless, sufficient conditions to verify reachability properties of hybrid systems are the best one can hope for, because the general reachability problem for hybrid systems has been shown to be undecidable ([14]). The main advantage of the approach described in this paper is its computability. The algorithm consists of a co-reachability algorithm for discrete event systems, combined with the verification of the solvability of sets of (bi)linear inequalities at each discrete state. Note that our solution method prevents the discrete event system from repeated switching between two or more discrete states. The outcome of the co-reachability algorithm guarantees that in all trajectories that reach the final discrete state q_f, each discrete state is visited at most once.

VI. Conclusions and Final Remarks

In this paper, necessary and sufficient conditions were derived for the existence of an affine feedback, that steers all trajectories of an affine system on a polytope in finite time to an a priori specified set of exit facets. The conditions consist of (bi)linear inequalities in the coefficients of the affine feedback law. For affine systems on simplices, an explicit solution procedure was presented, purely based on the solution of linear inequalities.

The result was used for control synthesis for piecewise-affine hybrid systems. An algorithm was presented to construct a piecewise-affine control law that solves a reachability problem, meanwhile guaranteeing safety. The algorithm is conservative, in that it may fail to find a suitable control law, even if one exists. However, since in general reachability of hybrid systems is undecidable, sufficient conditions combined with a synthesis procedure are of interest.

More research is required for a detailed elaboration of the algorithms; in particular, efficiency questions should be considered.

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