Reachability and Control Synthesis for Piecewise-Affine Hybrid Systems on Simplices

L. C. G. J. M. Habets, P. J. Collins, and J. H. van Schuppen

***Abstract*—In this paper, we consider the synthesis of control laws for piecewise-affine hybrid systems on simplices. The construction is based on the solution to the *control-to-facet* problem at the continuous level, and on dynamic programming at the discrete level. The construction is given as an explicit algorithm using only linear algebra and reach-set computations for automata; no numerical integration is required. The method is conservative, in that it may fail to find a control law where one exists, but one cannot hope for a sharp algorithm for control synthesis since reachability for piecewise-affine hybrid systems is undecidable.**

***Index Terms*—Discrete-event system (DES), exit facet, piecewise-affine hybrid systems, reachability, simplex, stability.**

I. INTRODUCTION

IN THE LAST decade, the study of hybrid systems has received considerable attention. One reason for this growing interest is the increasing number of engineering systems that is controlled by computers, thus creating an interaction between the continuous dynamics of a physical system and the discrete dynamics of a computer. But also the dynamics of many engineering systems is inherently discontinuous, or becomes so after modelling. Examples of hybrid control systems include electric power networks, car engines, air traffic control, chemical processes, robots, etc.

Recently, a specific subclass of hybrid systems, so-called piecewise-affine hybrid systems, introduced by Sontag in [27], [28], [30], has been studied quite extensively (see e.g. [3], [5], [6], and several papers in the conference proceedings [7], [23], and [24]). A piecewise-affine hybrid system consists of a discrete automaton, with a continuous-time affine system on a polyhedral set at each mode, and a switching mechanism between discrete and continuous dynamics. In the control of this type of hybrid systems one is concerned with the design of control laws such that the closed-loop system meets certain control objectives, like stability, safety, performance optimization, and robustness. In these control problems, the notion of reachability often plays a central role: does there exist an input trajectory that guarantees that the system transits from a given initial state to a required terminal state? Unfortunately, the reachability problem for piecewise-affine hybrid systems is, in full generality, undecidable (see [2], [17], and [29]).

The purpose of this paper is to present a set of sufficient conditions for the reachability of a subclass of piecewise-affine hybrid systems whose continuous state sets are assumed to be full-dimensional simplices. At the same time a procedure is developed to synthesize feedback control laws for the problems of reachability and stability while guaranteeing safety. Note that the approach proposed in this paper is conservative, because it is based on sufficient conditions for reachability. Nevertheless the method can be useful for many control engineering problems.

The approach to control synthesis presented in this paper is to decompose the problem into two reachability problems, one at the continuous level and one at the discrete level. The reachability problem at the continuous level reduces to determining a control law for an affine system on a simplex such that from any initial state a particular exit facet or a subset of exit facets is reached in finite time. Conditions for the existence of such a control law are stated in terms of linear inequalities on the continuous inputs at each vertex of a state simplex. These results are based on ideas from affine geometry; especially the convexity of the problem plays an important role. In this way computation of state trajectories by integrating the continuous dynamics can be avoided. At the discrete level, the reachability problem is to determine a path from the initial discrete state to the terminal discrete state in a (non-deterministic) finite state automaton. Both the conditions at the continuous and at the discrete level can be checked using elementary algorithms.

A comparison of the proposed control synthesis approach with the literature follows. The approach to reachability analysis and control synthesis for piecewise-affine hybrid systems based on the idea of decomposition originates from [25]. The same idea also led to an increased interest in some detailed control problems for affine systems on polytopes that appeared useful in a hybrid systems context (see e.g. [12]–[15]). The procedure described in this paper seems somewhat related to the bisimulation approach to verification of hybrid systems proposed by Henzinger [1], [16], [18]. However, in those papers no control is involved, whereas control synthesis is an essential part in the approach presented in this paper. In this respect, the approach is related to the procedure to compute the reachable set for a nonlinear system on the basis of optimal control in [31]. A different approach to control of hybrid systems has been developed by Morari e.a. (see e.g. [5], [6]). In these papers, the main emphasis

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is on performance optimization by computational methods for discrete-time piecewise-affine hybrid systems. Although the results in the present paper are not concerned with optimization issues, they also do not require a time-discretization step, and work directly in continuous time. Also, unlike in [21], simulation is not involved.

This paper is organized as follows. In the next section, the class of piecewise-affine hybrid systems on simplices is introduced, and the problems of reachability and control synthesis are formulated. In Section III some technical results are presented concerning fixed points of autonomous affine systems. Section IV is concerned with the continuous control problems of steering an affine system to a facet of a simplex or to a fixed point. Section V presents an algorithm to compute a feedback control law for steering a hybrid system to the required target state, while guaranteeing a priori specified safety conditions.

II. PROBLEM FORMULATION

In this paper, reachability and control synthesis is studied for a class of piecewise-affine hybrid systems. Before presenting a formal definition of a piecewise-affine hybrid system, we first introduce some terminology.

A *polyhedral set* is a subset of $\mathbb{R}^N$, $(N \in \mathbb{N})$, described by a finite number of linear inequalities. A bounded polyhedral set is called a *polytope*. A polytope can alternatively be characterized as the convex hull of a finite number of points, the vertices of the polytope. An N-dimensional polytope with exactly $N+1$ vertices is called a *simplex*. A *face* of a polyhedral set P is the intersection of P with one of its supporting hyperplanes. If a polyhedral set P has dimension N, the faces of P of dimension $N-1$ are called *facets*.

We now define piecewise-affine hybrid systems, based on a formalism in [12].

Definition 2.1: A *(continuous-time) piecewise-affine hybrid system* consists of an automaton (Q, E, f) in combination with a $|Q|$-tuple of affine systems $\mathcal{A}_q = (A_q, B_q, a_q)$, defined on polytopes X_q, $(q \in Q)$, with input in the polytope U. The automaton and the affine system interact via guard sets, $G_q(e) \subset \partial X_q$, each of which is a union of finitely many facets, and affine reset maps $\mathcal{R}_q(e) : G_q(e) \to X_{f(q,e)}$. The hybrid system $\mathcal{H}$ is therefore characterized by the tuple

$$\begin{aligned}\mathcal{H} = (Q, E, f, U, \{(X_q, \mathcal{A}_q) | q \in Q\}, \\ \{(G_q(e), \mathcal{R}_q(e)) \,|(q,e) \in \mathrm{dom}(f)\}).\end{aligned}$$

If, additionally, each continuous state set X_q is a simplex, then $\mathcal{H}$ is a *piecewise-affine hybrid system on simplices*.

The evolution of the hybrid system is defined in the following way:

Given a discrete location $q \in Q$, the continuous state x_q satisfies the affine differential equation

$$\dot{x}_q(t) = A_q x_q(t) + B_q u(t) + a_q \tag{1}$$

with $x_q \in X_q$ and $u \in U$. Whenever the continuous state leaves the polyhedral set X_q, a discrete event e will occur, corresponding to the guard set $G_q(e)$ that is crossed, and a discrete transition takes place according to the transition function $f :\subset Q \times E \to Q$:

$$\text{if } x_{q^-}^- := \lim_{s \nearrow t_0} x_{q^-}(s) \in G_{q^-}(e), \text{ then } q^+ = f(q^-, e).$$

In the new discrete location q^+, the evolution of the new continuous state x_{q^+} is described by differential equation (1), with q replaced by q^+, and with initial value $x_{q^+}(t_0) = x_{q^+}^+$ determined by the affine reset map $\mathcal{R}_{q^-}(e) : G_{q^-}(e) \to X_{q^+}$:

$$x_{q^+}^+ = \left[\mathcal{R}_{q^-}(e)\right] \left(x_{q^-}^-\right) = R_{q^-}(e) x_{q^-}^- + r_{q^-}(e).$$

The precise definition of "crossing a guard set" is technical, and is deferred to Definition 4.5.

In order to make trajectories of hybrid system $\mathcal{H}$ well defined, we assume that:

1) On any finite time interval only a finite number of discrete transitions can occur (non-Zenoness).
2) The system is non-blocking, i.e. all points on the boundary ∂X_q of a continuous state set X_q, through which it is possible to leave X_q by a suitable choice of $u \in U$, belong to a guard set $G_q(e)$ for some $e \in E$.

An *admissible piecewise-affine control law* is a family $\mathcal{K} = \{k_q | q \in Q\}$, where each $k_q : X_q \to U$ is an affine function characterized by $k_q(x) = F_q x + g_q$.

At first sight, studying problems like reachability and control synthesis for piecewise-affine hybrid systems seems hopeless because it was shown in [29] that for this class of systems, the reachability problem is undecidable, in general. We therefore restrict ourselves in this paper to only finding sufficient conditions for reachability, leading to some conservatism in the results obtained.

As defined, the next event occurring in the evolution of a piecewise-affine hybrid system only depends on the facet through which the state simplex X_q is left. However, the definition does raise a difficulty: what happens if the state simplex X_q is left through a face that belongs to two or more different facets? One option to overcome this difficulty is to allow the interior of each face of X_q to have a different event. In this paper we choose for another possibility, and allow some non-determinism in the hybrid system. If a point $x \in \partial X_q$ belongs to more than one guard set, the event occurring upon leaving X_q by crossing the guard sets $G_q(e_1), \ldots, G_q(e_n)$ is not determined uniquely, but can be any of the events $e_1, \ldots, e_n$. An explicit description of the exact switching condition will be given in Property 4.8.

Remark 2.2: In general frameworks for hybrid systems, e.g. in the formulation in [12], one distinguishes between two different types of discrete events: discrete events generated by the continuous dynamics and input events that can be applied by an operator at any time-instant. In Definition 2.1 only events generated by the continuous dynamics are considered. This limitation is only made to focus attention on control of hybrid systems using continuous inputs. Note however that in the approach described in this paper, input events can be incorporated without any problem.

The main results derived in this paper are concerned with reachability problems and control synthesis for piecewise-affine hybrid systems on simplices. These problems can be stated in different ways; especially the level of detail may be varied considerably. So, instead of defining exactly one control problem, we introduce a class of related control problems.

Problem 2.3: (Reach-avoid problem) Consider a piecewise-affine hybrid system on simplices $\mathcal{H}$, and assume that (Q, E, f) denotes the automaton underlying $\mathcal{H}$. Let $Q_u \subset Q$ be the subset of unsafe locations, that should be avoided during operation. Let $Q_s \subset Q \setminus Q_u$ be a set of possible starting locations, and $q_t \in Q \setminus Q_u$ a target location. The problem is to find an admissible piecewise-affine control law $\mathcal{K}$ that guarantees that in the resulting closed-loop hybrid system every hybrid state trajectory (q, x_q), starting in a location in Q_s reaches the target location q_t in finite time, after a finite number of discrete transitions, and without visiting any unsafe location $q \in Q_u$.

One may further require constraints on the affine control law at the target location.

a) *(Stabilize)*: Find an affine control law at the target location that guarantees that the continuous state never leaves the target location.

b) *(Stabilize to given fixed point)*: Find an affine control that stabilizes the target location, and additionally guarantees that the continuous state converges to a given fixed point.

Problems 2.3, 2.3(a) and 2.3(b) are motivated by many engineering problems, such as in chemical process control, automotive, and robot control (in particular motion planning). The strategy to solve Problem 2.3, of reaching a target location while avoiding unsafe locations, using continuous state feedback on each state simplex is based on the idea of decomposition as proposed in [12], [25]. In the latter paper the notions of arrival and departure sets were introduced to describe the control objective for the continuous state at one discrete location. To solve this problem, integration of the continuous dynamics is required, in combination with a refinement technique. This is a difficult computational problem, which involves nonlinear equations and a possibly non-terminating iteration. In general, this even leads to undecidable problems, unless the hybrid system satisfies some additional properties, such as O-minimality [22]. The approach developed in this paper considers reachability from a rather pragmatic point of view. Since integration and refinement lead to intractable problems, we avoid the use of these techniques. Instead, we consider conditions under which it is possible to block trajectories from crossing certain facets, which can be studied purely in terms of affine conditions.

The main advantage of the approach presented in this paper is its computational tractability. The necessary computations are limited to finding solutions of sets of linear inequalities, in combination with elementary dynamical programming at the discrete state level.

The main disadvantage is that the result is conservative. The method is based on sufficient conditions, and if these are satisfied, it leads to the construction of a piecewise-affine control law that realizes the control objective. However, if the sufficient conditions are not satisfied, the problem may still be solvable, e.g. by taking the affine reset maps and the initial values of the continuous state at each discrete location into account. Since this technique requires integration of the continuous dynamics, it is not further considered in this paper.

III. EXISTENCE OF FIXED POINTS OF AFFINE SYSTEMS ON POLYTOPES

In this section, an autonomous affine system on a polytope is considered. It is shown that all state trajectories of the affine system leave the polytope in finite time if and only if the polytope does not contain a fixed point for the given affine dynamics. In a hybrid systems context this result is used to guarantee whether at a given discrete location a discrete event will occur in finite time, transferring the hybrid system to the next discrete location.

Theorem 3.1: Let P be a closed polytope in $\mathbb{R}^N$, and let $A \in \mathbb{R}^{N\times N}$ and $a \in \mathbb{R}^N$. Let the solution trajectory of the autonomous affine system

$$\dot{x} = Ax + a \tag{2}$$

with initial value $x(0) = x_0$ be denoted by $x(t, x_0)$. Then there exists an $\hat{x} \in P$ such that $A\hat{x} + a = 0$, i.e. polytope P contains a fixed point of (2), if and only if there exists an $x_0 \in P$ such that $x(t, x_0) \in P$ for all $t \geq 0$.

If P contains a fixed point $\hat{x}$, then trajectory $x(t, \hat{x}) \equiv \hat{x}$ remains in P forever. The proof of the converse requires the following intermediate result.

Lemma 3.2: Let D be a closed convex set in $\mathbb{R}^N$, and consider the affine system $\dot{x} = Ax + a$ on $\mathbb{R}^N$. Assume that there exists an $x_0 \in D$ such that $x(t, x_0) \in D$ for all $t \geq 0$. Define

$$X = \mathrm{Conv}\left(\{x(t, x_0) | t \geq 0\}\right) \tag{3}$$

where Conv denotes the convex hull. Then $X \subset D$, and X is a positively invariant set for system (2), i.e. for all $x_1 \in X$ and all $t \geq 0$: $x(t, x_1) \in X$.

Proof: Since $x(t, x_0) \in D$ for all $t \geq 0$, and D is a convex set, it is obvious that $X = \mathrm{Conv}(\{x(t, x_0)|t \geq 0\}) \subset D$.

Let $x_1 \in X$. According to a result of Carathéodory (see e.g. [11, p. 15], there exist $w_0, w_1, \ldots, w_N \in \{x(t, x_0)|t \geq 0\}$ and $\alpha_0, \alpha_1, \ldots, \alpha_N \in [0, 1]$ such that $\sum_{i=0}^{N} \alpha_i = 1$ and $\sum_{i=0}^{N} \alpha_i w_i = x_1$. Since the differential equation (2) is affine, it follows that

$$x(t, x_1) = \sum_{i=0}^{N} \alpha_i x(t, w_i)$$

for all $t \geq 0$. Furthermore, there exist $t_0, t_1, \ldots, t_N \geq 0$ such that $w_i = x(t_i, x_0)$. Since the differential equation (2) is time-invariant this implies that $x(t, w_i) = x(t + t_i, x_0)$ for all $t \geq 0$. Hence

$$x(t, x_1) = \sum_{i=0}^{N} \alpha_i x(t, w_i) = \sum_{i=0}^{N} \alpha_i x(t + t_i, x_0) \in X$$

by definition of X. □

Proof of Theorem 3.1: Let $x_0 \in P$ be such that $x(t, x_0) \in P$ for all $t \geq 0$, and denote by V the affine space of lowest dimension that contains the trajectory $\{x(t, x_0) | t \geq 0\}$. Since the dynamics $\dot{x} = Ax + a$ is affine, the affine space V is invariant under these dynamics, i.e. for all $v \in V$ and all $t \geq 0$ we have $x(t, v) \in V$.

Let $\hat{P} := P \cap V$. Then $\hat{P}$ is a compact convex set. Furthermore, $x_0 \in \hat{P}$ and the trajectory $x(t, x_0) \in \hat{P}$ for all $t \geq 0$. Hence

$$X = \mathrm{Conv}\left(\{x(t, x_0) | t \geq 0\}\right)$$

is a subset of $\hat{P}$, and, according to Lemma 3.2, X is a positively invariant set with respect to the differential equation (2). Obviously, the closure $\overline{X}$ of X is a compact convex set, contained in $\hat{P}$. Moreover, $\overline{X}$ is positively invariant with respect to the dynamics (2). Indeed, since the affine function $f(x) = Ax + a$ satisfies the Lipschitz condition, solutions of (2) on finite time intervals depend continuously on their initial conditions (see e.g. [19, Section 3.1]).

Since $\overline{X}$ is a convex compact positively invariant set in the affine space V, the Brouwer Fixed Point Theorem guarantees the existence of a point $\hat{x} \in \overline{X} \subset P$ such that $A\hat{x} + a = 0$ (see e.g. [8, p. 82] or [10, pp. 202–203]). ☐

In [26, Section 1.6] a result similar to Theorem 3.1 was obtained for the special case that the polytope P is a hyper rectangular region, and the matrix A is a diagonal matrix with strictly negative eigenvalues.

IV. CONTROL OF THE CONTINUOUS STATE

In this section we focus on the affine dynamics of a hybrid system at one discrete location q, and study how affine feedback can be used to steer the continuous state to one or more specific facets of the state simplex. In a hybrid setting this corresponds to the enabling or disabling of certain events in the discrete automaton.

Throughout this section, let $N \in \mathbb{N}$, and let S_N denote a closed full-dimensional simplex in $\mathbb{R}^N$ with vertices $v_1, \ldots, v_{N+1}$. Let $F_1, \ldots, F_{N+1}$ denote the facets of S_N, and assume that the facets are numbered in such a way that for $i = 1, \ldots, N+1$, v_i is the only vertex of S_N not belonging to facet F_i. For $i = 1, \ldots, N+1$, let n_i denote the outward unit normal vector of facet F_i. Define

$$\Lambda_{N+1} := \left\{ (\lambda_1, \ldots, \lambda_{N+1}) \in [0,1]^{N+1} | \sum_{k=1}^{N+1} \lambda_k = 1 \right\}.$$

A. Problem Description

We consider the affine system $\mathcal{A}$ given by

$$\dot{x} = Ax + Bu + a \quad x(0) = x_0 \tag{4}$$

on the full-dimensional simplex $S_N \subset \mathbb{R}^N$, with state $x \in S_N$, and input $u \in U$, where U denotes a polytope in $\mathbb{R}^m$. In particular it is assumed that the differential equation (4), described by the matrices $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times m}$ and the vector $a \in \mathbb{R}^N$ remains valid as long as the state x is contained in the state simplex S_N.

We attempt to construct admissible affine control laws

$$k : S_N \longrightarrow U : k(x) = Fx + g \tag{5}$$

where $F \in \mathbb{R}^{m \times N}$ and $g \in \mathbb{R}^m$, such that, independent of the initial state $x_0 \in S_N$, the closed-loop system

$$\dot{x} = (A + BF)x + (a + Bg) \quad x(0) = x_0 \tag{6}$$

solves one of the following control problems.

Problem 4.1: Given a subset $\mathcal{I}$ of the index set $\{1, \ldots, N+1\}$, find an admissible affine control law (5) which guarantees that all trajectories of the closed-loop system (6) leave the simplex S_N in finite time, and do so by crossing one of the facets F_i, $i \in \mathcal{I}$.

The facets F_i, $i \in \mathcal{I}$, are called *admissible exit facets*. In a hybrid system, a solution to Problem 4.1 guarantees that in location q one of the events corresponding to an admissible exit facet will occur in finite time.

We additionally consider the following problems

Problem 4.2:

a) Find an admissible affine control law (5) such that for every initial state $x_0 \in S_N$, the corresponding state trajectory $x(t, x_0)$ of closed-loop system (6) satisfies $\forall t \geq 0 : x(t, x_0) \in S_N$.

b) Given $x_f \in S_N$, solve (a), and additionally guarantee that $\lim_{t \to \infty} x(t, x_0) = x_f$.

A solution to Problem 4.2a) guarantees that the continuous state never leaves the state simplex S_N. Hence, in the hybrid system no discrete event can occur, and the discrete automaton remains in location q forever. A solution to Problem 4.2b) additionally guarantees that the continuous state is stabilized to the given fixed point x_f.

Remark 4.3: Problem 4.1 looks similar to the control problem that was solved in [15], but there are two major differences. First of all, in [15] the exit facet is assumed to be unique, whereas in Problem 4.1 the set of admissible exit facets may contain more than one element. Secondly, in [15] it is required that every state trajectory starting on the exit facet leaves the state simplex instantaneously. In Problem 4.1 this assumption is dropped; state-trajectories starting on an exit facet are allowed to enter the interior of the simplex first, and leave the simplex later through one of the admissible exit facets.

B. Crossing a Facet

Up to now, we have been rather vague about the meaning of “crossing a facet”. The reason for this is that when a trajectory leaves the simplex through a face which is the intersection of two or more facets, it is not necessarily clear which facet has been crossed, and hence which event is triggered. We now rectify this situation by giving a precise definition of when a trajectory crosses a facet.

Definition 4.4 (Exit Set): Let $\dot{x} = \tilde{A}x + \tilde{a}$ be an autonomous affine system on a simplex S_N, and let F be a facet of S_N. Then the *exit set* of F is the set

$$\mathrm{cl}\left\{ x \in F | n^T(\tilde{A}x + \tilde{a}) > 0 \right\} \tag{7}$$

where n is the outward unit normal vector of F. We say that a facet is *blocked* if its exit set is empty.

It is trivial that F is a blocked facet if and only if

$$\forall x \in F : n^T(\tilde{A}x + \tilde{a}) \leq 0. \tag{8}$$

Definition 4.5 (Crossing a Facet): Let F be a facet of simplex S_N. A trajectory $x(t, x_0)$ of an autonomous affine system is said to leave the simplex S_N at time $T \geq 0$ if

1) $\forall t \in [0, T] : \ x(t, x_0) \in S_N$;
2) $\exists \varepsilon > 0, \forall t \in (T, T + \varepsilon) : \ x(t, x_0) \notin S_N$.

The point $x(T, x_0)$ is an *exit point* of the dynamics. We say $x(t, x_0)$ *crosses* facet F at time T if additionally:

3) $x(T, x_0)$ is in the exit set of F.

Remark 4.6: A trajectory may leave the simplex S_N by crossing more than one facet. Also, it is not necessarily the case that every point of an exit set is an exit point, as the boundary of an exit set may contain points through which the state simplex S_N is entered instead of left.

The next lemma shows that every exit point is in the exit set of at least one facet.

Lemma 4.7: If the state trajectory $x(t)$ of an autonomous affine system $\dot{x} = \tilde{A}x + \tilde{a}$ on S_N leaves S_N through the point $\hat{x}$, then there exists a facet F of S_N for which $\hat{x}$ is in the exit set of F.

Proof: Suppose $\hat{x}$ is an exit point and that for all facets F of S_N, $\hat{x}$ is not in the exit set of F. Let $\mathcal{J} \subset \{1, \ldots, N+1\}$ be the index set such that $\hat{x} \in F_j$ if and only if $j \in \mathcal{J}$. Then there exists $\delta > 0$ such that

i) $\forall j \in \mathcal{J}, \forall x \in F_j$ s.t. $\|x - \hat{x}\| < \delta$: $n_j^T(\tilde{A}x + \tilde{a}) \leq 0$, and
ii) $\forall j \in \{1, \ldots, N+1\} \setminus \mathcal{J} : \ \mathrm{dist}(\hat{x}, F_j) > \delta$.

Let p be a point in the interior of S_N, and define $g(x) = (p - x)/(1 + \|p - x\|)$. Then $\forall j \in \{1, \ldots, N+1\}$, $\forall x \in F_j$, $n_j^T g(x) < 0$.

Let $x_\varepsilon(t, \hat{x})$ denote the solution of the perturbed initial value problem

$$\dot{x}_\varepsilon = \tilde{A}x_\varepsilon + \tilde{a} + \varepsilon g(x_\varepsilon) \quad x_\varepsilon(0) = \hat{x}$$

for $\varepsilon > 0$. Then for all $x \in \partial S_N$ with $\|x - \hat{x}\| < \delta$, the vector field of the perturbed system points strictly into S_N.

Choose $t_1, \varepsilon_1 > 0$ such that $\|x_\varepsilon(t, \hat{x}) - \hat{x}\| < \delta$ whenever $t \in [0, t_1)$ and $\varepsilon \in [0, \varepsilon_1)$. Then $x_\varepsilon(t, \hat{x}) \in S_N$ whenever $t \in [0, t_1)$ and $\varepsilon \in (0, \varepsilon_1)$. Since $x_\varepsilon(t, \hat{x})$ is continuous in ε, t, and since S_N is closed, $x_\varepsilon(t, \hat{x}) \in S_N$ whenever $t \in [0, t_1]$ and $\varepsilon \in [0, \varepsilon_1]$. In particular, $x(t, \hat{x}) \in S_N$ for all $t \in [0, t_1]$. Hence $\hat{x}$ is not an exit point, a contradiction. □

Definitions 2.1 and 4.5 in combination with Lemma 4.7 imply that the discrete switching of a closed-loop piecewise-affine hybrid system on simplices satisfies the following property:

Property 4.8: Let q be a discrete location of a piecewise-affine hybrid system on simplices $\mathcal{H}$ in closed-loop with an admissible piecewise-affine controller $\mathcal{K}$. Let $x_q(t)$ be a continuous state trajectory at location q, leaving simplex X_q through the exit point $\hat{x}$. Then the event that is triggered at that time instant corresponds to a facet F of X_q that is crossed by the state trajectory, i.e. $\hat{x}$ is an element of the exit set of F.

If the exit point $\hat{x}$ belongs to exactly one facet F, Lemma 4.7 and Property 4.8 guarantee that at $\hat{x}$ the event corresponding to this facet does occur. If $\hat{x}$ is on the boundary of several facets, Property 4.8 restricts the events which may occur to those which correspond to facets through which it is possible to leave the state simplex in a neighborhood of $\hat{x}$ by crossing the facet transversely. Note that the switching condition described in Property 4.8 is non-deterministic, because an exit point may belong to the exit set of more than one facet. According to Property 4.8, an event corresponding to a blocked facet cannot occur, since a blocked facet has an empty exit set, and so cannot be crossed.

Remark 4.9: The switching condition described in Property 4.8 enhances the robustness of the hybrid behavior, by making the event that is triggered upon leaving a state simplex X_q through a point $\hat{x}$ not solely dependent on the exit point $\hat{x}$, but also on boundary points of X_q in a neighborhood of $\hat{x}$.

C. Disabling Exit Through a Facet

In Problem 4.1, a state trajectory of the closed-loop system is not allowed to leave S_N by crossing a non-admissible exit facet F_j. In Problem 4.2, a state trajectory is not allowed to leave at all. Before solving these two problems, we first consider the following question.

Problem 4.10: Given a (possibly empty) subset $\mathcal{I}$ of the index set $\{1, \ldots, N+1\}$, find an admissible affine control law (5) such that every facet F_i, $i \in \{1, \ldots, N+1\} \setminus \mathcal{I}$ is a blocked facet of the closed-loop system (6).

A solution to Problem 4.10 does not guarantee that every trajectory of the closed-loop system leaves simplex S_N in finite time. However, if the simplex is left, the exit point does not lie in the exit set of a non-admissible facet. So, although it is allowed that the exit point belongs to the boundary of a blocked facet, Property 4.8 assures that blocked facets cannot be crossed and that the event that is triggered upon leaving the state simplex always corresponds to an admissible exit facet.

The equivalence stated in (8) immediately yields necessary and sufficient conditions for the solvability of Problem 4.10.

Lemma 4.11: An admissible affine control law (5) solves Problem 4.10 for index set $\mathcal{I}$ if and only if

$$\forall i \in \{1, \ldots, N+1\} \setminus \mathcal{I}, \forall x \in F_i : \quad n_i^T\left((A + BF)x + (a + Bg)\right) \leq 0. \tag{9}$$

The necessary and sufficient condition of Lemma 4.11 is stated in terms of the affine feedback (5). In order to make these conditions on the solvability of Problem 4.10 easier to verify, they are reformulated in terms of the inputs at the vertices of the simplex. For this, we use the fact that an affine function is uniquely determined by its values at the vertices of a full-dimensional simplex.

Theorem 4.12: Let $\mathcal{I}$ be a (possibly empty) subset of the index set $\{1,\ldots,N+1\}$. There exists an admissible affine control law (5) that solves Problem 4.10 with admissible exit facets $\{F_i|i\in\mathcal{I}\}$ if and only if there exist inputs $u_1,\ldots,u_{N+1}\in U$ at the vertices of S_N such that

$$\begin{aligned}&\forall i\in\{1,\ldots,N+1\}\setminus\mathcal{I}\\ &\forall j\in\{1,\ldots,N+1\}\setminus\{i\}: n_i^T(Av_j+Bu_j+a)\leq 0.\end{aligned} \quad (10)$$

Furthermore, if $u_1,\ldots,u_{N+1}\in U$ satisfy (10), then a solution of Problem 4.10 is given by $u=Fx+g$, with $F\in\mathbb{R}^{m\times N}$ and $g\in\mathbb{R}^m$ the unique solution of the equation

$$(F|g)\left(\frac{v_1\cdots v_{N+1}}{1\cdots 1}\right)=(u_1\cdots u_{N+1}). \quad (11)$$

Proof: If (5) is an admissible affine feedback that solves Problem 4.10, then condition (9) of Lemma 4.11 implies that the inputs $u_j:=Fv_j+g$, $(j=1,\ldots,N+1)$, satisfy (10).

To prove sufficiency, let $u_1,\ldots,u_{N+1}\in U$ be such that condition (10) is satisfied. According to [15, Algorithm 4.3], (11) has unique solution $F\in\mathbb{R}^{m\times N}$, $g\in\mathbb{R}^m$, and it satisfies $Fv_j+g=u_j$ for all $j=1,\ldots,N+1$. Since $k(x):=Fx+g$ is affine and its values at the vertices of S_N belong to the polytope U, we have $k(x)\in U$ for all $x\in S_N$, i.e. k is an admissible affine feedback.

Let $i\in\{1,\ldots,N+1\}\setminus\mathcal{I}$ and $x\in F_i$. Then there exists $(\lambda_1,\ldots,\lambda_{N+1})\in\Lambda_{N+1}$, with $\lambda_i=0$, such that $x=\sum_{k=1}^{N+1}\lambda_k v_k$. Hence (10) implies

$$\begin{aligned}&n_i^T\left((A+BF)x+(a+Bg)\right)\\ &\quad=n_i^T\sum_{k=1}^{N+1}\lambda_k\left(Av_k+B(Fv_k+g)+a\right)\\ &\quad=\sum_{k=1}^{N+1}\lambda_k n_i^T(Av_k+Bu_k+a)\leq 0\end{aligned}$$

because $\lambda_k\geq 0$ and $n_i^T(Av_k+Bu_k+a)\leq 0$. □

Remark 4.13: If Problem 4.10 is solvable for an index set $\mathcal{I}_1$ by the affine control law $u=Fx+g$, then the same control law also solves Problem 4.10 for index sets $\mathcal{I}_2$ that contain $\mathcal{I}_1$.

Remark 4.14: It is for existence and uniqueness of the solution of (11) that we only consider affine systems on simplices, and not on more general polytopes. Other N-dimensional polytopes have more vertices, leading to a linear system like (11) with more equations than unknowns. In this case it is better to restrict the inequalities (9) to the vertices of every facet. In this way, one large system of linear inequalities in the coefficients of the feedback matrix F and the vector g is obtained, that has to be solved for F and g.

Remark 4.15: By interchanging the universal quantifiers in (10), it is possible to reorganize the inequalities in such a way that the inequalities for input u_j at vertex v_j are completely decoupled from the inequalities for all other inputs. For $j=1,\ldots,N+1$ we define the polytopes

$$\begin{aligned}U_j:=\{u\in U|\forall i\in\{1,\ldots,N+1\}\setminus(\mathcal{I}\cup\{j\}):\\ n_i^T(Av_j+Bu+a)\leq 0\}.\end{aligned} \quad (12)$$

Then condition (10) of Theorem 4.12 can be restated as $u_j\in U_j$ for all $j=1,\ldots,N+1$.

D. Ensuring Departure Through an Admissible Exit Facet

Throughout this subsection it is assumed that the index set $\mathcal{I}\subset\{1,\ldots,N+1\}$ is non-empty; otherwise Problem 4.1 is not solvable.

Since a trajectory cannot leave through a blocked facet by Lemma 4.7, and every trajectory of an autonomous system on a polytope leaves in finite time if, and only if, there are no fixed points by Theorem 3.1, we have the following theorem.

Theorem 4.16: An admissible affine feedback $k(x)=Fx+g$ solves Problem 4.1 for the set $\{F_i|i\in\mathcal{I}\}$ of admissible exit facets if and only if

i) $\forall i\in\{1,\ldots,N+1\}\setminus\mathcal{I},\forall x\in F_i$: $n_i^T((a+BF)x+(a+Bg))\leq 0$;

ii) $\forall x\in S_N$: $((A+BF)x+(a+Bg))\neq 0$.

Both for the question of existence of a solution to Problem 4.1, and for the construction of a solution, the result of Theorem 4.16 is not directly applicable. In order to obtain verifiable conditions, we again reformulate the necessary and sufficient conditions into requirements on the inputs at the vertices of the state simplex.

As in Theorem 4.12, condition i) of Theorem 4.16 is equivalent with the linear inequalities (10) on the inputs $u_1,\ldots,u_{N+1}$ at the vertices of S_N, yielding the polytope U_j of admissible controls at vertex v_j as given in (12). In terms of the inputs at the vertices condition ii) of Theorem becomes

$$\forall(\lambda_1,\ldots,\lambda_{N+1})\in\Lambda_{N+1}:\sum_{j=1}^{N+1}\lambda_j(Av_j+Bu_j+a)\neq 0. \quad (13)$$

Indeed, closed-loop system (6) has no fixed point in S_N if and only if for all $(\lambda_1,\ldots,\lambda_{N+1})\in\Lambda_{N+1}$: $(A+BF)\sum_{j=1}^{N+1}\lambda_j v_j+(a+Bg)\neq 0$. Since $u_j=Fv_j+g$, and $(A+BF)\sum_{j=1}^{N+1}\lambda_j v_j+(a+Bg)=\sum_{j=1}^{N+1}\lambda_j(Av_j+Bu_j+a)$, this criterion is equivalent to (13). So, Problem 4.1 is solvable if and only if for $j=1,\ldots,N+1$ there exist inputs $u_j\in U_j$ such that (13) is satisfied. To verify this condition it turns out that it is sufficient to check it only at the vertices of the polytopes U_j.

For $j=1,\ldots,N+1$, let W_j denote the set of all vertices of the polytope U_j; in particular, $W_j=\varnothing$ if and only if $U_j=\varnothing$.

Theorem 4.17: For a given set $\{F_i|i\in\mathcal{I}\}$ of admissible exit facets Problem 4.1 is solvable if and only if for every $j\in\{1,\ldots,N+1\}$ there exists $w_j\in W_j$ such that

$$0\notin\mathrm{Conv}\left(\{Av_j+Bw_j+a|j=1,\ldots,N+1\}\right). \quad (14)$$

In particular, if $w_j \in W_j$ satisfy (14), then a solution to Problem 4.1 is obtained by solving F and g from the linear equation (11) with right-hand side $(u_1\ u_2 \cdots u_{N+1})$ replaced by $(w_1\ w_2 \cdots w_{N+1})$.

Proof: Assume that for $j = 1, \ldots, N+1$ there exist $w_j \in W_j$ such that (14) is satisfied. Then $w_j \in U_j$, $j = 1, \ldots, N+1$, and according to Theorem 4.12, the feedback $u = Fx + g$, obtained by solving (11) with right-hand side $(u_1\ u_2 \cdots u_{N+1})$ replaced by $(w_1\ w_2 \cdots w_{N+1})$ satisfies condition i) of Theorem 4.16. At the same time, this feedback satisfies condition ii) of Theorem 4.16. Indeed, (14) implies that the inputs $w_1, \ldots, w_{N+1}$ satisfy (13), and therefore closed-loop system (6) has no fixed point in S_N. So, according to Theorem 4.16, this feedback solves Problem 4.1.

Next, assume that $u = Fx + g$ is an admissible feedback that solves Problem 4.1. By taking $u_j = Fv_j + g$, $j = 1, \ldots, N+1$, condition i) of Theorem 4.16 implies that $u_j \in U_j$, $j = 1, \ldots, N+1$. According to condition ii) of Theorem 4.16, closed-loop system (6) has no fixed points. Hence (13) holds, indicating that $0 \notin P := \mathrm{Conv}(\{Av_j + Bu_j + a | j = 1, \ldots, N+1\})$. Then there exist $n \in \mathbb{R}^N$ and $\alpha > 0$ such that the hyperplane $n^T x = \alpha$ separates 0 from the polytope P, i.e. $\forall x \in P : n^T x > \alpha$. Every polytope U_j, $j = 1, \ldots, N+1$ contains at least one point u_j, such that $Av_j + Bu_j + a$ is located in the halfspace $n^T x > \alpha$. This implies that each polytope U_j, $(j = 1, \ldots, N+1)$ has a vertex $w_j \in W_j$ such that $Av_j + Bw_j + a$ is contained in the halfspace $n^T x > \alpha$. Since 0 does not belong to this halfspace, this choice of $w_j \in W_j$, $(j = 1, \ldots, N+1)$ satisfies (14). ☐

The result of Theorem 4.17 yields a constructive method for the design of an affine feedback solution to Problem 4.1. In principle, the vertices of the polytopes U_j, $j = 1, \ldots, N+1$ can be computed using existing software for polyhedral sets, such as [20] and [32]. Subsequently, the necessary and sufficient condition in Theorem 4.17 can be checked in finitely many steps. For any combination of vertices $w_j \in W_j$, $j = 1, \ldots, N+1$, it has to be verified whether the following system of N equations

$$\sum_{j=1}^{N+1} \lambda_j (Av_j + Bw_j + a) = 0 \tag{15}$$

has a solution $(\lambda_1, \ldots, \lambda_{N+1}) \in \Lambda_{N+1}$. If for at least one combination $w_1, \ldots, w_{N+1}$ of vertices (15) has no solution $(\lambda_1, \ldots, \lambda_{N+1}) \in \Lambda_{N+1}$, then (14) is satisfied and Problem 4.1 is solvable; an affine control law is obtained by solving (11) with right-hand side $(w_1\ w_2 \cdots w_{N+1})$. If for all possible combinations of vertices $w_j \in W_j$, $j = 1, \ldots, N+1$, system (15) has a solution in Λ_{N+1}, then Problem 4.1 is not solvable. Note that the procedure described above may not be very efficient. Especially the computation of all vertices of the polytopes U_j may become very time-consuming. Also checking the solvability of (15) for every possible combination of vertices is a possible bottleneck, although this computation can be stopped as soon as a linear system has been found without solution $(\lambda_1, \ldots, \lambda_{N+1}) \in \Lambda_{N+1}$. To enhance robustness it may also be preferable to replace the inputs w_j, $j = 1, \ldots, N+1$ at the vertices by suitable inputs u_j from the interior of U_j, before constructing the affine feedback law with (11). More research is required to find efficient algorithms for checking the necessary and sufficient conditions of Theorem 4.17.

E. Stabilization to a Given Fixed Point

By taking $\mathcal{I} = \emptyset$, Theorem 4.12 in combination with Lemma 4.7, yields necessary and sufficient conditions for the solvability of Problem 4.2(a):

Theorem 4.18: There exists an admissible affine feedback law (5) solving Problem 4.2(a) if and only if there exist inputs $u_1, \ldots, u_{N+1} \in U$ such that

i) $\forall i, j \in \{1, \ldots, N+1\}, i \neq j : n_i^T (Av_j + Bu_j + a) \leq 0$.

For the additional requirement in Problem 4.2(b) of stabilizing to a given fixed point, some additional conditions are needed.

Theorem 4.19: Let $x_f \in S_N$, and $(\mu_1, \ldots, \mu_{N+1}) \in \Lambda_{N+1}$ be such that $\sum_{j=1}^{N+1} \mu_j v_j = x_f$. There exists an admissible affine feedback law (5) solving Problem 4.2b) with fixed point x_f if and only if there exist inputs $u_1, \ldots, u_{N+1} \in U$ such that condition i) of Theorem 4.18 is satisfied, and additionally

ii) $B \sum_{j=1}^{N+1} \mu_j u_j = -Ax_f - a$;

iii) $\mathrm{span}(\{Av_j + Bu_j + a | j = 1, \ldots, N+1\}) = \mathbb{R}^N$.

Proof: (Necessity) Assume that Problem 4.2(b) is solvable by the admissible affine control law $u = Fx + g$. For $j = 1, \ldots, N+1$ we define $u_j := Fv_j + g$. Then Lemma 4.11 with $\mathcal{I} = \emptyset$ implies that condition i) of Theorem 4.18 is satisfied. Condition ii) states that x_f is a fixed point of the closed-loop system. The necessity of condition iii) is shown by contradiction. If iii) is not satisfied, then there exists a vector $m \in \mathbb{R}^N \setminus \{0\}$, such that

$$\forall j \in \{1, \ldots, N+1\} : m^T (Av_j + Bu_j + a) = 0.$$

Because of convexity it follows that $m^T((A + BF)x + (a + Bg)) = 0$ for all $x \in S_N$. Hence, every trajectory $x(t)$ of the closed-loop system satisfies $m^T x(t) =$ constant. Since the simplex S_N is full-dimensional, there exist $x_1, x_2 \in S_N$ such that $m^T x_1 \neq m^T x_2$. Then both the sets

$$P_1 = S_N \cap \{x \in \mathbb{R}^N | m^T x = m^T x_1\}$$
$$P_2 = S_N \cap \{x \in \mathbb{R}^N | m^T x = m^T x_2\}$$

are positively invariant under the closed-loop dynamics, and Theorem 3.1 states that both P_1 and P_2 contain a fixed point of the closed-loop dynamics. This contradicts the fact that the fixed point x_f is unique.

(Sufficiency) Assume that there exist $u_1, \ldots, u_{N+1} \in U$ such that conditions i), ii), and iii) are satisfied. Let $F \in \mathbb{R}^{m \times N}$ and $g \in \mathbb{R}^m$ be the unique solution of the set of linear equations (11). Combining Theorem 4.12 with $\mathcal{I} = \emptyset$ and Lemma 4.7, condition i) implies that all trajectories of closed-loop system (6) remain in S_N. Condition ii) states that x_f is a fixed point of

(6). Next, condition iii) is used to show that this fixed point is unique. According to iii), the $N \times (N+1)$ matrix

$$C := (Av_1 + Bu_1 + a| \cdots |Av_{N+1} + Bu_{N+1} + a)$$

has rank N, so $\ker(C)$ is a one-dimensional space, spanned by $m := (\mu_1, \ldots, \mu_{N+1})^T \neq 0$. Suppose that $\hat{x} = \sum_{j=1}^{N+1} \lambda_j v_j \in S_N$, with $\ell := (\lambda_1, \ldots, \lambda_{N+1}) \in \Lambda_{N+1}$, is a fixed point of (6). Then $\ell \in \ker(C) \cap \Lambda_{N+1}$, and it follows that $\ell = m$, and thus $\hat{x} = x_f$.

Finally, to show that for every $x_0 \in S_N$, the corresponding solution $x(t, x_0)$ tends to x_f for $t \to \infty$, we consider for all $\beta \in [0, 1]$ the simplex

$$\beta S_N := \mathrm{Conv}\left(\{\beta v_i + (1-\beta)x_f | i = 1, \ldots, N+1\}\right).$$

βS_N is just a shrunken version of simplex S_N, obtained by multiplying S_N from the fixed point x_f by the factor β. In every vertex $\beta v_i + (1-\beta)x_f$ of βS_N the velocity vector of closed-loop system (6) is just the β-multiple of the velocity vector in the original vertex v_i. Hence, closed-loop system (6) is positively invariant for all simplices βS_N with $\beta \in [0, 1]$. Since in all vertices of S_n the vector field of closed-loop system (6) is pointing into S_N, there exist $t_0 > 0$ and $\hat{\beta} \in [0, 1)$ such that $x(t_0, v_i) \in \hat{\beta} S_N$ for all $i = 1, \ldots, N+1$. Then $x(t, x_0) \in \hat{\beta} S_N$ for all $x_0 \in S_N$ and $t \geq t_0$. By repeating the same argument, we obtain $x(t, x_0) \in \hat{\beta}^n S_N$ for $t \geq n t_0$, hence $x(t, x_0) \to x_f$ for $t \to \infty$. □

Remark 4.20: For the construction of an admissible affine feedback that solves Problem 4.2(a) or (b), one first has to find inputs $u_1, \ldots, u_{N+1} \in U$ that satisfy the conditions of Theorem 4.18 or Theorem 4.19. With these input values in the right-hand side, linear equation (11) is solved for F and g to find the corresponding control law.

V. CONTROL OF THE HYBRID SYSTEM

In this section we show how to use the solutions to Problems 4.1, 4.2a) and 4.2b) to solve Problems 2.3, 2.3a) and 2.3(b) for the original hybrid system.

Let $\mathcal{D} = (Q, E, f)$ be the underlying discrete event system of the hybrid system $\mathcal{H}$. The trajectories of $\mathcal{D}$ simulate the discrete behavior of $\mathcal{H}$, but do not take into account the continuous dynamics. We cannot control $\mathcal{D}$ directly, but instead control $\mathcal{D}$ indirectly by choosing an appropriate affine control law k_q in location q. By considering all possible control laws in each discrete location, we can obtain a supervisory controller [9] for $\mathcal{D}$. A more efficient approach to solving the reach-avoid problem is to use dynamic programming to construct affine controllers k_q. Using Theorem 4.17, we can restrict the possible events which can occur, and also guarantee that the solution does not remain stuck in a discrete state.

Problem 5.1: Let $q \in Q$ and $Q_c \subset Q$. Find a control law k_q solving Problem 4.1 with $S_N = X_q$, $\mathcal{A} = \mathcal{A}_q$ and

$$\mathcal{I} = \{i \in 1, \ldots, N+1 | F_i \cap G_q(e) = \emptyset \text{ whenever } f(q, e) \notin Q_c\}.$$

Problem 5.1 can be solved using Theorem 4.17, and can be used to solve Problem 2.3 as follows:

Algorithm 5.2

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0) Set $Q_0 := \{q_t\}$ and $j := 0$.

1) Look for $q \in Q \setminus (Q_j \cup Q_u)$, for which Problem 5.1 is solvable with $Q_c = Q_j$.

2) If Problem 5.1 with $Q_c = Q_j$ has solution k_q, for some $q \in Q \setminus (Q_j \cup Q_u)$, then increment j, take $Q_j := Q_{j-1} \cup \{q\}$, and go back to 1.

3) If Problem 5.1 with $Q_c = Q_j$ has no solution for all $q \in Q \setminus (Q_j \cup Q_u)$, then the algorithm halts, with $Q_c := Q_j$.

If the algorithm halts with $Q_s \subset Q_c$, then the output is the piecewise-affine control law $\mathcal{K} = \{k_q | q \in Q_c \setminus Q_0\}$.

a) To solve Problem 2.3a), additionally solve Problem 4.2a) for discrete state q_t using Theorem 4.18.

b) To solve Problem 2.3b), additionally solve Problem 4.2b) for discrete state q_t using Theorem 4.19.

Theorem 5.3: If Algorithm 5.2 halts with $Q_s \subset Q_c$, then the reach-avoid problem 2.3 is solvable, and a solution is given by the control law $\mathcal{K} = \{k_q | q \in Q_c\}$.

Proof: By construction, if $q \in Q_i$ with $i > 0$, then the control law k_q ensures that any continuous state trajectory in X_q leaves X_q in finite time, and does so by crossing a facet $F \subset G_q(e)$ such that $f(q, e) \in Q_{i-1}$. Hence, by induction on i, Algorithm 5.2 solves Problem 2.3.

It is clear that Algorithms 5.2(a) and 5.2(b) solve Problems 2.3(a) and 2.3(b). □

Remark 5.4: Algorithm 5.2 finds a solution $\mathcal{K}$ for which *every* continuous trajectory of the closed-loop system with initial point (q, x_q), with $q \in Q_c$ eventually leaves X_q through a guard set $G_q(e)$ with $f(q, e) \in Q_{i-1}$. Therefore, the piecewise-affine control law $\mathcal{K}$ that is a solution to Problem 2.3 for starting locations Q_s in fact solves Problem 2.3 for all starting locations in Q_c.

Remark 5.5: The reset maps $\mathcal{R}_q(e)$ are not needed by Algorithm 5.2, since they do not occur in the construction of the control laws k_q, and are irrelevant at the discrete-event level. Therefore, if Algorithm 5.2 succeeds in finding a piecewise-affine control law $\mathcal{K}$, then $\mathcal{K}$ is valid independently of the reset maps $\mathcal{R}_q(e)$.

Remark 5.6: Algorithm 5.2 is conservative, in that it may be possible to solve Problem 2.3 even if the algorithm finds no solutions. This happens when the reset maps ensure that the hybrid state trajectory enters a location q in a set $Z_q \subset X_q$, and all continuous trajectories starting in Z_q can be made to leave through the desired guard sets, but not every trajectory can be made to do so.

In the worst possible case, we need to check all states in $Q \setminus \{Q_j \cup Q_u\}$ each time we perform step 1 of Algorithm 5.2, giving a total of $n(n-1)/2$ attempts to solve Problem 5.1, where $n = |Q \setminus Q_u|$. Hence the algorithm is quadratic in the number of safe discrete states. The difficulty of Problem 4.1 decreases as the number of admissible exit facets increases. The overall cost is likely to be dominated by the difficulty of finding control laws without fixed points in the state simplex, as described in Theorem 4.17.

The next example illustrates the use of Algorithm 5.2.

Example 5.7: Let $\mathcal{H} = (Q, E, f, U, \{(X_q, \mathcal{A}_q) | q \in Q\}, \{(G_q(e), \mathcal{R}_q(e)) | (q, e) \in \mathrm{dom}(f)\})$ be a piecewise-affine hybrid system. Let $Q = \{q_1, \ldots, q_5\}$ and $E = \{e_1, \ldots, e_5\}$, and take $f(q_i, e_j) = q_j$ whenever $G_{q_i}(e_j) \neq \emptyset$.

Suppose each continuous state space X_q is the two-dimensional simplex

$$S_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1\right\}$$

with facets

$$\begin{aligned} F_1 &= \{(x_1, x_2) \in S_2 | x_1 = 0\} \\ F_2 &= \{(x_1, x_2) \in S_2 | x_2 = 0\} \\ F_3 &= \{(x_1, x_2) \in S_2 | x_1 + x_2 = 1\} \end{aligned}$$

vertices

$$v_1 = (1, 0) \quad v_2 = (0, 1) \quad v_3 = (0, 0)$$

and outward normals

$$n_1 = (-1, 0)^T \quad n_2 = (0, -1)^T \quad n_3 = (1, 1)^T.$$

Let $U = [-1, 1]$, and define continuous-state dynamics by

$$\begin{aligned} \dot{x}_{q_1} &= (3x_1 + 2u, -x_1 + 2x_2 + 2u) \\ \dot{x}_{q_2} &= (-1, -1) \quad \dot{x}_{q_3} = (-x_1 + u, x_1 - x_2 + u) \\ \dot{x}_{q_4} &= (0, 0), \text{ and } \dot{x}_{q_5} = (1 - x_1 + 2u, x_1 - x_2 + u). \end{aligned}$$

Take guard sets

$$\begin{aligned} G_{q_1}(e_2) &= F_3 \quad G_{q_1}(e_3) = F_2 \quad G_{q_1}(e_4) = F_1 \\ G_{q_2}(e_1) &= F_3 \quad G_{q_2}(e_3) = F_2 \quad G_{q_2}(e_5) = F_1 \\ G_{q_3}(e_1) &= F_1 \quad G_{q_3}(e_4) = F_2 \quad G_{q_3}(e_5) = F_3. \end{aligned}$$

The reset maps are not needed by the algorithm, so we do not give them here. Consider the reach-avoid-stabilize problem 2.3(a) with $Q_s = \{q_1\}$, $Q_u = \{q_4\}$ and $q_t = q_5$.

To stabilize the continuous dynamics in discrete location q_5, we need to prevent exit from all the facets F_1, F_2 and F_3. Taking u_i to be the input at vertex v_i, by Theorem 4.18 we have the inequalities

$$F_1 : \begin{cases} 1 + 2u_2 \geq 0 \\ 1 + 2u_3 \geq 0; \end{cases} \quad F_2 : \begin{cases} 1 + u_1 \geq 0 \\ u_3 \geq 0; \end{cases}$$
$$F_3 : \begin{cases} 1 + 3u_1 \leq 0 \\ 3u_2 \leq 0. \end{cases}$$

A solution to these inequalities is given by $u_1 = -1/2$, $u_2 = -1/4$ and $u_3 = 1/2$, giving control law

$$u = k_{q_5}(x_1, x_2) := \frac{1}{2} - x_1 - \frac{3}{4}x_2$$

and closed-loop dynamics

$$\dot{x}_{q_5} = \left(2 - 3x_1 - \frac{3}{2}x_2, \frac{1}{2} - \frac{7}{4}x_2\right).$$

There is a unique fixed-point in S_2 at (11/21,2/7).

We now try to drive the system to discrete state q_5 without passing through q_4. It is impossible to reach q_5 in one step from q_1. In discrete mode q_2, the system reaches q_5 if the trajectory leaves through facet F_1. However, there is no control in this discrete mode, and we cannot prevent the system leaving through facet F_2 and reaching q_3.

In location q_3, we arrive at q_5 on exiting through facet F_3. To block facet F_1 and F_2, we have the inequalities

$$F_1 : \begin{cases} u_2 \geq 0 \\ u_3 \geq 0 \end{cases} \quad F_2 : \begin{cases} u_1 \geq -1 \\ u_3 \geq 0. \end{cases}$$

It is easy to see that the control law $u = k_{q_3}(x) := 1$ solves these inequalities, giving closed-loop system

$$\dot{x}_{q_3} = (1 - x_1, 1 + x_1 - x_2).$$

Since the unique fixed point (1,2) does not lie in S_2, by Theorem 3.1, all trajectories leave S_2 in finite time by crossing F_3, hence $q_3 \in Q_c$.

In discrete location q_1 we can now try to enforce a transition to discrete location q_3, by controlling the continuous dynamics to leave through facet F_2. The inequalities to block facets F_1 and F_3 are

$$F_1 : \begin{cases} -2u_2 \leq 0 \\ -2u_3 \leq 0 \end{cases} \quad F_3 : \begin{cases} 2 + 4u_1 \leq 0 \\ 2 + 4u_2 \leq 0 \end{cases}$$

which are inconsistent at vertex v_2, because we require $u_2 \geq 0$ and $2 + 4u_2 \leq 0$. Hence there is no control law which forces all trajectories to leave through F_2.

In discrete location q_2, all trajectories exit through F_1 or F_2, and hence enter discrete states q_5 or q_3, respectively, so $q_2 \in Q_c$.

To transfer the discrete location from q_1 to either q_2 or q_3, the continuous state has to leave S_2 through F_2 or F_3. So we need to block facet F_1, which means that $u_2 \geq 0$ and $u_3 \geq 0$. Taking control law $u = k_{q_1}(x) := 1$ blocks facet F_1, yielding the closed-loop system

$$\dot{x}_{q_1} = (3x_1 + 2, -x_1 + 2x_2 + 2). \tag{16}$$

This system has fixed point $(-2/3, -4/3)$ which is outside S_2. Hence, all trajectories leave S_2 in finite time.

The resulting control law $\mathcal{K} = \{k_{q_1}, k_{q_3}, k_{q_5}\}$ solves Problem 2.3(a).

It is easily verified that the control law $k_{q_1}(x) = 1$ blocks facet F_1 as well as F_2, and hence forces the system (16) to leave S_2 by crossing F_3, transferring the hybrid system to discrete location q_2. There are other control laws, such as $k_{q_1}(x) := 1/4$, for which only facet F_1 is blocked. In this case, the initial condition of the continuous state in discrete location q_1 determines whether facet F_2 or F_3 is reached first, and hence whether the hybrid system transfers to discrete location q_2 or to q_3.

We now give an example of a piecewise-affine hybrid system $\mathcal{H}$ for which Problem 2.3 has a solution as a piecewise-affine control law, but the solution is not found by Algorithm 5.2.

Example 5.8: Let $Q = \{q_1, q_2, q_3, q_4\}$, $E = \{e_2, e_3, e_4\}$, and $f :\subset Q \times E \to Q$ be given by $f(q_1, e_2) = q_2$, $f(q_2, e_3) = q_3$, and $f(q_2, e_4) = q_4$.

Let $U = [0,1]^2$, $X_{q_1} = S_1 := [0,1]$ and $X_{q_2} = S_2$, where simplex S_2 is as in Example 5.7, with boundary facets F_1, F_2 and F_3. Take flows

$$\dot{x}_{q_1} = -1 \text{ and } \dot{x}_{q_2} = (-2 - u_1, -2 - u_2).$$

Let

$$\begin{aligned} G_{q_1}(e_2) &= \{0\} \text{ with } [\mathcal{R}_{q_1}(e_2)](0) = (1/3, 1/3) \\ G_{q_2}(e_3) &= F_1, \text{ and } G_{q_2}(e_4) = F_2. \end{aligned}$$

The control objective is to reach discrete state q_3 from discrete state q_1 while avoiding discrete state q_4.

This problem is clearly solvable; every initial point in X_{q_1} leaves X_{q_1} through $G_{q_1}(e_2)$ and is reset to $(1/3, 1/3) \in X_{q_2}$. In X_{q_2}, we can take $u = k_{q_2}(x) := (1, 0)$, yielding the closed-loop system $(\dot{x}_1, \dot{x}_2) = (-3, -2)$. The initial point (1/3,1/3) reaches point (0,1/9) in facet F_1 after time $t = 1/9$, triggering event e_3 which takes the system to discrete state q_3.

However, facet F_2 cannot be blocked whatever control law (u_1, u_2) is chosen in X_{q_2}, so it cannot be guaranteed that every trajectory starting in discrete location q_2 reaches q_3 and not q_4. Hence Algorithm 5.2 fails to find a solution.

VI. CONCLUDING REMARKS

The contribution of the paper is an algorithm to construct a control law for the problem of reaching a target state from a starting location for the class of piecewise-affine hybrid systems defined on simplices. The algorithm depends only on the solution of a finite set of linear inequalities in a finite-dimensional space, and on the solution of a control problem for a discrete-event system. The algorithm is conservative, in that it may fail to find a control law even if one exists. Hence the conditions under which the algorithm succeeds in finding a control law are sufficient, but not necessary, for the existence of such a control law.

There are a number of possible directions for further research. Of particular importance are generalizations to piecewise-affine systems defined on polyhedra which are not necessarily simplices, and on generalizations to particular classes of nonlinear hybrid systems, such as piecewise multi-affine hybrid systems on rectangles (see e.g. [4]). The development of algorithms which can compute control laws under weaker reachability conditions is also of interest.

The authors plan to implement the algorithms to allow the synthesis of control laws for realistically-sized piecewise-affine hybrid systems of practical interest.

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