

# **Heavy-traffic analysis of the M/PH/1 discriminatory processor sharing queue with phase-dependent weights**

**Maaike Verloop (CWI)**

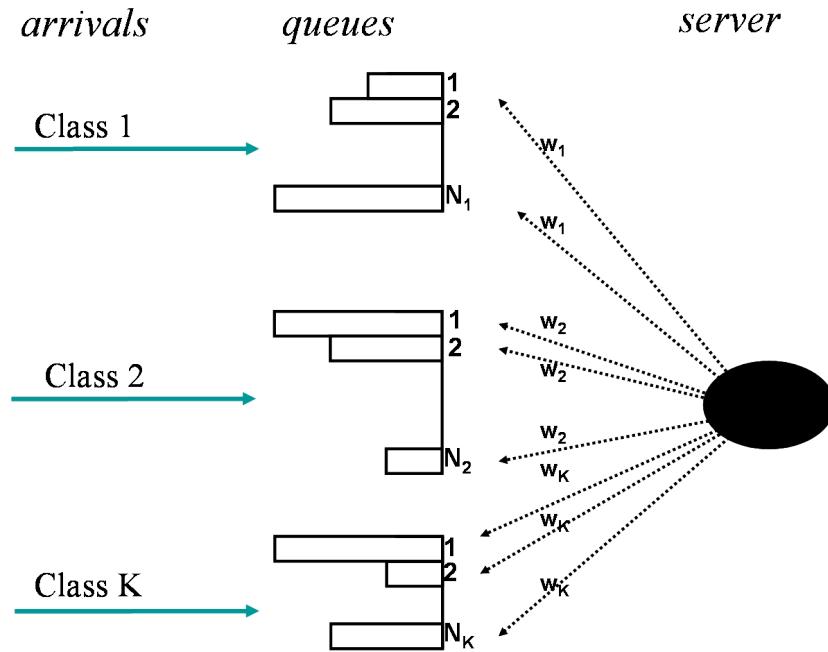
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**Sindo Núñez-Queija (University of Amsterdam & CWI)**

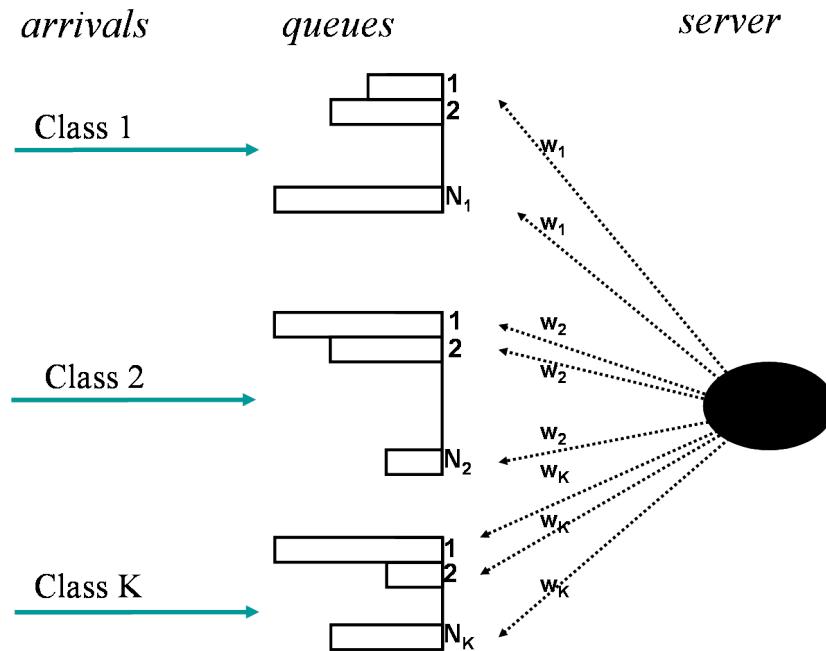
# Heavy-traffic analysis of the M/PH/1 discriminatory processor sharing queue with phase-dependent weights

- Discriminatory Processor Sharing  
vs Egalitarian processor Sharing
- Dynamics in heavy traffic
- Proof for phase type distributions

# Model description



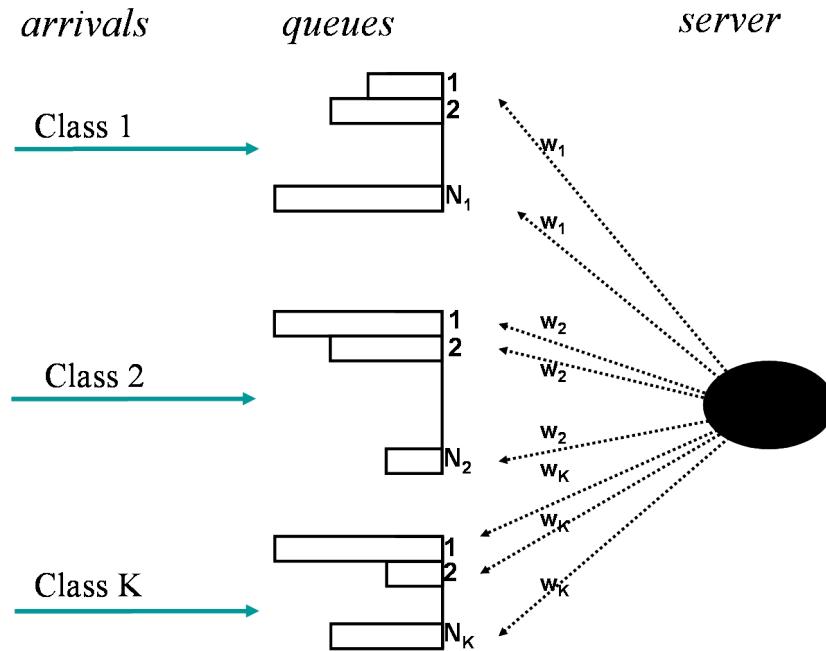
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Class  $k$

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- Service  $\mathbb{B}_k(x) := \mathbb{P}(B_k \leq x)$
- Load  $\rho_k = \lambda_k \beta_k$

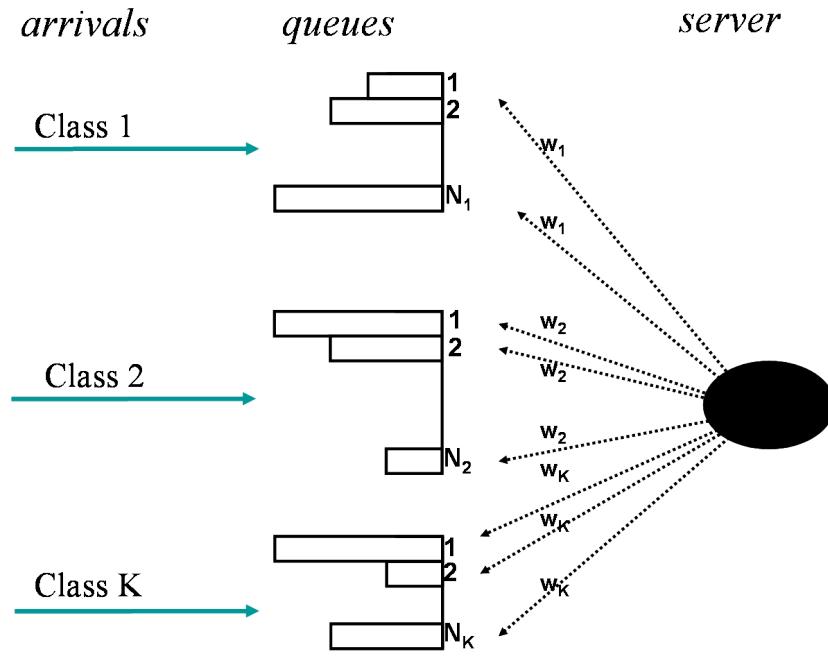
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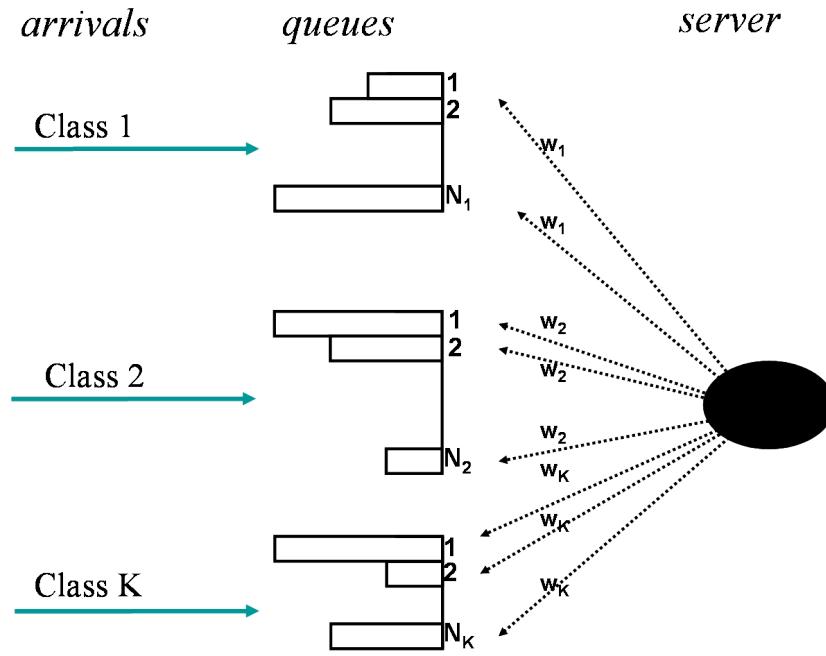
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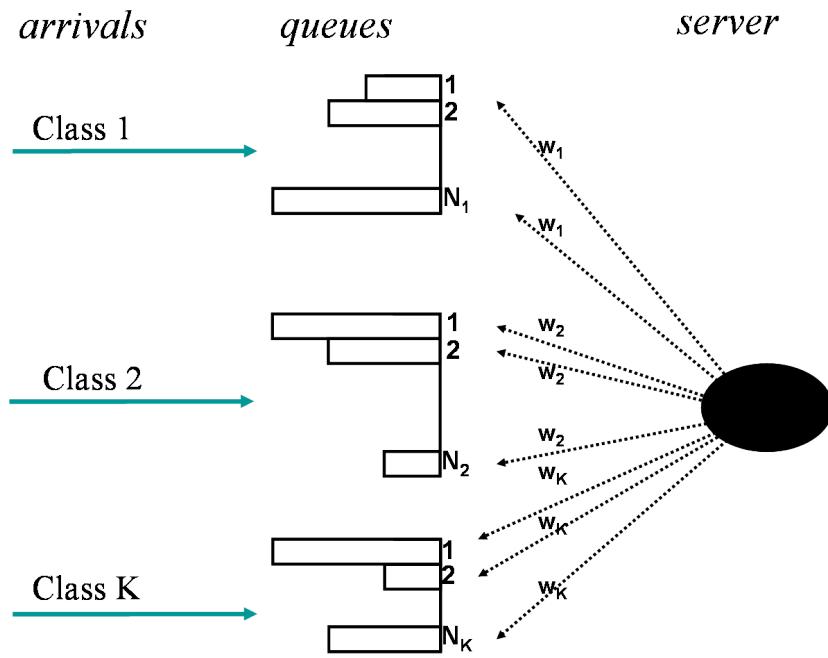


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Applications: time-shared computing systems, TCP, ADSL

# Literature

- Kleinrock [1967] ('Priority Processor-shared Model')
- Fayolle, Mitrani & Iasnogorodski [1980]
- Grishechkin [1992, 1994]
- Rege & Sengupta [1994, 1996]
- Borst, Van Ooteghem & Zwart [2003]
- Altman, Jimenez & Kofman [2004]
- Avrachenkov, Ayesta, Brown, N-Q [2005]

# Discriminatory PS versus (Egalitarian) PS

PS: Mean queue lengths are entirely “insensitive” as opposed to non-preemptive disciplines like FCFS

$$\mathbb{E}N_k = \frac{\rho_k}{1 - \rho}$$

For DPS:

$\mathbb{E}N_k$  are all finite under the usual stability condition  
(regardless of the higher-order moments of the service requirements)

Other insensitivity properties of PS only carry over to DPS in asymptotic regimes

# Discriminatory PS versus Egalitarian PS

PS: Expected sojourn time for jobs of given size

$$\frac{\mathbb{E}T_k(x)}{x} = \frac{1}{1 - \rho}$$

DPS: true in the limit as  $x \rightarrow \infty$  [Fayolle, Mitrani & Iasnogorodski]

and the "bias" is also insensitive

$$\lim_{x \rightarrow \infty} \left( \mathbb{E}T_k(x) - \frac{x}{1 - \rho} \right) = \frac{\sum_j \lambda_j (1 - \frac{w_k}{w_j}) \mathbb{E}((B_j)^2)}{2(1 - \rho)^2}.$$

# Discriminatory PS versus Egalitarian PS

## Tails of the sojourn time distributions

PS and DPS:

For regularly varying service requirement distributions with finite variance (conditions can be relaxed):

$$\frac{\mathbb{P}\{T_k > x\}}{\mathbb{P}\{B_k > (1 - \rho)x\}} \rightarrow 1, \quad \text{as } x \rightarrow \infty$$

Again, the “scaling factor”  $1 - \rho$  is insensitive and common to all classes

# Discriminatory PS versus Egalitarian PS

## Time-scale separation (1):

Class  $k$  operates on a much faster time scale than class  $k + 1$ , for all  $k = 1, 2, \dots, K - 1$

- arrival rates  $\lambda_k f_k(r)$
- service requirements of class  $k$  distributed as  $B_k / f_k(r)$
- with  $f_{k+1}(r) / f_k(r) \rightarrow 0$  as  $r \rightarrow \infty$

# Discriminatory PS versus Egalitarian PS

Time-scale separation (2):

The limiting distribution of the slow class is geometric

$$\mathbb{P}_r\{N_2 = n_2\} \rightarrow \left(1 - \frac{\rho_2}{1 - \rho_1}\right) \left(\frac{\rho_2}{1 - \rho_1}\right)^{n_2}$$

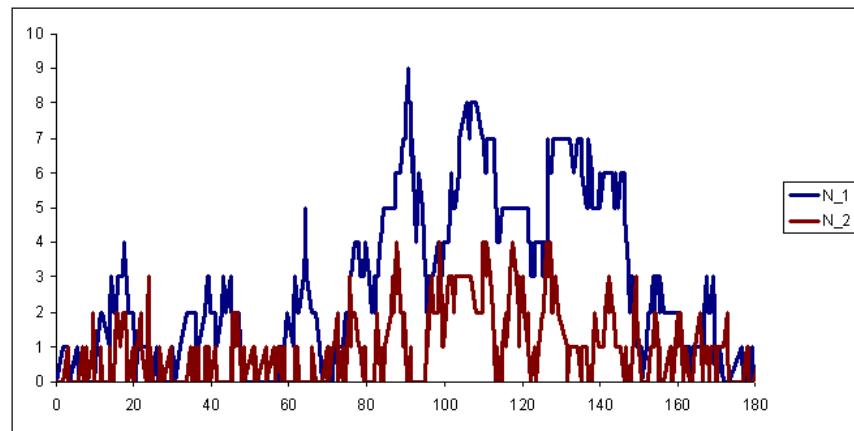
and

$$\begin{aligned} & \mathbb{P}_r\{N_1 = n_1 | N_2 = n_2\} \\ & \rightarrow \frac{\Gamma(n_1 + \frac{n_2 w_2}{w_1} + 1)}{\Gamma(n_1 + 1)\Gamma(\frac{n_2 w_2}{w_1} + 1)} \rho_1^{n_1} (1 - \rho_1)^{\frac{n_2 w_2}{w_1} + 1} \end{aligned}$$

For PS all limits can be replaced with equalities

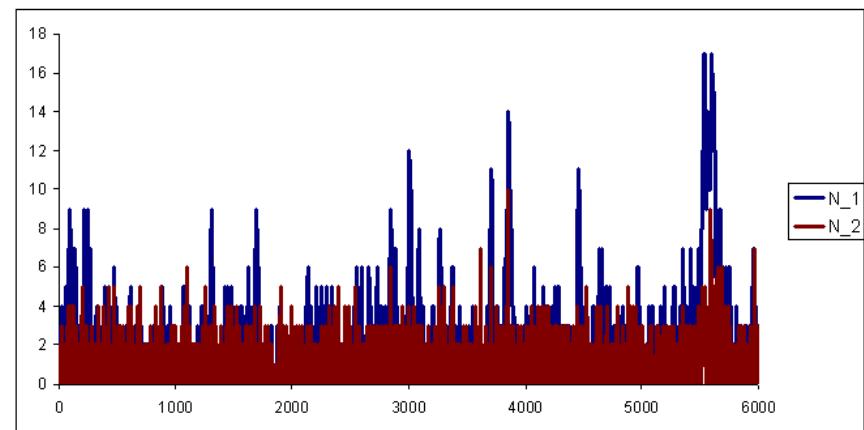
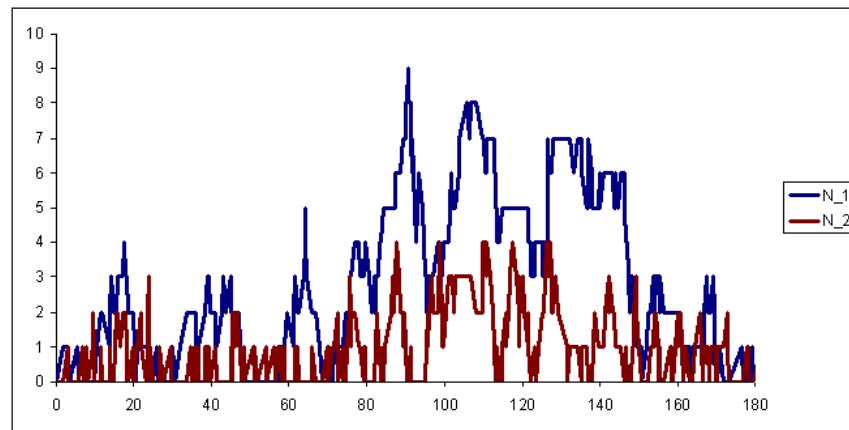
# Dynamics

lambda_1	lambda_2	mu_1	mu_2	rho_1	rho_2	rho	w_1	w_2
0.4	0.6	1	2	0.4	0.3	0.7	1	4



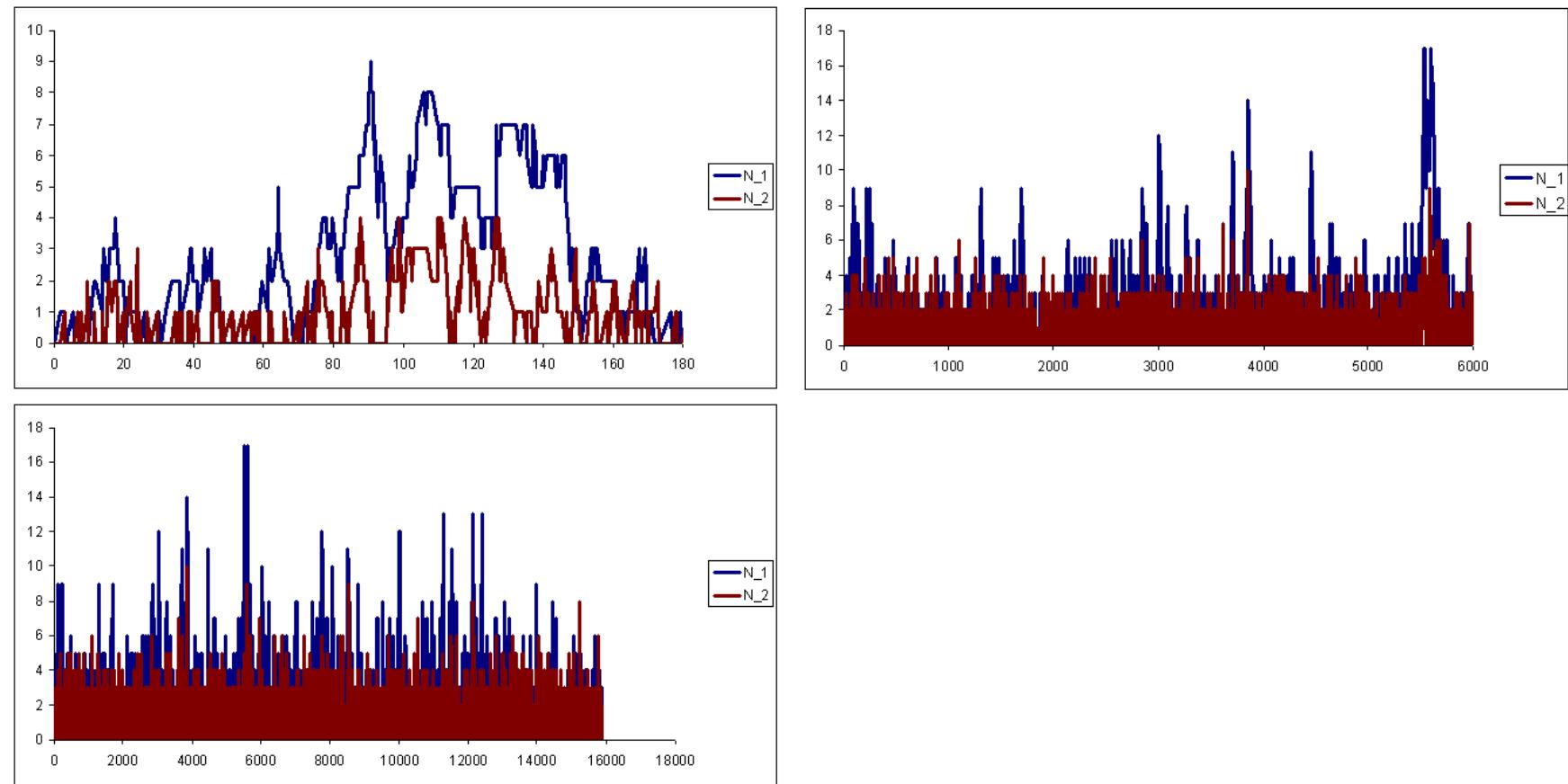
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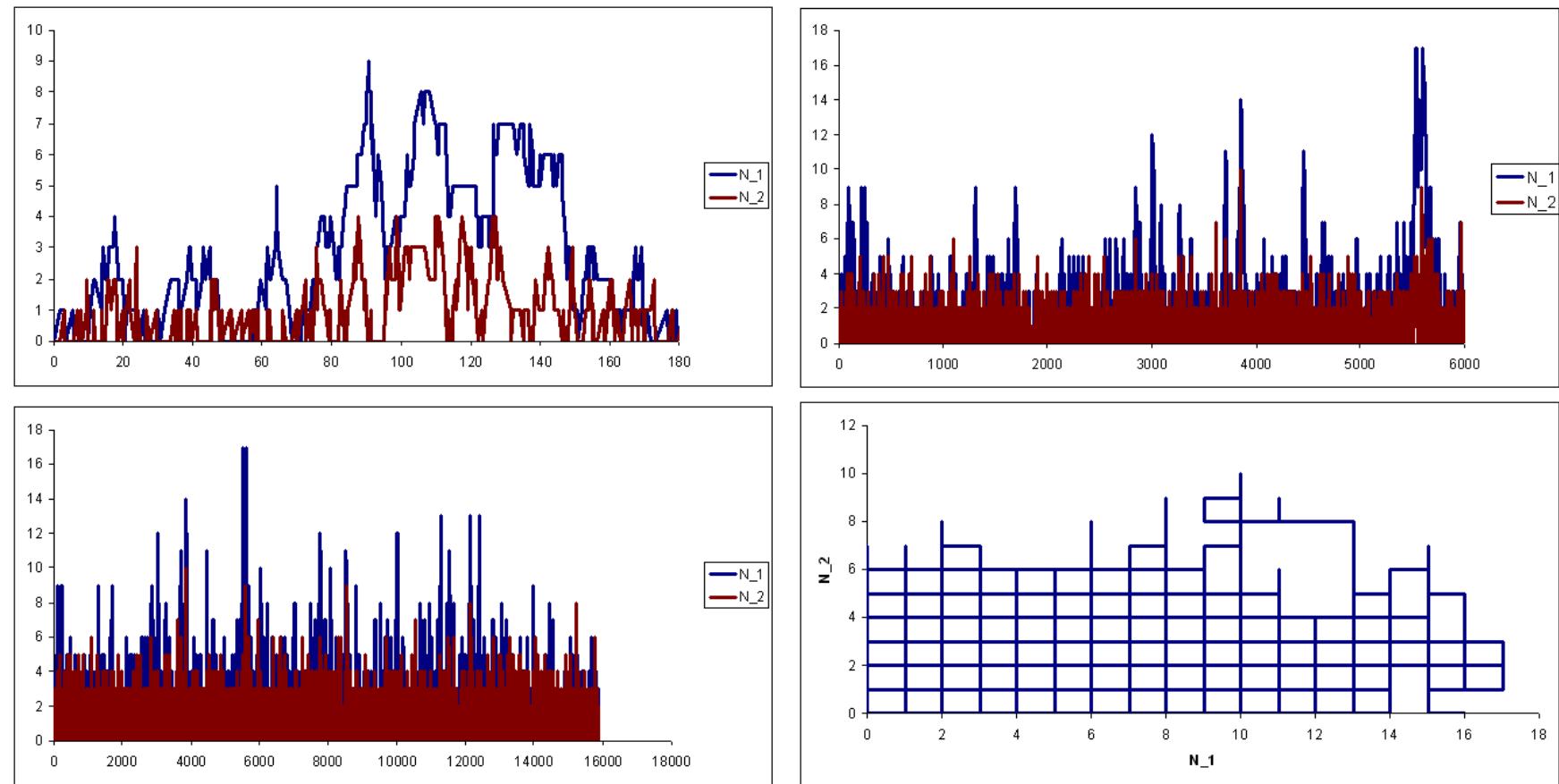
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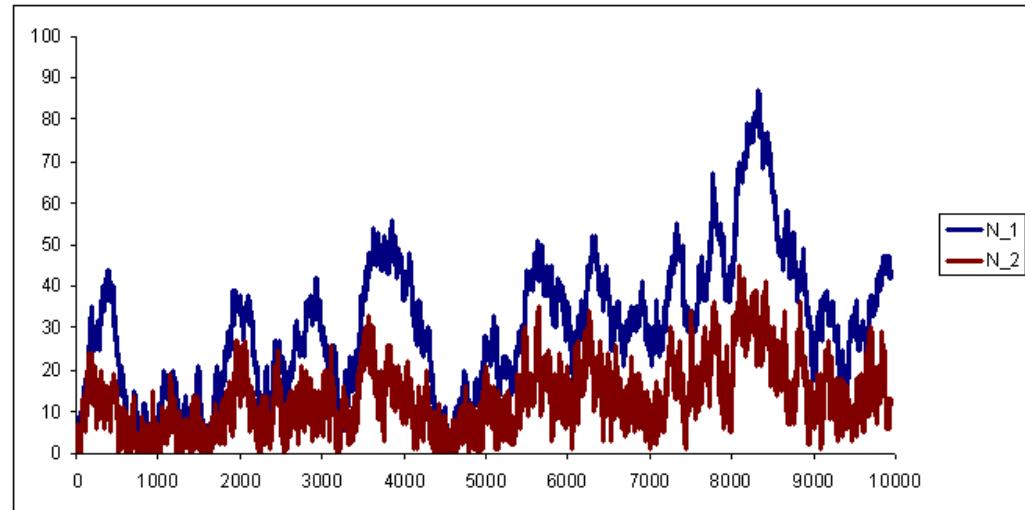
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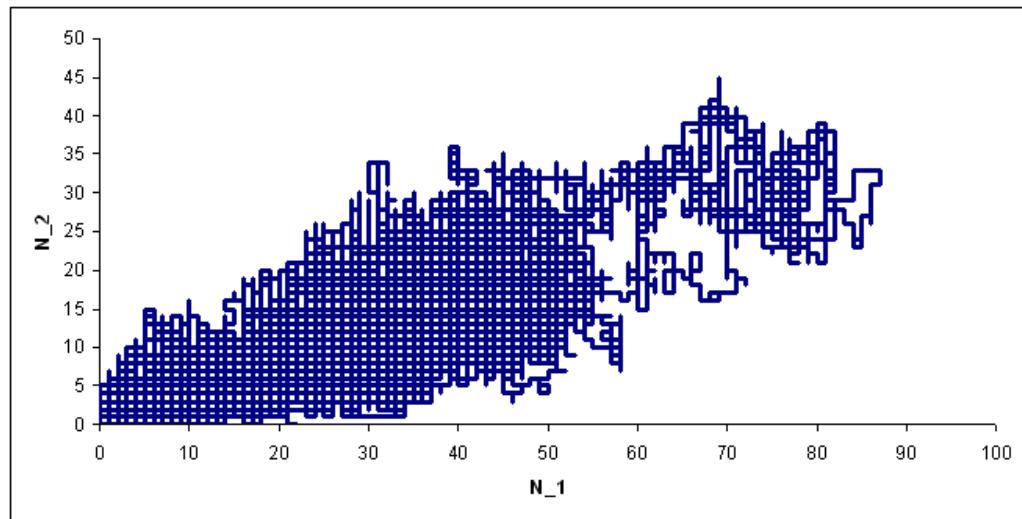
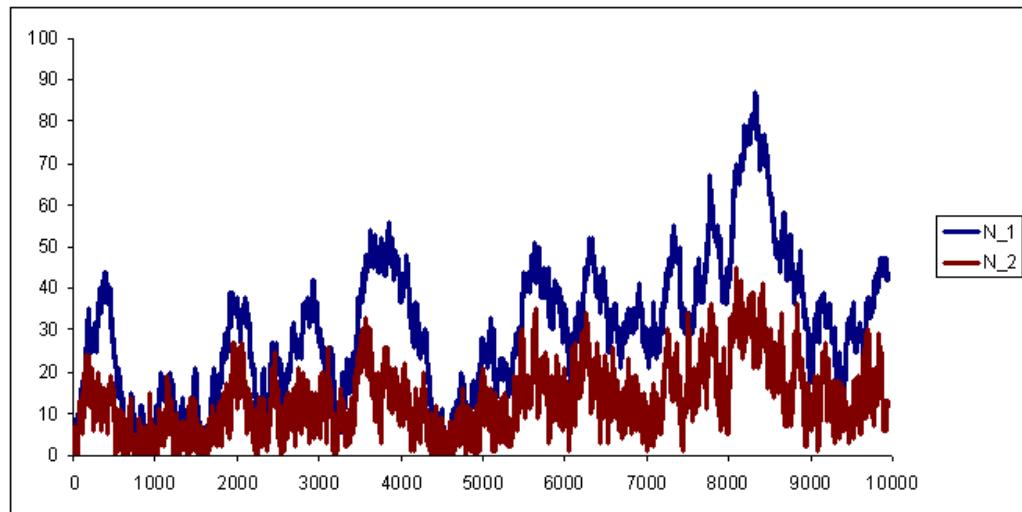
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# Heavy traffic: Theorem

For phase-type distributions

$$(1 - \rho)(N_1, N_2, \dots, N_K) \xrightarrow{d} E \cdot \left( \frac{\bar{\rho}_1}{w_1}, \frac{\bar{\rho}_2}{w_2}, \dots, \frac{\bar{\rho}_K}{w_K} \right)$$

where  $E$  is exponential with mean

$$\frac{\sum_k p_k \mathbb{E}[(B_k)^2] / \sum_k p_k \mathbb{E}B_k}{\sum_k \frac{1}{w_k} \bar{\rho}_k \mathbb{E}[(B_k)^2] / \mathbb{E}B_k}$$

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State-space collapse:

“In heavy traffic  $(1 - \rho)N_1, \dots, (1 - \rho)N_K$  are proportional to a common exponentially distributed random variable”

## Heavy traffic: Interpretation

$$(1 - \rho)(N_1, N_2, \dots, N_K) \xrightarrow{d} E \cdot \left( \frac{\bar{\rho}_1}{w_1}, \frac{\bar{\rho}_2}{w_2}, \dots, \frac{\bar{\rho}_K}{w_K} \right)$$

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Assume that  $N'_i/N'_j = n_i/n_j$  for some constants  $n_k$  (as in the exponential case), then

$$\frac{w_j n_j}{\sum_{i=1}^J w_i n_i} = \mu \sum_{i=1}^J p_{0i} a_{ij}$$

normalizing  $\sum_{i=1}^J w_i n_i = 1$  gives the result up to a multiplicative factor

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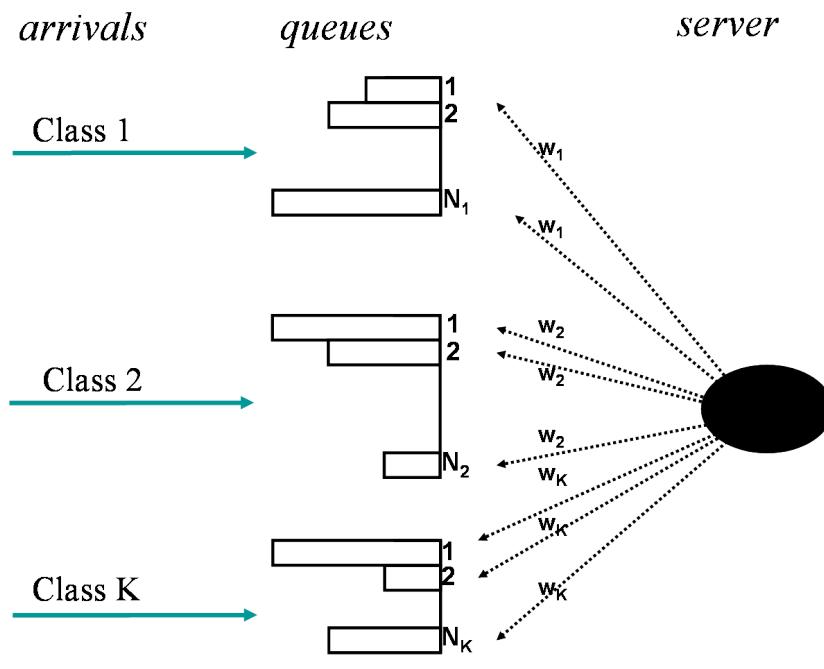
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Work-conserving and non-idling:

$$\mathbb{E}V = \frac{\rho}{2(1 - \rho)} \sum_k p_k \mathbb{E}[(B_k)^2] / \sum_k p_k \mathbb{E}B_k$$

# Phase-type service requirements (1)

## Recall



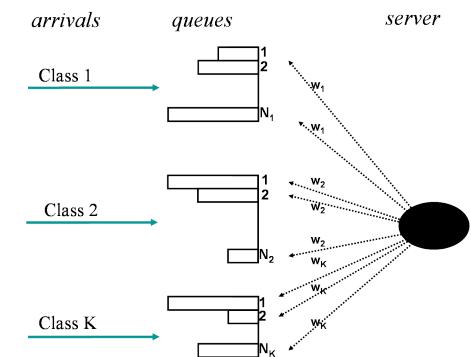
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## Phase-type service requirements (2)

Class  $k$  has  $m_k$  service phases

total # service phases:  $\sum_{k=1}^K m_k := J$

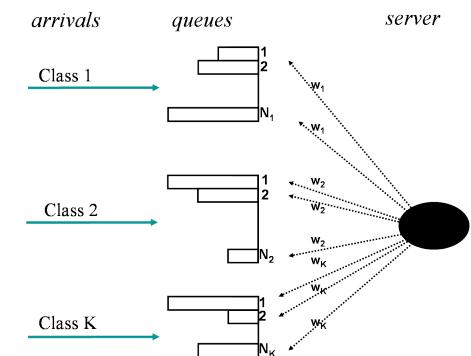


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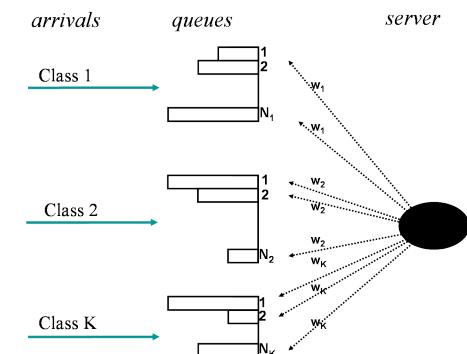
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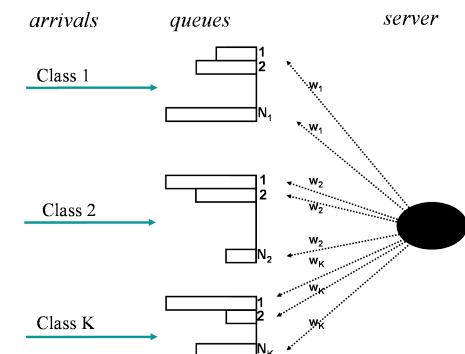
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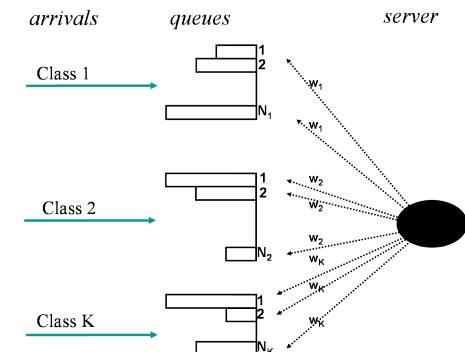
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$p_{ij}$  = phase transition probabilities



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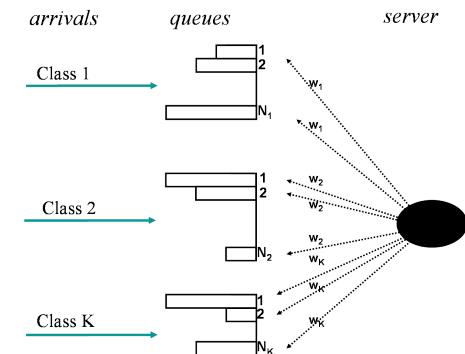
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$p_{ij}$  = phase transition probabilities

$p_{i0}$  = probability of completing service after phase  $i$



## Phase-type service requirements (3)

$$\begin{aligned} [\Lambda + \sum_{i=1}^J g_i(\bar{n})\mu_i]P(\bar{n}) &= \sum_{i=1}^J [\Lambda p_{0i}\delta_{n_i}P(\bar{n} - \bar{e}_i) + g_i(\bar{n} + \bar{e}_i)\mu_i p_{i0}P(\bar{n} + \bar{e}_i) \\ &\quad + \sum_{j=1}^J g_i(\bar{n} + \bar{e}_i - \bar{e}_j)\mu_i p_{ij}\delta_{n_j}P(\bar{n} + \bar{e}_i - \bar{e}_j)] \end{aligned}$$

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**Transformation:**  $R(\bar{n}) = \frac{P(\bar{n})}{\sum_{j=1}^J n_j g_j}$ , if  $\bar{n} \neq \bar{0}$

$$p(\bar{z}) = \sum z_1^{n_1} \dots z_J^{n_J} P(\bar{n}) = \mathbb{E}[z_1^{N'_1} \dots z_J^{N'_J}]$$
$$r(\bar{z}) = \sum z_1^{n_1} \dots z_J^{n_J} R(\bar{n}) = \mathbb{E} \left[ \frac{z_1^{N_1} \dots z_J^{N_J}}{\sum_{i=1}^J N_i g_i}; \sum_{j=1}^J N_j > 0 \right]$$

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$$p(\bar{z}) = \sum_{i=1}^J g_i z_i \frac{\partial r}{\partial z_i} + 1 - \rho$$

# Phase-type service requirements (4)

## Partial differential equation

$$\begin{aligned} & \Lambda(1 - \rho)(1 - \sum_{j=1}^J p_{0j}z_j) \\ &= \sum_{i=1}^J \{\mu_i g_i(p_{i0} + \sum_{j=1}^J p_{ij}z_j - z_i) - \Lambda g_i z_i (1 - \sum_{j=1}^J p_{0j}z_j)\} \frac{\partial r}{\partial z_i} \end{aligned}$$

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Can be used for analysis of

- Moments of the queue length distribution  
[Van Kessel, N-Q, Borst, 2004]
- Heavy traffic (today)

# Heavy-traffic scaling

## Notation

- **Change of variables**  $z_i = e^{-s_i}$
- $e^{-(1-\rho)\bar{s}} := (e^{-(1-\rho)s_1}, \dots, e^{-(1-\rho)s_J})$

## Goal

$$\begin{aligned}\lim_{\rho \uparrow 1} p(e^{-(1-\rho)\bar{s}}) &:= \lim_{\rho \uparrow 1} \mathbb{E}(e^{-(1-\rho)s_1 N_1} \dots e^{-(1-\rho)s_J N_J}) \\ &= \mathbb{E}(e^{-s_1 \hat{N}_1} \dots e^{-s_J \hat{N}_J})\end{aligned}$$

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## Investigate

$$\hat{r}(\bar{s}) = \mathbb{E} \left( \frac{1 - e^{-s_1 \hat{N}_1} \dots e^{-s_J \hat{N}_J}}{\sum_{j=1}^J \hat{N}_j g_j} \cdot \mathbf{1}_{(\sum_{j=1}^J \hat{N}_j > 0)} \right)$$

# Heavy traffic analysis

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## Lemma

The function  $\hat{r}(\bar{s})$  satisfies the following partial differential equation:

$$0 = \sum_{i=1}^J F_i(\bar{s}) \frac{\partial \hat{r}(\bar{s})}{\partial s_i} \quad \forall \bar{s} \geq 0,$$

# Heavy traffic analysis

## Investigate

$$\hat{r}(\bar{s}) = \mathbb{E} \left( \frac{1 - e^{-s_1 \hat{N}_1} \dots e^{-s_J \hat{N}_J}}{\sum_{j=1}^J \hat{N}_j g_j} \cdot \mathbf{1}_{(\sum_{j=1}^J \hat{N}_j > 0)} \right)$$

## Lemma

The function  $\hat{r}(\bar{s})$  satisfies the following partial differential equation:

$$0 = \sum_{i=1}^J F_i(\bar{s}) \frac{\partial \hat{r}(\bar{s})}{\partial s_i} \quad \forall \bar{s} \geq 0,$$

where

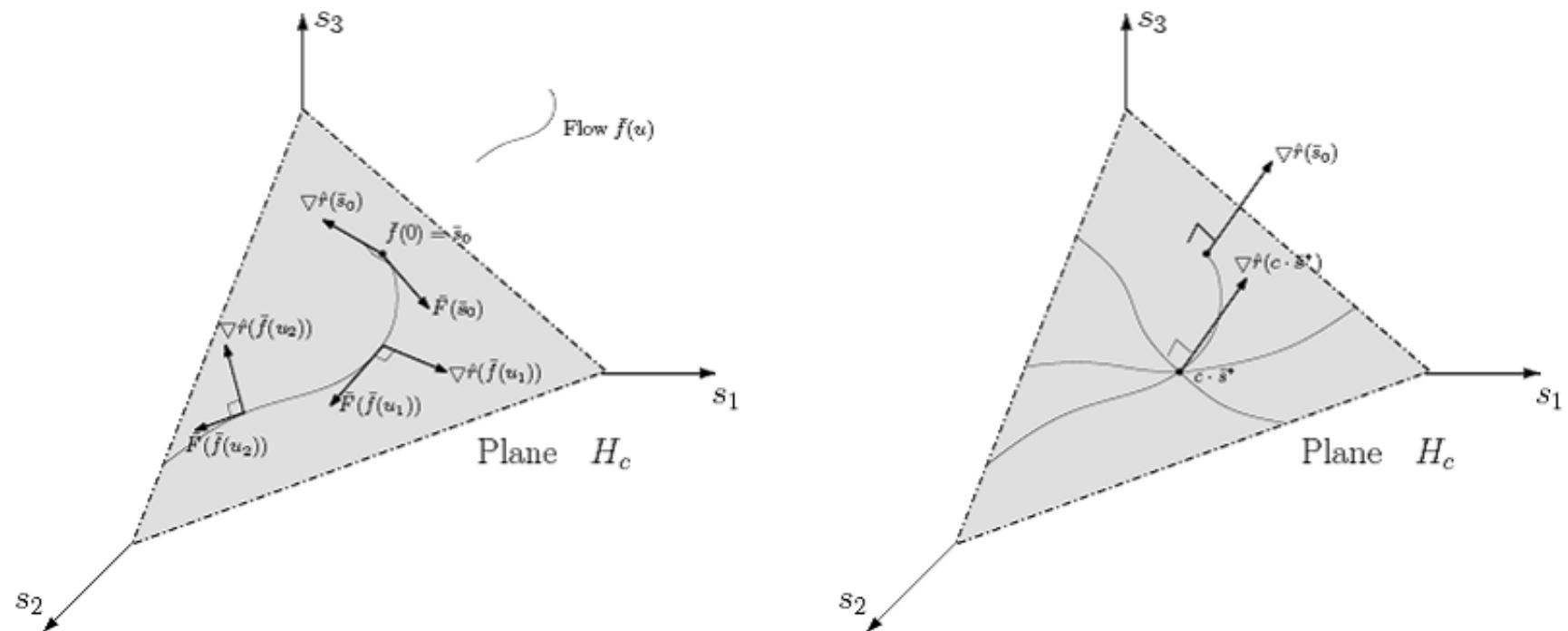
$$F_i(\bar{s}) = g_i \left( \mu_i (-s_i + \sum_{j=1}^J p_{ij} s_j) + \hat{\lambda} \sum_{j=1}^J p_{0j} s_j \right),$$

with  $\hat{\lambda}$  equal to the limiting arrival rate

# State space collapse in heavy traffic

The function  $\hat{r}(\bar{s})$  is constant on the  $J - 1$  dimensional set

$$H_c := \{\bar{s} \geq \bar{0} : \sum_{j=1}^J \frac{\hat{\rho}_j}{g_j} s_j = c\}, \quad c > 0,$$



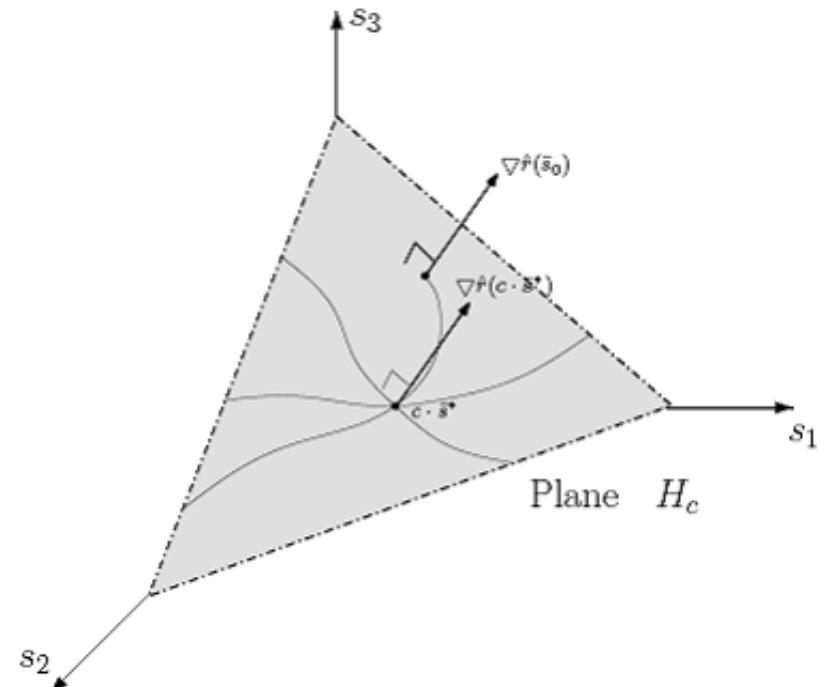
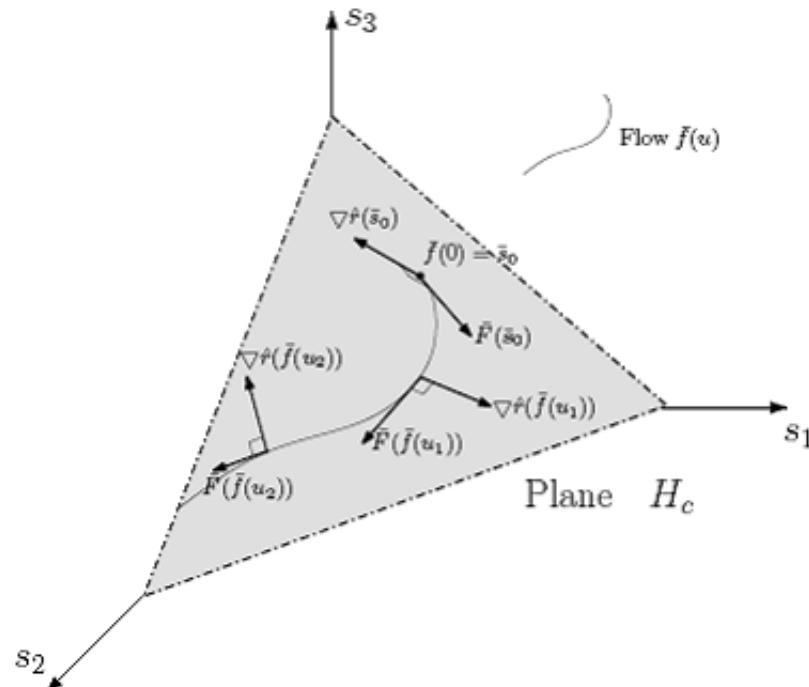
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hence

$$(\hat{N}_1, \hat{N}_2, \dots, \hat{N}_J) \stackrel{d}{=} \left( \frac{\hat{\rho}_1}{g_1}, \frac{\hat{\rho}_2}{g_2}, \dots, \frac{\hat{\rho}_J}{g_J} \right) \cdot X,$$



# Residual service requirements in heavy traffic

For phase-type distributed service requirements

$$\begin{aligned} & \lim_{\rho \uparrow 1} \mathbb{E} \left( e^{-\sum_{l=1}^K s_l (1-\rho) N_l - \sum_{l=1}^K \sum_{h=1}^{y_l} s_{l,h} B_{l,h}^r} \right) \\ &= \mathbb{E} \left( e^{-\sum_{l=1}^K s_l \hat{N}_l} \right) \cdot \prod_{l=1}^K \prod_{h=1}^{y_l} \mathbb{E} \left( e^{-s_{l,h} B_l^{fwd}} \right) \end{aligned}$$

for  $y_l \in \{0, 1, \dots\}$  and  $s_{l,h}, s_l > 0$ ,  $l = 1, \dots, K$ ,  $h = 1, \dots, y_l$

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For PS, the limit can be replaced with an equality (for all loads)

# Concluding remarks

- Joint queue length distribution of DPS in heavy traffic
  - ◊ Closed form analysis
  - ◊ State space collapse
  - ◊ Sensitive to second moments, but not "too much" (see paper)
- Product form for residual service requirements
- More
  - ◊ Size based scheduling
  - ◊ Monotonicity with respect to the weights

# **Heavy-traffic analysis of the M/PH/1 discriminatory processor sharing queue with phase-dependent weights**

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