Analysis of Two Competing TCP/IP Connections

Eitan Altman[1,2]

Tania Jiménez[2]

Sindo Núñez-Queija[3,4]

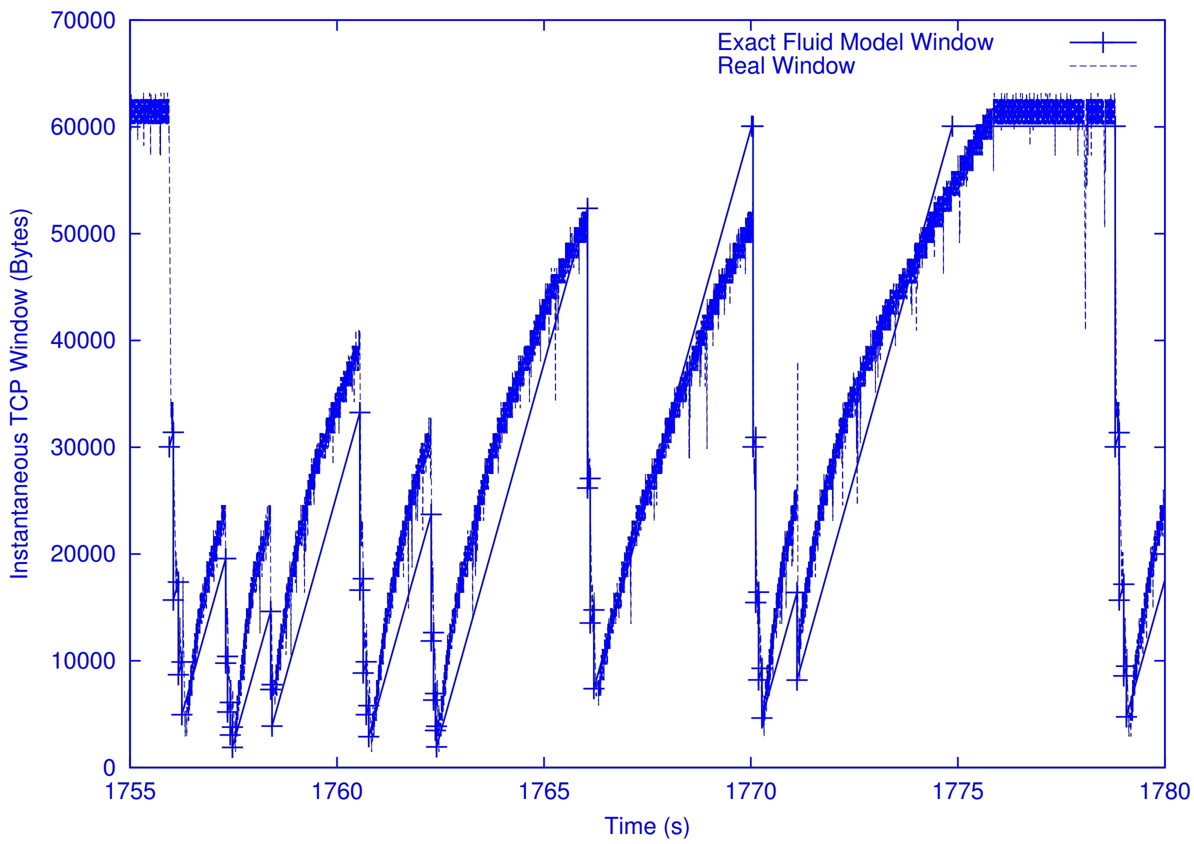
[1] INRIA Sophia Antipolis, France
[2] CESIMO, Venezuela
[3] CWI, The Netherlands
[4] Eindhoven University of Technology, The Netherlands

Analysis of Two Competing TCP/IP Connections

- Motivation & Model
- Analysis
- Approximation
- Asymptotic analysis
- Proofs

Motivation

Transmission Control Protocol (TCP)



Overview of models

- **Many flows [Field+, Padhye+, Mathis+, Altman+]**
  - ◇ **constant packet loss – throughput** $\sim \frac{1}{RTT}$
  - ◇ **exogenous loss process – throughput** $\sim \frac{1}{RTT^2}$

- **Small number of flows [Ait-Hellal&Altman, Brown, Lakshman&Madhow]**
  - ◇ **complete synchronization**
    - ⋆ **throughput** $\sim \frac{1}{RTT^\alpha}$ **with** $1 < \alpha < 2$**,**
    - ⋆ **tail drop buffers and similar RTTs**

overview of models ...

- No synchronization (asymmetry, RED)
  - ◇ fixed loss probability [Baccelli&Hong]
  - ◇ share of bandwidth [Altman+]
  - ◇ discretized Markov chain
  - ◇ throughput $\approx RTT^{-0.85}$
  - ◇ TCP more fair than assuming synchronization

Goals

- **Substantiate qualitative conclusions (fairness)**

- **Throughput** $\sim \frac{1}{RTT}$

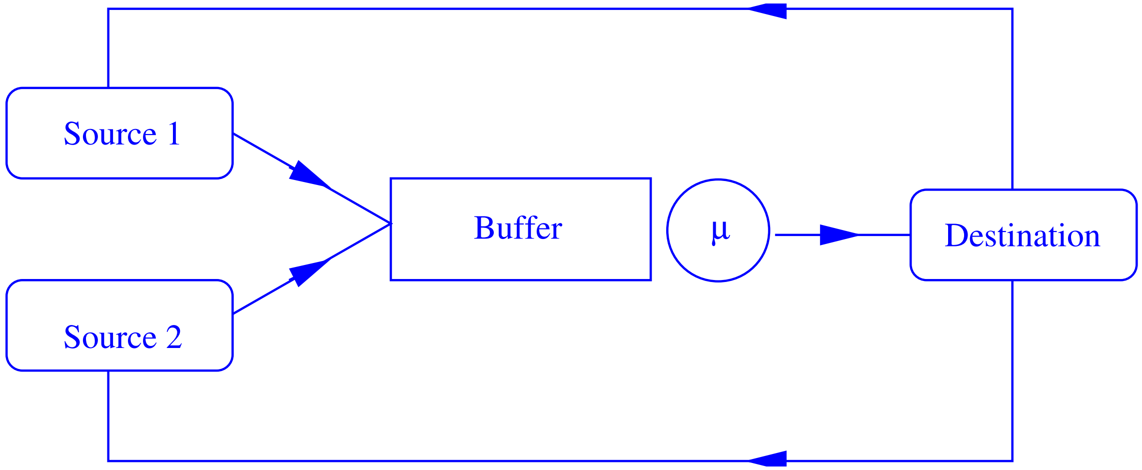
- **Proof through bounds**

- **Approximation that "matches"** $RTT^{-0.85}$ **for moderate RTTs**

Model description

- 2 saturated TCP sources

- Bandwidth μ



- RTT_k

- $W_k(t)$ window size

- Rate $X_k(t) = W_k(t)/RTT_k$

- **No time-out/slow-start (SACK, New-Reno)**

- **Negligible queueing delay; RTT constant (AQM)**

- **Increase 1 packet per RTT**

- **Linear increase**

$$\frac{dW_k(t)}{dt} = \frac{dW_k(t)}{dack_k} \times \frac{dack_k}{dt} = \frac{1}{W_k(t)} \times \frac{W_k(t)}{RTT_k} = \frac{1}{RTT_k}.$$

- **Congestion when** $X_1(t) + X_2(t) = \mu$ **(no queueing delay)**

- **Loss probability** $X_k(t)/\mu$

Markov model

- **Congestion epochs** t_n

- $X_n := X_1(t_n)$; $X_2(t_n) = \mu - X_n$

- $a := (RTT_1)^2$, $b := (RTT_2)^2$, $c := (a^{-1} + b^{-1})^{-1}$, **and** $r := \frac{b}{a}$

- $S_{n+1} := t_{n+1} - t_n$

- **Connection** 1 **suffers from congestion at time** t_n
  - ◇ $S_{n+1} = \frac{c}{2} X_n$
  - ◇ $X_{n+1} = \frac{1+2r}{2(1+r)} X_n$

- **Connection 2 suffers from congestion at time t_n**
  - $S_{n+1} = \frac{c(\mu - X_n)}{2}$
  - $X_{n+1} = \frac{\mu r}{2(1+r)} + \frac{2+r}{2(1+r)} X_n$
- **Markov process**

$$X_{n+1} = \begin{cases} \frac{1+2r}{2(1+r)} X_n & \textbf{w.p. } \frac{X_n}{\mu} \\ \mu - \frac{2+r}{2(1+r)} (\mu - X_n) & \textbf{w.p. } 1 - \frac{X_n}{\mu} \end{cases}$$

- **Assume stationarity**

Moments at loss instants

$$E[X^k] = E[E[(X_{n+1})^k | X_n]]$$

$$= E\left[\frac{X_n}{\mu}\left[\frac{1+2r}{1+r}\cdot\frac{X_n}{2}\right]^k + \frac{\mu - X_n}{\mu}\left[\frac{\mu r + (2+r)X_n}{2(1+r)}\right]^k\right]$$

$$= Z_1(k) + Z_2(k)$$

- **where**

$$Z_1(k) = \frac{2(1+r)}{\mu(1+2r)}\left(\frac{1+2r}{2(1+r)}\right)^{k+1} E[X^{k+1}]$$

$$Z_2(k) \;=\; \frac{2(1+r)}{2+r}E\left[\left(\frac{\mu r + (2+r)X_n}{2(1+r)}\right)^k\right] - \frac{2(1+r)}{\mu(2+r)}E\left[\left(\frac{\mu r + (2+r)X_n}{2(1+r)}\right)^{k+1}\right]$$

- **Recursion, but $E[X]$ unknown**

Symmetric case

- $r = 1$

$$\begin{aligned}
(1-r)E[X^2] &= \mu r\,(\mu - 2E[X]),\\
(1-r)E[X^3] &= \frac{\mu r\left(\mu^2 r + \mu(4+r)E[X] - (8+5r)E[X^2]\right)}{3(1+r)},\\
(1-r)E[X^4] &= \frac{\mu r}{7r^2+13r+7} \times \big(- 2(5r^2+15r+12r)E[X^3]\\
&\quad +6\mu(2+r)E[X^2] + 2\mu^2 r(3+r)E[X] + \mu^3 r^2\big).
\end{aligned}$$

- $E[X] = \mu/2$

- $E[X^2] = 7\mu^2/26$

- $E[X^3] = 2\mu^3/13$

- **at loss instants!**

Throughput distribution at loss instants

- **Define**

$$\beta := \frac{1+2r}{2(1+r)}, \quad u := \frac{\mu r}{2(1+r)}, \quad v := \frac{2+r}{2(1+r)}$$

- **Rewrite**

$$Z_1(k) = \frac{E[(\beta X)^{k+1}]}{\beta\mu}$$

$$Z_2(k) = \frac{1}{v}(E[(u+vX)^k] - \mu^{-1}E[(u+vX)^{k+1}])$$

- **Laplace Stieltjes transform**

$$F(s) := E[\exp(-sX)] = \sum_{k=0}^{\infty} \frac{(-s)^k E[X^k]}{k!}$$

- **From recursion**

$$\begin{aligned} F(s) &= -\frac{1}{\mu}F'(\beta s) + \frac{1}{v}\exp(-us)F(vs) \\ &\quad +\frac{1}{\mu}\exp(-us)F'(vs) - \frac{u}{v\mu}e^{-us}F(vs) \end{aligned}$$

- **Inversion**

$$\begin{aligned} f(x) &= \frac{1}{\beta^2\mu}xf\left(\frac{x}{\beta}\right) + \frac{1}{v^2}f\left(\frac{x-u}{v}\right) \\ &\quad -\frac{1}{\mu v^2}(x-u)f\left(\frac{x-u}{v}\right) - \frac{u}{v^2\mu}f(\frac{x-u}{v}) \end{aligned}$$

Bounds on $E[X]$ when $r \neq 1$

- $E[X^2] = r\,E[(\mu - X)^2]$

- **If** $r \leq 1$ **(**$RTT_1 \geq RTT_2$**)**

$$\begin{aligned} E[X]^2 &= (1-r)E[X]^2 + rE[X]^2 \\ &\leq (1-r)E[X^2] + rE[X]^2 \\ &= rE[(\mu - X)^2 - X^2] + rE[X]^2 \\ &= r\,(\mu - E[X])^2 \end{aligned}$$

$$\Rightarrow \frac{E[X]}{\mu - E[X]} \leq \sqrt{r} = \frac{RTT_2}{RTT_1}$$

- $If r \geq 1$

$$\frac{\mu - E[X]}{E[X]} \leq \frac{1}{\sqrt{r}}$$

- **First bounds**

$$E[X] \begin{cases} \leq & \frac{\mu\sqrt{r}}{1+\sqrt{r}} \quad \textbf{if } r \leq 1 \\ \geq & \frac{\mu\sqrt{r}}{1+\sqrt{r}} \quad \textbf{if } r \geq 1 \end{cases}$$

- **Coincide when** $r = 1$

- **Complementary bounds for** $E[X]$ **using** $E[X^2] < \mu^2$**:**

$$\frac{\mu/2 - E[X]}{1-r} = \frac{E[X^2]}{2\mu r} \leq \frac{\mu}{2r}$$

$$\Rightarrow \left| \frac{\mu}{2} - E[X] \right| \leq |1-r| \frac{\mu}{2r}$$

$$\rightarrow \frac{\mu}{2(1+\sqrt{r})^2} < \frac{\mu/2 - E[X]}{1-r} \leq \frac{\mu}{2r}$$

- **Symmetric expression for** $r > 1$

Time average throughput

- $\overline{X}_k := \mathrm{C}\lim_{t\to\infty} X_k(t)$

- **Inversion formula** $\overline{X}_1 = E\left[\int_{t_{n-1}}^{t_n} X_1(t)dt\right] / E\left[S_n\right]$

$$\overline{X}_1 = \frac{E\left[\frac{cX^2}{2\mu}\left(\frac{X}{2}+\frac{cX}{4a}\right)+\frac{c(\mu-X)^2}{2\mu}\left(X+\frac{c(\mu-X)}{4a}\right)\right]}{E\left[\frac{cX^2}{2\mu}+\frac{c(\mu-X)^2}{2\mu}\right]}$$

$$= \frac{E\left[\frac{3}{2}X^3-\mu(2+\frac{3r}{4(1+r)})X^2+\mu^2(1+\frac{3r}{4(1+r)})X+\mu^3\frac{r}{4(1+r)}\right]}{E\left[2X^2-2\mu X+\mu^2\right]}$$

- $r=1$; $E[X]=\mu/2$, $E[X^2]=7\mu/26$ **and** $E[X^3]=2\mu/13$

$$\overline{X}_1 = \overline{X}_2 = \frac{3}{7}\mu$$

Asymmetric case

- $r \neq 1$: $\overline{X_1} = \mu\, h_1(E[X])$

  with

$$h_1(x) := \frac{(1+r)(4+9r)x - \mu r(7+6r)}{4(1-r)(1+r)(\mu - 2x)}$$

- $h_1(x)$ **increasing in** $x \neq \frac{1}{2}\mu$

$$\overline{X}_1 \begin{cases} \leq & \mu\sqrt{r}\frac{4-3\sqrt{r}+3r-3r\sqrt{r}}{4(1-r)(1+r)}, & \textbf{if } r < 1 \\ \geq & \mu\sqrt{r}\frac{4-3\sqrt{r}+3r-3r\sqrt{r}}{4(1-r)(1+r)}, & \textbf{if } r > 1 \end{cases}$$

- **Bounds not useful for** $r \approx 1$

- $\overline{X}_1/(\mu\sqrt{r}) \leq 1$ **when** $r \to 0$

- $\overline{X}_2 \to \frac{3}{4}\mu$ **when** $r \to 0$

Fairness

- $r \neq 1$: $\frac{\overline{X}_1}{\overline{X}_2} = h(E[X])$

  where

$$h(x) := \frac{(1+r)(4+9r)x - \mu r(7+6r)}{(1+r)(4r+9)(\mu - x) - \mu(7r+6)}.$$

- h **increasing in** x **for all values of** $x \neq \mu\frac{3+6r+4r^2}{(1+r)(4r+9)}$

$$\frac{\overline{X}_1}{\overline{X}_2} \begin{cases} \leq \sqrt{r}\left(\frac{4-3\sqrt{r}+3r-3r\sqrt{r}}{3-3\sqrt{r}+3r-4r\sqrt{r}}\right), & \textbf{if } r < r_0 \\ \geq \sqrt{r}\left(\frac{4-3\sqrt{r}+3r-3r\sqrt{r}}{3-3\sqrt{r}+3r-4r\sqrt{r}}\right), & \textbf{if } r > 1/r_0 \end{cases}$$

- $r_0 \approx 0.32$ **unique root in** $(0,1)$ **of** $-3+7\sqrt{x}-6x+7x\sqrt{x}-3x^2$

- **Right order of magnitude when** $r \to 0$ **or** $r \to \infty$

$$\liminf_{r \to 0} \frac{1}{\sqrt{r}} \cdot \frac{\overline{X}_1}{\overline{X}_2} \geq \frac{2}{3}$$

- **From bound on** $\overline{X}_1 / \overline{X}_2$

$$\limsup_{r \to 0} \frac{1}{\sqrt{r}} \cdot \frac{\overline{X}_1}{\overline{X}_2} \leq \frac{4}{3}$$

- $r \to 0$**:** $\overline{X}_1 \sim \sqrt{r} = RTT_2 / RTT_1$

Asymptotic bound

$$X_{n+1} = \begin{cases} \frac{1+2r}{2(1+r)} X_n & \textbf{w.p. } \frac{X_n}{\mu} \\ \mu - \frac{2+r}{2(1+r)} (\mu - X_n) & \textbf{w.p. } 1 - \frac{X_n}{\mu} \end{cases}$$

- $x_0 = x_0(r) := \mu\sqrt{r} / (1 + \sqrt{r}) \sim \mu\sqrt{r}$, $r \to 0$
- **Drift is positive if $X_n < x_0$, negative if $X_n > x_0$**
- X_n **"tends to be in the neighborhood of x_0"**
- Y_n **mimics** X_n

$$Y_{n+1} = \begin{cases} \frac{(1+2r)Y_n}{2(1+r)}, & \textbf{w.p. } \begin{cases} t/\mu, & \textbf{if } Y_n \le t \\ 1, & \textbf{if } Y_n > t \end{cases} \\ \frac{(2+r)Y_n + r\mu}{2(1+r)}, & \textbf{w.p. } \begin{cases} \mu - t/\mu, & \textbf{if } Y_n \le t \\ 0, & \textbf{if } Y_n > t \end{cases} \end{cases}$$

- $t \in (0, \mu)$ **arbitrary threshold (later $t = 2\mu\sqrt{r}$)**

Process Y_n: lower bound

- **Coupling:** $Y_0 \le X_0 \Rightarrow Y_n \le X_n$

- $E[X] \ge E[Y] = \dfrac{(1 - t/\mu)\,(r\mu - (1-r)\,E[Y\,\mathbf{1}(Y > t)])}{r + (1-r)t/\mu}$

- $E[Y\,\mathbf{1}(Y > t)] \le \left(\frac{(r+2)t}{2(r+1)} + \frac{r\mu}{2(r+1)}\right) P(Y > t)$

- **Bound $P(Y > t)$ from above**

- τ_t **is the return time to the set $\{Y > t\}$**

  (if $Y_n > t$ then $Y_{n+1} < t$)

- $P(Y > t) = 1/(1 + \tau_t)$

Bounding $P(Y > t) = 1/(1+\tau_t)$

$$
\begin{aligned}
\tau_t \;\geq\; \hat{\tau}_t \;&:=\; K + \frac{t}{\mu}\sum_{k=1}^{K} a_k, \\
K \;&:=\; \frac{t - \frac{2r+1}{2(r+1)}\left(t + \frac{r(\mu-t)}{2(r+1)}\right)}{r\mu/(2(r+1))}, \\
a_k \;&:=\; \left(\frac{2r+1}{2(r+1)}\left(t + \frac{r(\mu-t)}{2(r+1)}\right) + \frac{kr\mu}{2(r+1)}\right) \times \frac{(2r+1)/(2(r+1))}{r\mu/(2(r+1))}.
\end{aligned}
$$

- K **= minimum number of steps to get back above level** t **after it has dropped below**

  Proof

  ◇ **After dropping below** t **the process is surely below level** $\frac{2r+1}{2(r+1)}\left(t + \frac{r(\mu-t)}{2(r+1)}\right)$

  ◇ $E[Y_{n+1} - Y_n] \leq r\mu/(2(r+1)) \Rightarrow$ **at least** K **steps**

bounding $P(Y > t) = 1/(1 + \tau_t)$...

- **At each step new reduction with probability** t/μ

- **Reduction at** k**-th step: at least** a_k **additional steps to "recover"**

- $E[X] \geq \frac{r(\mu-t)}{r\mu+(1-r)t}\left(\mu - \frac{r\mu+(1-r)\left(t+\frac{r(\mu-t)}{2(r+1)}\right)}{r(1+\widehat{\tau}_t)}\right).$

- **Choose** $t = t(r) = c\sqrt{r}$

$$\lim_{r\to 0} \frac{r\widehat{\tau}_{t(r)}}{t(r)} = \frac{1}{\mu}\left(1 + \frac{c^2}{4\mu^2}\right),$$

$$\Rightarrow \liminf_{r\to 0} \frac{1}{\sqrt{r}} E[X] \geq \frac{c\mu^2}{4\mu^2 + c^2} = \frac{1}{2}\mu,$$

Set $c = 2\mu$ **for sharpest bound**

Approximation for $\overline{X}_1/\overline{X}_2$

- **Approximate $\overline{X}_1$ and $\overline{X}_2$ by average throughput in between two consecutive losses:**

$$\begin{aligned}\overline{X}_1 &\approx \frac{1}{2}E[X] + \frac{1}{2}\left(E[X] - \frac{1}{2\mu}E[X^2]\right) \\ &= \frac{2-r}{2(1-r)}E[X] - \frac{r\mu}{4(1-r)}\end{aligned}$$

- **Use bound for $E[X]$ as approximation**

$$\frac{\overline{X}_1}{\overline{X}_2} \approx \frac{\sqrt{r}\,(4+3\sqrt{r})}{3+4\sqrt{r}}$$

- **"Explains" fairness ratio of $(RTT_2/RTT_1)^{0.85} = (\sqrt{r})^{0.85}$ for moderate values of r**

- **Approximation matches correct order of magnitude when $r \to 0$ or $r \to \infty$**

Summary

- Throughput of two TCP connections
- Dynamic loss probability proportional to bandwidth share
- Bounds and approximations for mean transmission rates and fairness ratio
- $\overline{X}_1/\overline{X}_2$ is of the order RTT_2/RTT_1
- TCP more fair than with synchronization
- Same order predicted by models for many competing TCP with constant packet loss probability
- Suggests that the order of magnitude R is valid throughout the whole spectrum: for many and for few competing connections

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