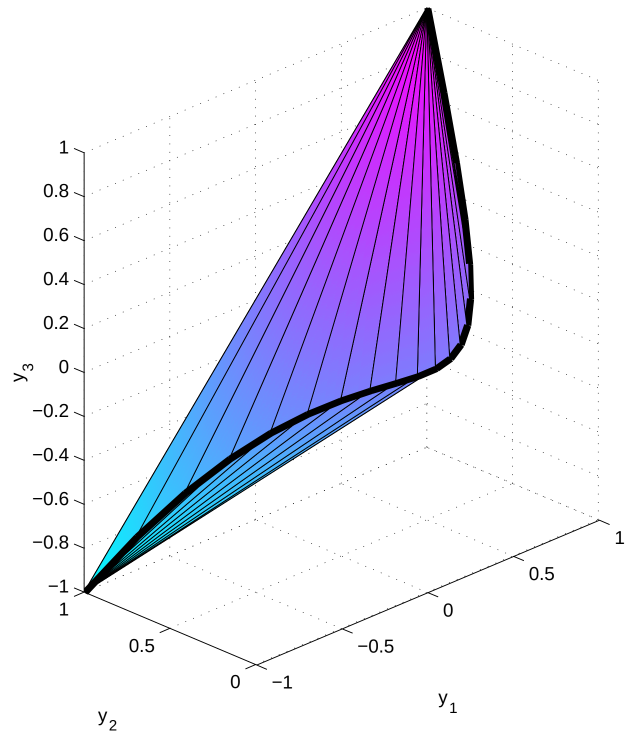
Theta Bodies for Polynomial Ideals



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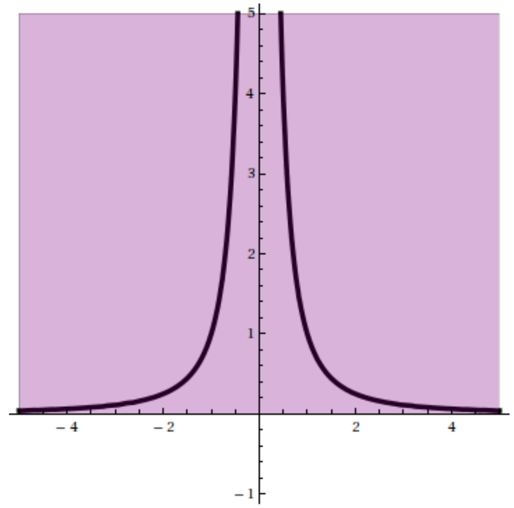
Hierarchy of relaxations for convex hulls of real algebraic varieties

- $I \subseteq \mathbb{R}[x_1, \ldots, x_n] =: \mathbb{R}[\mathbf{x}]$ ideal
- $\mathcal{V}_{\mathbb{R}}(I) := \{\mathbf{p} \in \mathbb{R}^n : f(\mathbf{p}) = 0 \; \forall f \in I\}$

  real variety of I, closed, semi-algebraic
- conv$(\mathcal{V}_{\mathbb{R}}(I))$ convex hull of $\mathcal{V}_{\mathbb{R}}(I)$

  convex, semi-algebraic

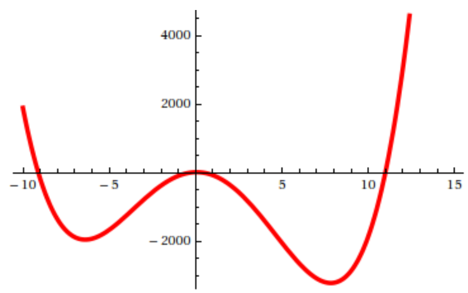


$I = \langle x^2 y - 1 \rangle$

Examples:

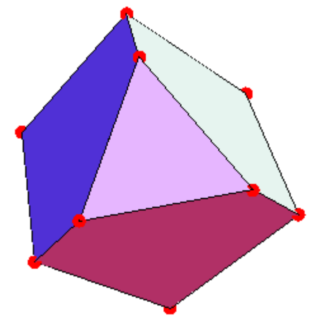
(1) univariate ideals: $I = \langle f \rangle$

conv$(\mathcal{V}_{\mathbb{R}}(f))$ – min and max real roots of f



(2) $\mathcal{V}_{\mathbb{R}}(I)$ finite:

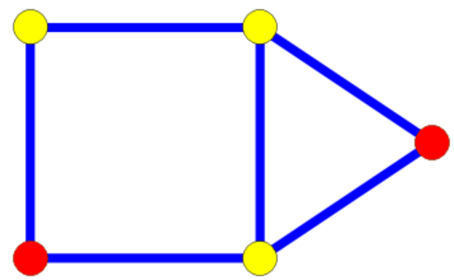
- conv$(\mathcal{V}_{\mathbb{R}}(I))$ is a polytope
- allows linear optimization over $\mathcal{V}_{\mathbb{R}}(I)$



contains $0/1$ integer programming: $\max\{\mathbf{c}\cdot\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \{0,1\}^n\}$

Ex: $G = ([n], E)$ graph

- $S_G := \{\chi^S : S$ stable set in $G\}$
- STAB(G) := conv(S_G)



max stable set problem: max$\{\sum x_i : \mathbf{x} \in \text{STAB}(G)\}$

Vanishing ideal of S_G: $I_G := \langle x_i - x_i^2 : i \in [n],\ x_i x_j : ij \in E\rangle$

(3) Polynomial Optimization: $p^* := \inf\{p(\mathbf{x}) : \mathbf{x} \in K\}$, K semi-algebraic

$$p(\mathbf{x}) = \sum_{\alpha \in S} p_\alpha \mathbf{x}^\alpha, \ |S| = s, \ p_\alpha \in \mathbb{R}$$

- $K = \mathbb{R}^n$ (unconstrained poly optimization):

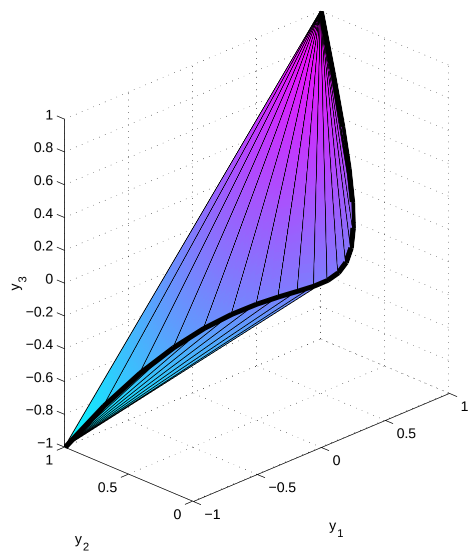
  $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^s, \ \mathbf{t} \mapsto (\mathbf{t}^{\alpha_1}, \dots, \mathbf{t}^{\alpha_s})$ toric variety

  $p^* = \min\{\sum p_\alpha y_\alpha : y \in \mathrm{conv}(\phi(\mathbb{R}^n))\}$ linear objective!

  $n = 1$ & $\deg(p) = d$: $\Rightarrow$

  $\phi(t) = (1, t, t^2, \dots, t^d)$

  rational normal curve



- Constrained Optimization:

similar, extensive applications

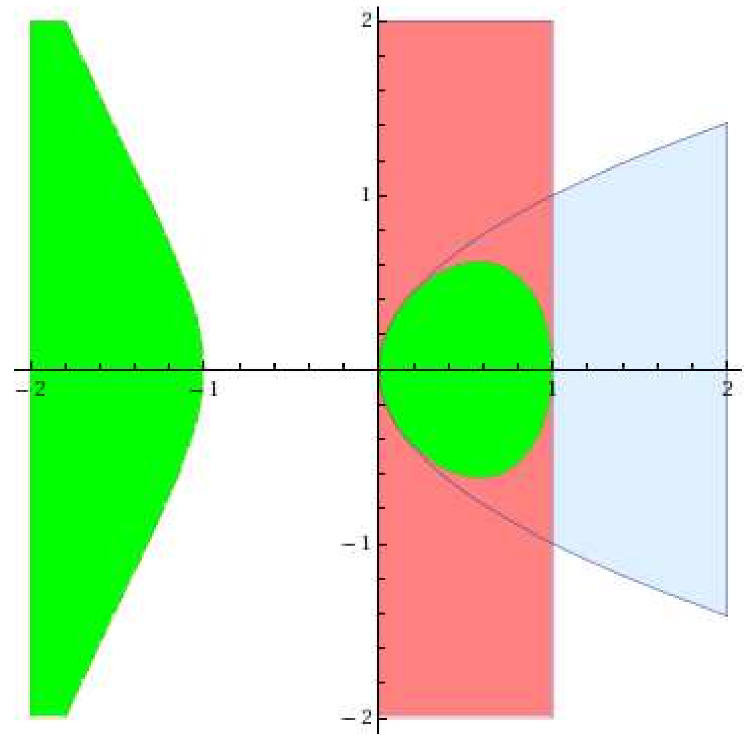
Spectrahedra

(feasible regions of semidefinite programs (sdp))

Example:

$$\left\{ (x,y) \in \mathbb{R}^2 : \begin{bmatrix} x & 0 & y \\ 0 & 1 & -x \\ y & -x & 1 \end{bmatrix} \succeq 0 \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{c} 0 \le x \le 1, \\ x \ge y^2, \\ x - x^3 - y^2 \ge 0 \end{array} \right\}$$



Open Problem: Can every convex semi-algebraic set (special case: $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$) be written as a spectrahedron or a projection of one?

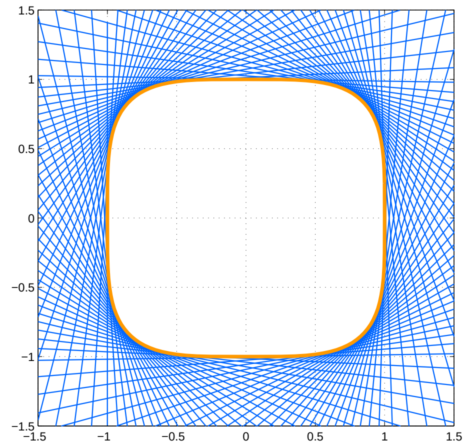
Approximating $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ via SDP

⋆ $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ cut out by all linear $f \in \mathbb{R}[\mathbf{x}]$ s.t.

$f(\mathbf{p}) \geq 0 \; \forall \, \mathbf{p} \in \mathcal{V}_{\mathbb{R}}(I)$

⋆ A Certificate of $f \geq 0$ on $\mathcal{V}_{\mathbb{R}}(I)$:

$f \equiv \sum h_j^2 \bmod I \Leftrightarrow f - \sum h_j^2 \in I$



Definitions: $f \in \mathbb{R}[\mathbf{x}]$, $I \subseteq \mathbb{R}[\mathbf{x}]$:

- If $f \equiv \sum h_j^2 \bmod I$ say f is sum of squares (sos) mod I &
  k-sos mod I if $\deg(h_j) \leq k$
- I is k-sos if $f \geq 0$ on $\mathcal{V}_{\mathbb{R}}(I) \Rightarrow f$ k-sos mod I
- I is $(1,k)$-sos if $f \geq 0$ on $\mathcal{V}_{\mathbb{R}}(I)$ & f linear $\Rightarrow f$ k-sos mod I

Lovász: Which ideals are $(1,1)$-sos, $(1,k)$-sos?

Lovász's motivation: stable set problem on $G = ([n], E)$

$I_G := \langle x_i - x_i^2 \,:\, i \in [n],\ x_i x_j \,:\, ij \in E \rangle$ is $(1,1)$-sos $\Leftrightarrow$ G is perfect

Examples:

- (Parrilo) I zero-dim & radical $\Rightarrow I$ is $|\mathcal{V}_{\mathbb{C}}(I)|$-sos $\Rightarrow (1, |\mathcal{V}_{\mathbb{C}}(I)|)$-sos.
- (radical is important:) $\langle x^2 \rangle$ not $(1,k)$-sos for any k — all $f \geq 0$ on $\{0\}$ are 1-sos mod I except $\pm x \geq 0$.
- $\langle x^2 y - 1 \rangle$ is $(1,2)$-sos.
- $\langle x^3 - y^2 \rangle$ has no linear polynomials that are sos mod I.
- $J_n := \langle \sum_{i=1}^{n} x_i^2 - 1 \rangle$ is $(1,1)$-sos for all n.

Theta bodies of polynomial ideals (Gouveia-Parrilo-T)

$$\mathrm{TH}_k(I) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq 0 \ \forall f \text{ linear \& k-sos mod } I\}$$

$$\mathrm{TH}_1(I) \supseteq \mathrm{TH}_2(I) \supseteq \cdots \supseteq \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))} \text{ —}(*)$$

Theorem (GPT):

(i) $\mathrm{TH}_k(I)$ is the projection of a spectrahedron (is a Lasserre relaxation)

(ii) Compute using combinatorial moment matrices (Laurent)

Definition: I is TH_k-exact if $\mathrm{TH}_k(I) = \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$

I is $(1,k)$-sos $\Rightarrow$ I is TH_k-exact

Examples:

- 0-dim radical ideals are TH_*-exact $(1,*)$-sos
- $\langle x^2 \rangle$ is TH_1-exact not $(1,k)$-sos for any k
- $\langle x^2 y - 1 \rangle$ is $(1,2)$-sos and TH_2-exact $(1,2)$-sos
- All theta bodies of $\langle x^3 - y^2 \rangle$ are just $\mathbb{R}^2$
- stable set problem in $G = ([n], E)$:

  theta body of G (Lovász):

  $$\mathrm{TH}(G) := \{\mathbf{x} \in \mathbb{R}^n : \begin{bmatrix} 1 & \mathbf{x}^t \\ \mathbf{x} & U \end{bmatrix} \succeq 0, \mathrm{diag}(U) = \mathbf{x}, U_{ij} = 0 \, \forall \{i,j\} \in E\}$$

  $$\mathrm{TH}_1(I_G) = \mathrm{TH}(G)$$

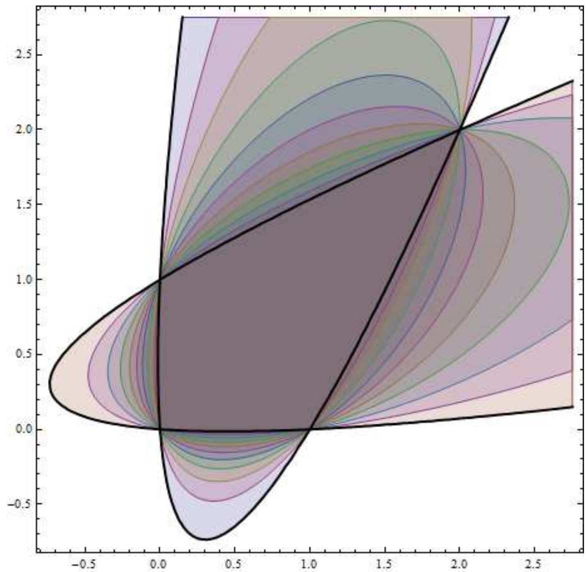
  $$I(G) \text{ is } (1,1)\text{-sos} \Leftrightarrow \mathrm{TH}_1\text{-exact} \Leftrightarrow G \text{ is perfect.}$$

  There is a hierarchy of theta bodies for G

Geometry of theta bodies

Theorem (GPT): $\text{TH}_1(I) = \bigcap \{\text{conv}(\mathcal{V}_{\mathbb{R}}(F)) : F \text{ convex quadric in } I\}$

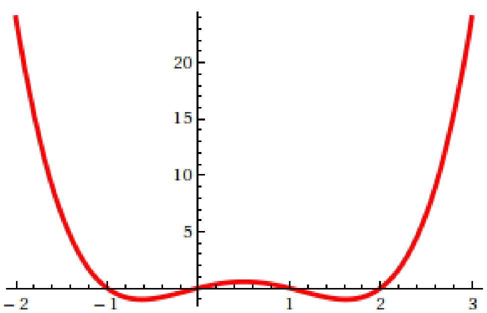
Ex. $I =$ Vanishing ideal of $\{(0,0),(1,0),(0,1),(2,2)\}$



Question: What is the geometry of higher theta bodies?

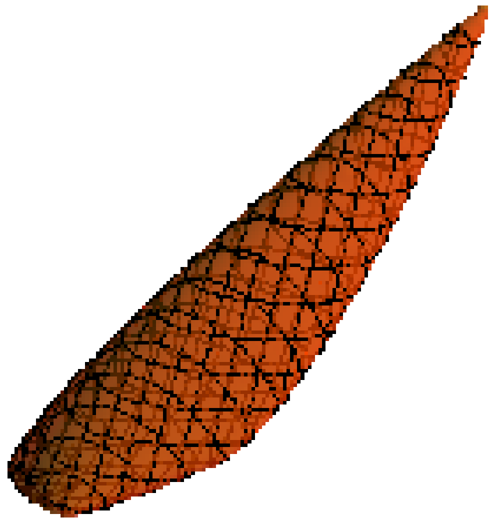
The spectrahedra upstairs

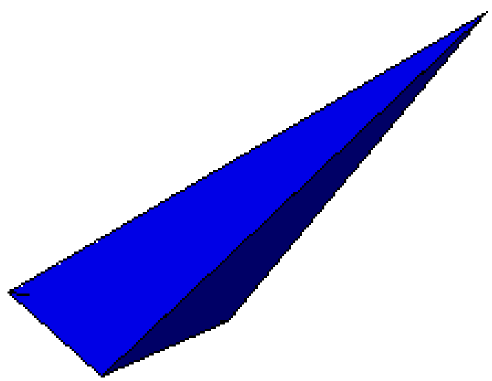
$y_2 \geq y_1^2$



$$x^4 - 2x^3 - x^2 + 2x = 0$$

roots: $-1, 0, 1, 2$





- $y_2 \geq y_1^2$
- $y_2 y_4 \geq y_3^2$
- $y_4 \geq y_2^2$,
- $2y_3y_2 + y_2^2$
  $-2y_2y_1 - y_3^2$
  $-2y_3y_1^2 - y_2y_1^2$
  $+2y_1^3 + 2y_3y_2y_1$
  $-y_2^3 \geq 0$

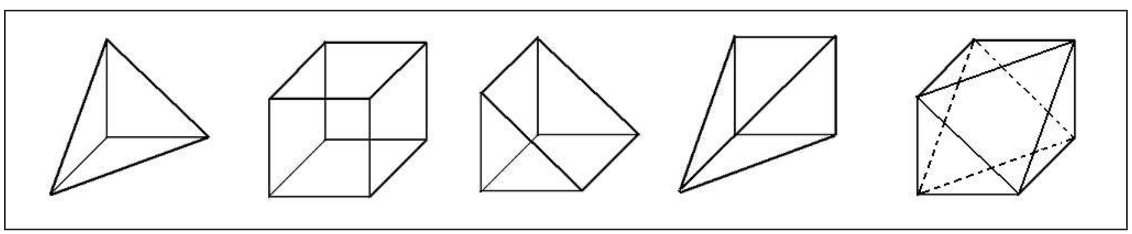
tetrahedron

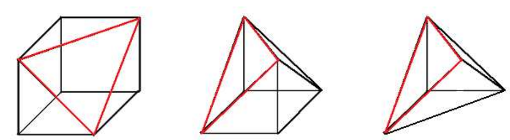
conv(roots lifted to $(1, x, x^2, x^3)$)

Real radical ideals

Definition: I real radical if it is the vanishing ideal of $\mathcal{V}_{\mathbb{R}}(I)$ (eg. I_G)

- Theorem (GPT): I real radical $\Rightarrow$ I is $(1,k)$-sos $\Leftrightarrow$ I is TH_k-exact
- Theorem (GPT): I real radical & 0-dimensional. Then TFAE:
  - I is TH_1-exact
  - conv$(\mathcal{V}_{\mathbb{R}}(I))$ has a (finite) linear inequality description in which $\forall$ $f(\mathbf{x}) \geq 0$, $\mathcal{V}_{\mathbb{R}}(I) \subseteq \{f(\mathbf{x}) = 0\} \cup \{f(\mathbf{x}) = 1\}$ ($\mathcal{V}_{\mathbb{R}}(I)$ is 2-level)





Corollary: G is perfect $\Leftrightarrow$ STAB(G) is 2-level

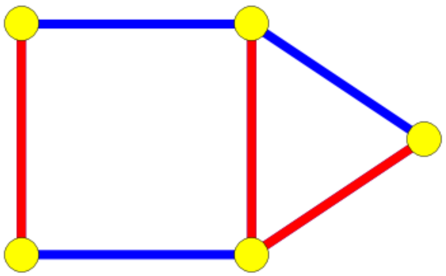
More corollaries:

If I is 0-dimensional, real radical and TH_1-exact then:

- conv$(\mathcal{V}_{\mathbb{R}}(I))$ affinely equivalent to a $0/1$-polytope.
- conv$(\mathcal{V}_{\mathbb{R}}(I))$ has at most 2^n vertices and facets. Both bounds are sharp — cube and cross-polytope.
- A down-closed $0/1$-polytope is 2-level if and only it is STAB(G) for a perfect graph.

Max Cut in $G=([n],E)$

$C\subseteq E$ cut in G, $\quad \chi^C : (\chi^C)_e = \begin{cases} -1 & \text{if } e\in C \\ 1 & \text{if } e\notin C \end{cases}$



$\text{CUT}(G) := \text{conv}\{\chi^C : C \text{ cut in } G\}$

max cut problem: $\max\{\sum \frac{1}{2}(1-x_{ij}) : \mathbf{x}\in \text{CUT}(G)\}$

$I_{\text{cut}} := I(\{\chi^C\}) = \langle \{x_{ij}^2-1\} \cup \{\mathbf{x}^A - \mathbf{x}^B : A\cup B \text{ circuit in } G\}\rangle$

Theorem (GPT+Laurent): G is cut-perfect (i.e., $\text{CUT}(G) = \text{TH}_1(IG)$) $\Leftrightarrow$ G has no K_5-minor and chordless circuits of length ≥ 5. (a Lovász qn)

generalization possible to certain binary matroids (GLPT)