A separated splitting technique for disconnected rare event sets

Wander Wadman
Daan Crommelin & Jason Frank
CWI Amsterdam

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Find $\gamma := \mathbb{P}(\tau_B < \zeta \mid X_0 = x_0)$.

$\tau_B := \inf\{t > 0 : X_t \in B\}$

using splitting

Model too complex for large deviations results

$\{X_t, t \geq 0\}$ discretization of a continuous stochastic process

B (possibly) disconnected
Example: power grid reliability

[Wadman, Crommelin and Frank, 2013]
Discrete-time Markov Chain \( \{X_t, t \geq 0\} \).

\[ \gamma := \mathbb{P}(\tau_B < \zeta | X_0 = x_0) \]

\[ \tau_B := \inf\{t > 0 : X_t \in B\} \]

Crude Monte Carlo estimator

\[ \hat{\gamma}_n := \frac{1}{n} \sum_{i=1}^{n} 1_{\{\tau_B < \zeta \text{ in sample } i\}} \]

\[ \frac{\text{Var} \hat{\gamma}_n}{\gamma^2} = \frac{1 - \gamma}{n\gamma} \to \infty \quad \text{as } \gamma \to 0 \]
Decompose
\[ \gamma = \prod_{k}^{m} \mathbb{P}(T_{k} < \zeta | T_{k-1} < \zeta), \]
with
\[ T_{k} := \min\{t > 0: h(X_{t}) \geq l_{k}\}, \]
importance function
\[ h: \mathbb{R}^{d} \mapsto \mathbb{R}, \]
and levels
\[ l_{0} < l_{1} < \ldots < l_{m}. \]

Splitting estimator
\[ \hat{\gamma} = \prod_{k} \frac{[\text{hits}]_{k}}{[\text{trials}]_{k}} \]
Optimal importance function and levels

\[
\frac{\text{Var}(\hat{y})}{\gamma^2} \sim \frac{(\log \gamma)^2}{n}
\]

What is optimal?

\[
h^*(x) := g(\mathbb{P}(\tau_B < \zeta | X_0 = x)),
\]

Ideally, the most likely path to the rare event set should coincide with the most likely path to any intermediate level.

[Amrein and Künsch, 2011][Glasserman et al., 1998][Garvels et al., 2000]
Barrier crossing probability

\[ B = [b, \infty) \]

Then

\[ h_p(x) := 1 - \frac{b - x}{b - x_0} \]

is increasing in

\[ \mathbb{P}(\tau_B < \zeta | X_0 = x) \]
Choosing the levels

[Amrein and Künsch, 2011]
Choosing the levels

[Amrein and Künsch, 2011]

If

\[ \mathbb{P}(T_k < \zeta | T_{k-1} < \zeta, (T_{k-1}, X_{T_{k-1}})) = p_k \quad \forall (T_{k-1}, X_{T_{k-1}}), \]

then

\[ \frac{\text{Var}(\hat{y})}{\gamma^2} \leq -1 + \prod_{k} \frac{r_k - 1}{r_k - 2}. \]
Disconnected rare event set

\[ B = (\infty, b_2] \cup [b_1, \infty) \]

\[ h_p(x) := \max_{i=1,2} h_i(x), \]

\[ h_i(x) = 1 - \frac{b_i - x}{b_i - x_0} \]
Disconnected rare event set

\[ dX(t) = \theta (\mu \text{ sgn } X(t) - X(t))dt + \sigma dW(t) \]
Separated splitting

\[ \gamma = \gamma_1 + \gamma_2, \]

\[ \gamma_1 := \mathbb{P}(\tau_{B_1} < \zeta \mid X_0 = x_0), \]

\[ \gamma_2 := \mathbb{P}(\tau_{B_2} < \zeta \cup \tau_{B_1} \geq \zeta \mid X_0 = x_0) \]
## Results

<table>
<thead>
<tr>
<th>Value</th>
<th>Estimate for SRE(·)</th>
<th>Bound for SRE(·)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\gamma}_{12} )</td>
<td>5.28 \times 10^{-4}</td>
<td>0.120</td>
</tr>
<tr>
<td>( \hat{\gamma}_1 )</td>
<td>8.75 \times 10^{-4}</td>
<td>0.043</td>
</tr>
<tr>
<td>( \hat{\gamma}_2 )</td>
<td>6.18 \times 10^{-8}</td>
<td>0.095</td>
</tr>
</tbody>
</table>

\[
\text{SRE}(\gamma) := \frac{\text{Var}(\hat{\gamma})}{\gamma^2} \leq -1 + \prod_k \frac{r_k - 1}{r_k - 2}
\]
Conclusion

Further research
Order of runs

Rare event (possibly) disconnected => use separated splitting runs!