# Optimal local interventions in the two-dimensional Abelian sandpile model

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# Motivating example

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# Wildfire prevention

The dynamics of wildfires can be modelled as follows:

- as a precaution, periodically one may remove all flammable material from a selected site;
- in the next time interval, new flammable material arrives at a random site;
- if the total amount of flammable material exceeds a threshold, a fire starts and spreads to neighbouring sites with enough flammable material.

Goal: which site should be selected for removal to minimise the impact of a fire?

# Outline

- Abelian sandpile models for wildfires
- Algorithmic wave representation of topplings
- Expected avalanche size
- Optimal local intervention
- Conclusions

# Abelian sandpile

Define graph G by vertex set $V = \{1, \dots, L\}^2$, horizontal and vertical neighbours and sink connected to boundary vertices.

A sandpile is a map

$$\eta : V \to \mathbb{Z}$$

and can be interpreted as a configuration of sand grains or units of flammable material.

A vertex $v$ is stable if $0 \leq \eta(v) < 4$, critical if $\eta(v) = 3$. The sandpile is stable if all its vertices are.

# Dynamics

Each time step,

- a new sand grain is dropped on a vertex $v$ uniformly;
- if $v$ is now unstable, it topples: each of the four neigbours receives a sand grain;
- if received by the sink, the sand grain disappears;
- unstable vertices in turn topple until a stable sandpile is reached.

Notes: Toppling commutes, the number of topplings is finite and the resulting stable sandpile is unique.

# Avalanches

The size of an avalanche is the number of topplings,

$$x(\eta, v_i) = \sum_{j=1}^{L^2} n(v_i, v_j; \eta),$$

where $n(v_i, v_j; \eta)$ is the number of times $v_j$ topples before the stable sandpile is reached when dropping a sand grain on $v_i$ in sandpile $\eta$.

If $X(\eta)$ is the size of the next avalanche,

$$\mathbb{P}(X(\eta) = k) = \frac{1}{L^2} \sum_{i=1}^{L^2} 1\{x(\eta, v_i) = k\}, \quad k \in \mathbb{N}_0.$$

Writing $Y$ for the vertex hit by the next sand grain, we focus on

$$\mathbb{P}(X(\eta) = k \mid Y \in A).$$

# Action: Removal of pile of sand grains

To reduce the impact of wild fires, a forester removes all flammable material from some appropriate vertex.

Set

$$(\gamma_i \eta)(v_j) = \begin{cases} 0 & \text{if } i = j, \\ \eta(v_j) & \text{otherwise.} \end{cases}$$

Goal: find the set of vertices $v_i$ that attain the minimum in

$$\lambda(\eta, A) = \min_{v_i \in A} \frac{\mathbb{E}[X(\gamma_i \eta) | Y \in A]}{\mathbb{E}[X(\eta) | Y \in A]}$$

for tractable $A \subset V$.

# Waves of topplings (Ivashkevich et al. (1994))

Suppose that for sandpile $\eta$, $\eta(v_i) = 3$ for some $v_i$ and $\eta(v_j) \leq 3$ for $j \neq i$. Drop a sand grain on $v_i$. Then

- topple $v_i$;
- topple unstable vertices in $V \neq \{v_i\}$ until all vertices are stable except possibly $v_i$.

These form the first wave.

Theorem: (Ivashkevich et al.) The set of these topplings is well-defined and unique, so no vertex topples twice.

If $v_i$ is unstable after the first wave, repeat to obtain the second wave and so on until a stable sandpile is reached.

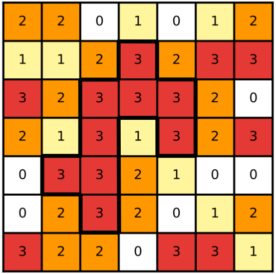
# Dorso and Dadamia algorithm (2002)

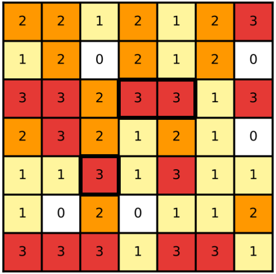
Let $A$ be a generator, i.e., a connected component of critical vertices in $\eta$ that contains $v_i$, and drop a sand grain on $v_i$.

Theorem: (Dorso and Dadamia) Given a stable sandpile configuration $\eta$ and a generator $A$, the first wave $W_A(\eta)$ does not depend on $v_i$, nor does its size $w_A(\eta)$.

The authors assume that the sizes of second and higher order waves are equal for all $v_i$ to derive an expression of avalanche size as the sum of all wave sizes.

# Counterexample




The generator (left, outlined in bold) splits into two components after the first wave (right), producing second waves of different size.

# Recursive expression for expected avalanche size

Let $\eta$ denote a stable sandpile configuration and let $A \subseteq V$ denote a generator in $\eta$.

Write $\eta^A$ for the configuration after the first wave and let $A_1, \ldots, A_l$ be the connected components of $\{v \in A : \eta^A(v) = 3\}$. Then

$$E[X(\eta)|Y \in A] = w_A(\eta) + \sum_{j=1}^{l} \frac{|A_j|}{|A|} E[X(\eta^A)|Y \in A_j]$$

if $\{v \in A|\eta^A(v) = 3\} \neq \emptyset$ and $w_A(\eta)$ otherwise.

# Algorithmic representation

Let $A$ be a generator, i.e., a connected component of critical vertices, that contains $v_i$ and drop a sand grain on $v_i$.

- Initialise a directed graph $(V, E)$ with $E = \emptyset$.
- For each $v \in A$, add an edge to $E$ from $v$ to all its neighbours and set $w = |A|$.
- While there is a $v \notin A$: $\eta(v) + \mathrm{indeg(v)} - \mathrm{outdeg(v)} \geq 4$,
  - for each $v \notin A$ for which $\eta(v) + \mathrm{indeg(v)} - \mathrm{outdeg(v)} \geq 4$, add an edge to $E$ from $v$ to all its neighbours and set $w := w + 1$.
- Set $\eta' = \eta(v) + \mathrm{indeg(v)} - \mathrm{outdeg(v)}$.
- If $\{v \in A : \eta'(v) = 3\} \neq \emptyset$,
  - let $A_1, \ldots, A_l$ denote its connected components and apply the algorithm to each $A_i$ in sandpile $\eta'$ to obtain $w_i$;
  - output $w + \sum_{i=1}^{l} \frac{|A_i|}{|A|} w_i$.
- If $\{v \in A : \eta'(v) = 3\} = \emptyset$,
  - output $w$.

# Expected avalanche size for square generator

Theorem: Let $A^{(N)}$ be a generator for a stable $\eta$ that is an $N \times N$ square. Then

$$\mathbb{E}[X(\eta)|Y \in A^{(N)}] = \frac{3N^4 + 15N^3 + 20N^2 - 8}{30N}.$$

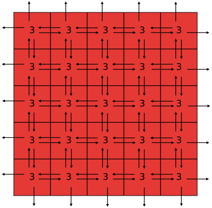
Proof: By induction.

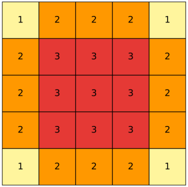
Basis: Since for $N = 1, 2$, the algorithm terminates in one step, $\mathbb{E}[X(\eta)|Y \in A^{(1)}] = 1$, $\mathbb{E}[X(\eta)|Y \in A^{(2)}] = 4$ and the assumption holds.

Induction hypothesis: The claim holds for $N = k - 2$ for some $k = 3, 4, \ldots$.

Induction step: Consider $A^{(k)}$.

Then the first wave has size $k^2$. The set of vertices with $\eta' = 3$ is a $(k-2)\times(k-2)$ square.




Hence,

$$\begin{aligned}\mathbb{E}[X(\eta)|Y \in A^{(k)}] &= k^2 + \frac{(k-2)^2}{k^2}\frac{3(k-2)^4+15(k-2)^3+20(k-2)^2-8}{30(k-2)} \\ &= \frac{3k^4+15k^3+20k^2-8}{30k}. \qquad \square\end{aligned}$$

# Optimal action

The impact of removing a pile of sand grains is quantified by the stability level

$$\lambda(\eta, A) = \min_{v_i \in A} \frac{\mathbb{E}[X(\gamma_i \eta) | Y \in A]}{\mathbb{E}[X(\eta) | Y \in A]}.$$

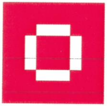
The cornerstone vertices are those $v \in A$ for which the minimum is attained. Denote their ensemble by $B_A^*(\eta)$.

Theorem: Let $A^{(N)}$ be a generator for a stable $\eta$ that is an $N \times N$ square. Then

$$B_{A^{(N)}}^*(\eta) = \begin{cases} R_1(A^{(N)}), & N = 1, 2 \\ R_2(A^{(N)}) \setminus C(A^{(N)}), & N = 3, 4 \\ R_3(A^{(N)}) \setminus C(\cup_{i=1}^{3} R_i(A^{(N)})), & N \geq 5 \end{cases}$$

where $R_i$ is the $i$-th ring from the center, and $C(\cdot)$ denotes corners.

# Expected size of avalanche after intervention



Claim: for $N \geq 3$ odd, stable sandpile $\eta$ and generator $A^{(N)}$, an $N \times N$ square, let $v_i$ be in the $k$-th ring without the corners for $k \geq 2$ (that is, not the centre). Then

$$\mathbb{E}[X(\gamma_i \eta) | Y \in A^{(N)}] =$$

$$\frac{3N^5 + 15N^4 + 15N^3 - 15N^2 - 18N + 10(4k^3 - 24k^2 + 23k - 6)}{30N^2}.$$

# Sketch of proof (ctd)

Write $\tilde{A}_R^{N,k}$ for the remainder of $A^{(N)}$ in $\gamma_i\eta$. Condition on the vertex on which the next grain lands,

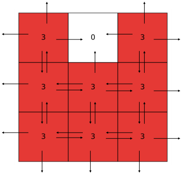
$$\mathbb{E}\left[X(\gamma_i(\eta))|Y \in A^{(N)}\right] = \frac{N^2-1}{N^2}\mathbb{E}\left[X(\gamma_i(\eta))|Y \in \tilde{A}_R^{N,k}\right]$$

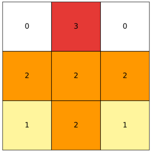
hence it suffices to prove that

$$\mathbb{E}[X(\gamma_i\eta)|Y \in \tilde{A}_R^{N,k}] = \frac{3N^5+15N^4+15N^3-15N^2-18N+10(4k^3-24k^2+23k-6)}{30(N^2-1)}.$$

## Sketch of proof (ctd)

$N = 3$ and $k = 2$.



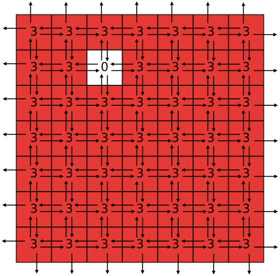

$$\mathbb{E}[X(\gamma_i \eta)|Y \in \tilde{A}_R^{3,2}] = 8$$

and the result is seen to hold.

Induction hypothesis: claim holds for $N = n - 2$ and all $2 \leq k \leq (n - 2 + 1)/2 = (n - 1)/2$.

Induction step: Consider $N = n$ and $k \geq 2$.

Case $k = l < (n+1)/2$ not the outer ring:




| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 | 1 |
| 2 | 3 | 0 | 3 | 3 | 3 | 2 |
| 2 | 3 | 3 | 3 | 3 | 3 | 2 |
| 2 | 3 | 3 | 3 | 3 | 3 | 2 |
| 2 | 3 | 3 | 3 | 3 | 3 | 2 |
| 2 | 3 | 3 | 3 | 3 | 3 | 2 |
| 1 | 2 | 2 | 2 | 2 | 2 | 1 |

After first step, for the generator obtained after deleting $v_i$, only $v_i$ generates edges and we arrive at situation for $N = n - 2$ and $k = l$.

# Sketch of proof (ctd)

By the induction hypothesis,

$$n^2 + \frac{(n-2)^2-1}{n^2-1} \times \{$$

$$\frac{3(n-2)^5 + 15(n-2)^4 + 15(n-2)^3 - 15(n-2)^2 - 18(n-2)}{30((n-2)^2-1)} +$$

$$\frac{10(4l^3 - 24l^2 + 23l - 6)}{30((n-2)^2-1)}\}$$

$$= \frac{3n^5 + 15n^4 + 15n^3 - 15n^2 - 18n + 10(4l^3 - 24l^2 + 23l - 6)}{30(n^2-1)}.$$

As the probability that the next sand grain does not hit $v_i$ is $(n^2-1)/n^2$, the claim holds for $N = n$ and all $2 \leq k \leq (n-1)/2$.

# Sketch of proof (ctd)

- Check the boundary case $k = (n+1)/2$ separately to finish the proof of the claim.
- Minimise $4k^3 - 24k^2 + 23k - 6$ over $k \geq 2$ to see that the optimal choice is $k = 3$.
- By similar arguments, calculate $\mathbb{E}[X(\gamma_i \eta) | Y \in A^{(N)}]$ for even $k$ to obtain

$$\frac{3N^5 + 15N^4 + 15N^3 - 15N^2 - 18N + 20k(k^2 - 9k + 1)}{30N^2}.$$

  Again $k = 3$ is optimal.
- Repeat the exercise for corner and for center vertices.
- Compare the results to complete the proof. $\square$

# Stability levels

Corollary: Let $A^{(N)}$ be a generator for a stable $\eta$ that is an $N \times N$ square. Then

$$\lambda(\eta, A^{(N)}) = \begin{cases} 9/16, & \text{if } N = 2, \\ 32/41, & \text{if } N = 3, \\ 15/17, & \text{if } N = 4, \\ \frac{3N^5+15N^4+15N^3-15N^2-18N-450}{N(3N^4+15N^3+20N^2-8)}, & \text{if } N \geq 5 \text{ odd}, \\ \frac{3N^5+15N^4+15N^3-15N^2-18N-480}{N(3N^4+15N^3+20N^2-8)}, & \text{if } N \geq 5 \text{ even}. \end{cases}$$

# Conclusions

We

- took a step towards understanding of interventions in the Abelian sandpile model,
- generalised the Dorso and Dadamia method for calculating expected avalanche size,
- investigated the effect of removing sand grains in square generators,
- and found optimal intervention locations.

Future work could focus on

- generators drawn from, e.g, the stationary distribution of the sandpile Markov chain,
- embedding of the approach within an MDP framework.

# Thank you for your attention!

Reference

M.C. de Jongh, R.J. Boucherie and M.N.M. van Lieshout (2025)
Preventing large-scale avalanches in the 2D Abelian sandpile model through strategic interventions.
Arxiv 2603.24459, March 2026.

## Markov decision process on the line

Let $V = \{1, \dots, L\}$ and $\eta$ a stable configuration.

- Before a new sand grain is dropped, a **decision maker** may remove a grain at some vertex $v \in V^+(\eta)$, the set of occupied vertices, or do nothing (0).
- Action space $A(\eta) = V^+(\eta) \cup \{0\}$.
- After action $a$, the new state is $\eta^a$, and the cost of this action is

$$c(\eta, a) = \mathbb{E}X(\eta^a) + \beta \mathbf{1}\{a \neq 0\}$$

for some $\beta \geq 0$.

# Decision rule

A decision rule is a mapping $d$ that assigns an action $d(\eta) \in A(\eta)$ to every stable $\eta$.

Aim: minimise the expected average cost

$$g^d(\eta) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[c(Y_t, d(Y_t) \mid Y_1 = \eta\right]$$

where $Y_t$ is the Markov chain induced by $d$.

# For large $\beta$, never intervening is best

Set

$$\Delta_{x,y} = \begin{cases} \mathrm{deg(x)} & \text{if } x = y \in V, \\ -\# \text{ edges between } x, y & \text{else.} \end{cases}$$

Write $G = \Delta^{-1}$ for the **Green's function** and set $\phi(x) = \sum_{y \in V} G(x, y)$.

**Theorem:** for stable $\eta$, the cost $g^0(\eta)$ under $d \equiv 0$ is

$$g^0(\eta) = \frac{1}{|V|} \sum_{x \in V} \phi(x).$$

The rule $d \equiv 0$ is optimal if and only if

$$\beta \geq \max\{\phi(v) : v \in V, \mathrm{deg(v)} \geq 2\}.$$

# The one-dimensional lattice

For $V = \{1, \ldots, L\}$,

$$G(x, y) = \begin{cases} \frac{x(L+1-y)}{L+1} & \text{if } x \leq y, \\ \frac{y(L+1-x)}{L+1} & \text{if } x > y, \end{cases}$$

so

$$\phi(x) = x(L + 1 - x)/2$$

and never intervening is optimal if

$$\beta \geq \begin{cases} \frac{(L+1)^2}{8} & \text{if } L \text{ is odd}, \\ \frac{L(L+2)}{8} & \text{if } L \text{ is even}, \end{cases}$$

with expected average cost $g^0(\eta) = (L + 1)(L + 2)/12$.

# Local intervention

**Definition:** A decision rule $d_W$, $W \subseteq V$, has only local interventions if, for stable $\eta$,

$$d_W(\eta) \in \begin{cases} W \cap V^+(\eta) & \text{if } W \cap V^+(\eta) \neq \emptyset, \\ \{0\} & \text{else.} \end{cases}$$

**Theorem:** Suppose that $L$ is odd and set, for $k = 0, \ldots, (L-1)/2$, $W_k = \{k+1, \ldots, L-k\}$. Then the Markov chain induced by $d_{W_k}$ has a unique recurrent class $R_k$ given by

$$R_0 = \{\eta \in \{0,1\}^L : \sum_{i=1}^{L} \eta(i) = 1\},$$

$$R_k = \left\{ \eta \in \{0,1\}^L : \sum_{i=1}^{k} \eta(i) \geq k-1, \sum_{i=L-k+1}^{L} \eta(i) \geq k-1, \sum_{i=k+1}^{L-k} \leq 1 \right\} \setminus \{\tilde{\eta}_{k-1}\},$$

where $\tilde{\eta}_{k-1}$ is the configuration with empty $W_{k-1}$ and ones elsewhere.

# Optimal policy

Suppose that $L$ is odd and set, for $k = 0, 1, \ldots, (L-1)/2$,

$$\beta_k = k(k+1)/2$$

and

$$\beta_{(L+1)/2} = (L+1)^2/8.$$

**Theorem:** For $\beta \in [\beta_k, \beta_{k+1}]$, any local intervention decision rule $d_{W_k}$ is optimal and has expected average cost

$$\frac{2k}{L} g_k^0 + \frac{L-k}{L}\beta = \frac{k(k+1)(k+2)}{6L} + \frac{L-k}{L}\beta.$$

# Example: L = 7

Any policy that removes a sand grain in the set

$$
\begin{aligned}
W_0 = V = \{1, \dots, 7\} &\quad \text{is optimal for } \beta \in [0, 1] \text{ with cost } \beta, \\
W_1 = \{2, \dots, 6\} &\quad \text{is optimal for } \beta \in [1, 3] \text{ with cost } (1 + 6\beta)/7, \\
W_2 = \{3, 4, 5\} &\quad \text{is optimal for } \beta \in [3, 6] \text{ with cost } (4 + 5\beta)/7, \\
W_3 = \{4\} &\quad \text{is optimal for } \beta \in [6, 8] \text{ with cost } (10 + 4\beta)/7,
\end{aligned}
$$

whereas for $\beta \geq 8$, doing nothing is best and yields expected average cost 6.