# Streamer Propagation as a Pattern Formation Problem: Planar Fronts

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Streamers often constitute the first stage of dielectric breakdown in strong electric fields: a nonlinear ionization wave transforms a nonionized medium into a weakly ionized nonequilibrium plasma. New understanding of this old phenomenon can be gained through modern concepts of (interfacial) pattern formation. As a first step towards an effective interface description, we determine the front width, solve the selection problem for planar fronts, and calculate their properties. Our results are in good agreement with many features of recent three-dimensional numerical simulations. [S0031-9007(96)01612-2]

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Transient discharges occur in various forms [1], e.g., as leaders in spark formation or as streamers in ac silent discharges [2]. A common feature is the creation of a nonequilibrium plasma through the propagation of a nonlinear ionization wave into a previously nonionized region. Although it is well known that, depending on the polarity of the field, discharge patterns on a larger scale may either be fractal [3] or form more regular nonfractal patterns [4], ionization fronts do not seem to have been analyzed before as a pattern forming system on scales resolving their internal structure. While the idea of a shock front or a thin ionization sheet has been formulated in the literature on streamers in the 1970s [5], the analytical treatment then frequently was based on *ad hoc* assumptions and on equilibrium concepts, e.g., on the assumption that the high electric field would raise the electron temperature and that subsequent ionization would be thermal. In the last 10 years, models incorporating nonequilibrium impact ionization of neutral molecules by free electrons have been investigated both numerically [6,7] and analytically [8]. Figure 1(a) shows a snapshot from a numerical study by Vitello *et al.* [7] of the streamer equations, Eqs. (1)–(4), below. Here, the evolution of the electron and ion densities between two planar electrodes with distance 0.5 cm and voltage difference 25 kV is integrated forward in time for parameter values describing $N_2$ under normal conditions. At time $t = 0$, the electron density was taken nonzero only in a small localized region near the upper negative electrode. The figure shows the electron density 5.5 ns later. Each contour line indicates the increase of the electron density by a decade. The lines enclose a fingerlike region (the body of the streamer), consisting of a nonequilibrium plasma; this region rapidly expands downwards towards the anode. In the region outside, the gas is essentially nonionized. The fact that the contour lines in the figure are very closely and about equidistantly spaced illustrates that the electron density within a zone of the order of a few $\mu$m grows about exponentially by a factor of about $10^{10}$. Since the total charge density is negligible before as well as behind the front, streamer dynamics can be viewed as the propagation of a thin charged ionization sheet separating a nonionized high field region from an ionized electrically screened region.

Obvious, and up to now unanswered, questions are: What determines the scale of the pattern (e.g., the lateral width of the fingerlike region), its velocity, the field-enhancement near the tip, and what effects do the boundary and initial conditions have? Triggered by the observation of interfacelike profiles in the simulations [6,7] and by the fact that these are precisely the questions that are

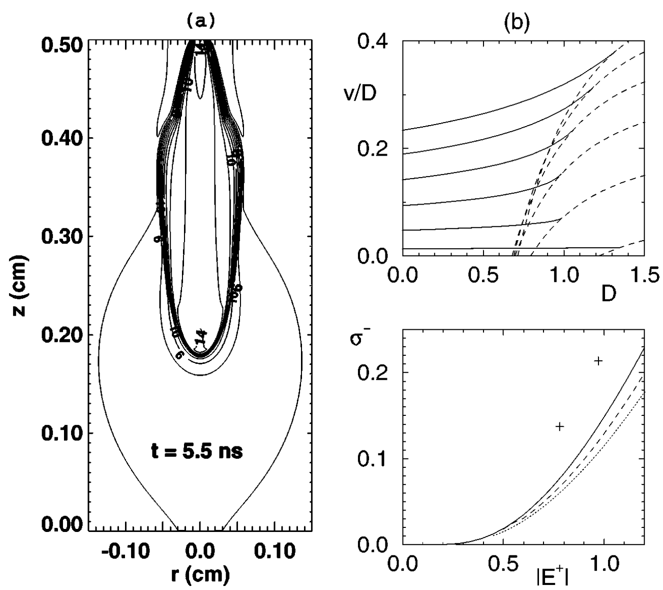


FIG. 1. (a) Electron density profile in a negative streamer from the 3D cylinder-symmetric numerical simulations [7] of Eqs. (1)–(4). Courtesy of P. A. Vitello. (b) Our predictions for planar fronts. Upper panel: $v^\dagger/D$ (solid) and $v^*/D$ (dashed lines) as a function of $D$ for positive fronts and for $E^+ = 0.4$, 0.6, 0.8, 1.0, 1.2, and 1.4, from bottom to top. Lower panel: Electron density $\sigma^- = n_e e/q_0$ behind a negative front as a function of the field $E^+$ before the front for $D = 0$ (solid line), 1 (dashes), 3 (dots). Crosses: values of $\sigma^-(E^+)$ on the symmetry axis in the 3D simulations [7] at times 4.75 ns and 5.5 ns, with $E^+$ the value of the outer field extrapolated towards the tip, in accord with the asymptotic matching prescription.

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studied in the field of interfacial pattern formation for, e.g., dendrites and viscous fingers [9], we show here that pattern formation concepts provide a systematic route to unravel precisely these aspects.

(i) For planar fronts, we trace the "great defect," "the inability of the theory to determine a value for the wave speed" [5(c)], to the fact that streamers are an example of *front propagation into unstable states* in virtually all models analyzed [5,8]. For such problems, it is well known that the velocity cannot be obtained just by analyzing uniformly translating fronts using standard methods. In the field of pattern formation, the mechanism of *dynamical front selection* has been understood in the last decade [10,11]. In this paper, we show that this allows us to derive all essential properties of planar fronts for the model of the recent simulations [6,7].

(ii) Clearly, an analysis of planar streamer fronts does not suffice to explain the *global* questions of pattern formation posed above, such as the field enhancement in front of the streamer head or the radius of curvature of the tip. However, both the simulations [6,7] and our analysis show that the propagating charge sheet is only a few $\mu$m thick, while the tip radius and the electrode spacing are of order mm or more. This separation of scales that makes simulations so demanding can be made into an analytical tool. Much of our present knowledge about similar problems like combustion fronts [12], thermal plumes [13], and chemical waves [14], etc., is based on an *effective interface description*. Such a physically appealing formulation can be systematically derived in a matched asymptotic expansion to lowest order in the ratio $\ell_{\rm in}/\ell_{\rm out}$, where the inner length scale $\ell_{\rm in}$ is the thickness of the front (here the thickness of the charge sheet), and $\ell_{\rm out}$ the scale of the pattern, e.g., the tip radius. In the effective interface approach that we propose for streamers, the charge sheet can be viewed as a *weakly curved* locally almost planar front, since the thickness of the charge sheet is much smaller than its radius of curvature. As in the other problems, the importance of our planar front analysis therefore lies in the fact that, apart from curvature corrections, it provides a complete solution of the so-called inner problem.

(iii) In the nonionized region outside the streamer, the electrical potential $\Phi$ obeys the Laplace equation, $\nabla^2\Phi = 0$. Moreover, our analysis shows that the normal velocity of a negatively charged planar streamer front ($v^*$ below) is a weakly nonlinear function of the field $E^+ = -\nabla\Phi$ just ahead of it. Both features are reminiscent of the equations for other interfacial pattern forming problems like dendrites—e.g., the enhanced diffusion in front of a dendrite tip is analogous to the field enhancement in front of a streamer. Streamers will therefore be amenable to the same type of analysis [9,12,13]. Physically, we expect that the interface equations will take the form of a conservation equation for a charge sheet (involving transport terms along the sheet, a stretch term due to interface curvature and a term associated with charge transport from the plasma behind), supplemented with an equation for the front speed that includes curvature corrections, and an equation for the degree of ionization created by the front which is not determined by any conservation law. The derivation of the appropriate equations is left to the future, as the analysis is far from trivial due to the coupling to the dynamics of the plasma, the fact that the electric field is typically not normal to the front, and the fact that in this fully nonequilibrium situation, the curvature corrections do not follow from simple thermodynamic considerations.

We now sketch our analysis [15] of planar fronts in the streamer model equations [6–8] that also underlie Fig. 1(a). The electron and ion densities $n_e$, $n_+$, and the electric field $\mathcal{E}$ obey the balance equations

$$\partial_t n_e + \nabla_{\mathbf{R}} \cdot \mathbf{j}_e = |n_e \mu_e \mathcal{E}| \alpha_0 e^{-E_0/|\mathcal{E}|}, \tag{1}$$

$$\partial_t n_+ + \nabla_{\mathbf{R}} \cdot \mathbf{j}_+ = |n_e \mu_e \mathcal{E}| \alpha_0 e^{-E_0/|\mathcal{E}|}, \tag{2}$$

and the Poisson equation

$$\nabla_{\mathbf{R}} \cdot \mathcal{E} = \frac{e}{\varepsilon_0}(n_+ - n_e). \tag{3}$$

The electron and ion current densities $\mathbf{j}_e$ and $\mathbf{j}_+$ are

$$\mathbf{j}_e = -n_e\mu_e\mathcal{E} - D_e\nabla_{\mathbf{R}} n_e, \quad \mathbf{j}_+ = 0, \tag{4}$$

so that $\mathbf{j}_e$ is the sum of a drift and a diffusion term, while the ion current $\mathbf{j}_+$ is neglected, since the ions are much less mobile than the electrons. The right-hand sides of Eqs. (1) and (2) are source terms due to the ionization reaction: In high fields free electrons can generate free electrons and ions by impact on neutral molecules. The source term is given by the magnitude of the electron drift current times the target density times the effective ionization cross section; the rate constant $\alpha_0$ has the dimension of an inverse length. The exponential function expresses that only in high fields electrons have a non-negligible probability to collect the ionization energy between collisions.

To identify the proper parameters for the behavior on the inner front scale, we note that in the simulations the fields just ahead of the front are of order of the threshold field $E_0 = 2 \times 10^5$ V/cm in Eqs. (1) and (2). The larger the rate parameter $\alpha_0$, the more rapid the impact ionization will be, and the thinner the front region. The natural length scale for the width of the front will indeed turn out to be $\alpha_0^{-1}$, which is about 2.3 $\mu$m in the simulations [6,7]. As the drift velocity of electrons in a field of order $E_0$ is $\mu_e E_0$, the natural time scale for the motion of fronts is then $t_0 = (\alpha_0\mu_e E_0)^{-1}$ ($\approx 3 \times 10^{-12}$ s in [6,7]) and the natural scale for the charge density is $q_0 = \varepsilon_0\alpha_0 E_0$ ($\approx 4.7 \times 10^{14}$ $e/\mathrm{cm}^3$ in [6,7]).

For analyzing planar fronts, we now introduce dimensionless variables, $x = X\alpha_0$, $\tau = t/t_0$, $E = \mathcal{E}/E_0$, the electron density $\sigma = en_e/q_0$, and the total charge density $q = (n_+ - n_e)e/q_0$. In these units, the *only* remaining dimensionless parameter is the dimensionless diffusion coefficient $D = D_e\alpha_0/\mu_e E_0$. In the simulations for $N_2$ [6,7], this value is about 0.1; for typical gases, $D$ ranges from 0.1 to 0.3 [15]. In these variables, the charge conservation equation becomes from (1), (2), and (4), $\partial_\tau q + \partial_x(\sigma E + D\partial_x\sigma) = 0$. Upon combining this with the Poisson equation $\partial_x E = q$ and integrating, we obtain

$$\partial_\tau E = -\sigma E - D\partial_x\sigma\,. \tag{5}$$

Here the integration constant is zero because on the inner time and spatial scale the charge and electron densities vanish for $x \to \infty$, while $E(x \to \infty) = E^+$ time independent. Equation (5) and the equation for the electron density,

$$\partial_\tau\sigma = \partial_x(\sigma E) + D\partial_x^2\sigma + \sigma|E|e^{-1/|E|}, \tag{6}$$

together constitute the one-dimensional streamer equations. These equations have two important classes of steady state solutions: the ones with $\sigma = 0$, $E^+$ arbitrary, correspond to the nonionized state of the gas into which the front propagates. The ones with $\sigma$ constant (denoted $\sigma^-$) and $E = 0$ correspond to the screened ionized state behind the front. It is straightforward to analyze the linear stability of these states with Fourier modes of the form $e^{\omega\tau + ikx}$. Physically, one expects the nonionized $(+)$ state to be *unstable:* Any small electron density drifts in the field $E^+$ and gets amplified due to impact ionization while the stabilization due to diffusion dominates only at short wavelengths. The corresponding dispersion relation $\omega^+ = ikE^+ + |E^+|e^{-1/|E^+|} - Dk^2$ confirms the long wavelength instability. It is easily checked that screening stabilizes the ionized $(-)$ states at all wavelengths, and that $\omega^- = -\sigma^- - Dk^2$.

Propagating streamer fronts are therefore an example of *front propagation into an unstable state.* We thus follow the common path for such problems [10,11].

(a) As usual, one can demonstrate the existence of a continuous family of uniformly translating front solutions of the form $\sigma(\xi)$ and $E(\xi)$ with $\xi = x - v\tau$, parametrized by the velocity $v$. This is done by formulating the equations for $\sigma(\xi)$ and $E(\xi)$ as a flow in the phase space $(\sigma, E, \sigma')$ with $\xi$ playing the role of a timelike variable. A front profile then corresponds to a trajectory connecting one $(-) = (\sigma^-, 0, 0)$ fixed point with one $(+) = (0, E^+, 0)$ fixed point, and the existence and multiplicity of these can be studied with counting arguments [11]. The family of solutions can be obtained explicitly for $D = 0$ by writing Eq. (5) as $v\partial_\xi \ln|E| = \sigma$, by inserting this form into Eq. (6) and integrating: we then get $\sigma[E] = v/(v + E)\int_{|E|}^{|E^+|} dx\, \exp(-1/x)$. This determines the flow in phase space for $D = 0$.

(b) Physically acceptable front solutions must satisfy the additional constraint that the number densities $n_e$ of electrons and $n_+$ of ions be positive, i.e., $\sigma(\xi) \geq 0$ for all $\xi$.

(c) We can show that the condition (b) entails a lower bound on the range of velocities. More precisely, one can show [15] that the velocity of physically admissible front solutions obey

$$v \geq v_f = \max[v^*, v^\dagger] > 0\,, \tag{7}$$

where $v^\dagger$ is the fastest *nonlinear front* [11] if it exists. Nonlinear fronts correspond to strongly heteroclinic orbits in phase space: they reach the $(+)$ fixed point along the eigendirection with the fastest contraction. The velocity $v^* = -E^+ + 2\sqrt{D|E^+|\exp(-1/|E^+|)}$ is the value of the velocity $v$ below which the eigenvalues describing the flow close to the $(+)$ fixed point become complex, so that the $\sigma(\xi)$ profiles violate (b) as they oscillate around zero far ahead of the front.

(d) Existing knowledge of front propagation [10,11] leads us to conjecture the following mechanism of front selection: *Fronts emerging from sufficiently localized initial conditions* [16] *converge asymptotically to the slowest physically acceptable front solution $v_f$ defined in* (7) [17].

We have investigated the existence of nonlinear ($v^\dagger$) fronts analytically and numerically and checked the above conjecture about dynamical selection by direct numerical integration of Eqs. (5) and (6). Both qualitatively and quantitatively, our predictions reflect the strong asymmetry between fronts moving parallel and antiparallel to the field.

Fronts propagating parallel to the electron drift, i.e., into a field $E^+ < 0$, are *negatively charged.* Numerically we find no $v^\dagger$ front solutions so that we predict the selected front velocity to be always the value $v^*$ given under (c). Here diffusion and ionization help to raise the front velocity to a value somewhat larger than the electron drift velocity $-E^+$. The degree of ionization $\sigma^-$ behind the front only weakly depends on $D$. The analytic result $\sigma^- = \sigma[E = 0]$ for $D = 0$ [see formula under (a)] [8] is independent of $v$ and a good approximation for all physical values of $D$, as the lower panel of Fig. 1(b) illustrates. Moreover, the values of $\sigma^-(E^+)$ extracted from the full 3D simulations of Vitello *et al.* [7] (crosses) are close to the values we calculate for planar streamer fronts.

Fronts screening a field $E^+ > 0$ are *positively charged.* They can propagate only if diffusion overcomes the drift. As a result, for small $D$ propagating fronts are extremely steep and slow. The front velocity vanishes like $D$, while both the spatial decay rate and the degree of ionization behind the front scale like $1/D$. In the limit $D \to 0$ this singular behavior can be derived analytically [15]. For general $D$, we have predicted the front velocities $v_f(E^+, D)$ numerically. They are shown in Fig. 1(b).

The numerical integration of the initial value problem fully supports all our predictions on the asymptotically

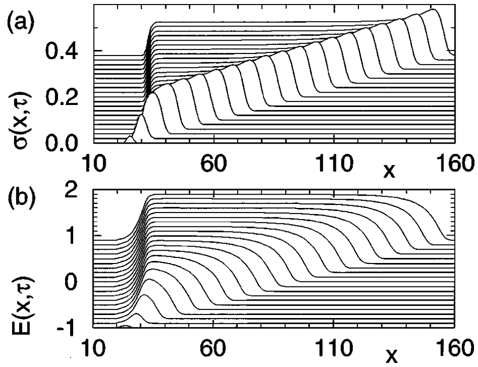


FIG. 2. Numerical integration of Eqs. (5) and (6) for $D = 0.1$ in a constant background field $E = -1$. Initial state at $t = 0$: lowest line. Each new line corresponds to a time step $\Delta t = 5$ and the upper line to $t = 100$. (a) Electron density, (b) electric field.

approached front for sufficiently localized initial conditions. As an example, Fig. 2 shows the spatiotemporal development of electron density (a) and field (b) of an initial state with $E = -1$ and a small charge-neutral Gaussian ionization seed. The diffusion constant is $D = 0.1$, and the field far from the ionized region is held constant. The ionized region initially grows exponentially and the electrons drift with the field, until field screening in the middle sets in. Then a negative front emerges to the right and asymptotically (after $\Delta t \approx 20$) approaches the $v^*$ ($= 1.38$) front with $\sigma^- = 0.144$. The positive front on the left initially recedes and then gets stuck by the combined action of drift and screening. This structure keeps slowly evolving in time, however, until after a time of order 4000, the predicted positive front with $v^\dagger = 0.0146$ and $\sigma^- = 6.23$ emerges (not shown).

In summary, we have solved the planar streamer front problem. Based on these results, we advocate that one should understand streamer dynamics as a two-scale problem: on the inner scale, we have a moving ionization sheet, whose thickness $\approx 10\ \mu$m is set by the ionization length $1/\alpha_0$. This interface plays the role of a free boundary for the outer dynamics, whose scale is set by the global geometry. *It is on this scale that the pattern formation problem should be studied.* The similarity with other well-known interfacial pattern forming problems (a Laplace equation for the potential $\Phi$ in the nonionized region with, apart from curvature corrections, a normal front velocity a function of $\nabla\Phi$) gives us confidence that properties of streamer patterns like field enhancement at the tip, velocity, and tip radius can be obtained in an analogous way [9] by including the curvature corrections in the resulting effective interface equations.

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[16] See [15] for details. For negatively charged fronts of type (a), sufficiently localized initial conditions obey $\sigma(x, \tau = 0) < C\ \exp(-\Lambda^* x)$ for $x \to \infty$ with $\Lambda^* = \sqrt{|E^+|\ \exp(-1/|E^+|)/D}$.
[17] For equations of the form $u_t = u_{xx} + f(u)$, the statement (iv) is consistent with exact results (see references in [10,11]). In the language of the marginal stability scenario [11], $v^*$ is the linear marginal stability and $v^\dagger$ the nonlinear marginal stability velocity.