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Numerical convergence of the branching time of negative streamers

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Discharge streamers in experiments branch frequently. Arrayás *et al.* [Phys. Rev. Lett. **88**, 174502 (2002)] presented simulations of branching streamers and interpreted them as physical branching events. The numerical results were criticized by Kulikovsky [Phys. Rev. Lett. **89**, 229401 (2002)]. Using an adaptive grid algorithm, we here present numerical experiments on the effect of grid resolution on streamer branching. The convergence of branching time with stepwise finer grid sizes provides a quantitative correction on the earlier, low-resolution results in overvolted gaps. Furthermore, streamers can branch even in undervolted, but sufficiently long gaps, but fewer branching modes are accessible than in higher fields.

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PROBLEM SETTING AND REVIEW

Streamers are transient weakly ionized plasma channels that rapidly grow into a non- or weakly ionized medium under influence of the self-enhanced electric field at their tip. They are widely used in technology [1,2] and ubiquitous in nature, where they play a role in creating the path of sparks, lightning [3], and blue jets above thunderclouds. Streamers are also directly observed as so-called sprites [4,5], which are very large discharge structures in the higher parts of the atmosphere that are composed of tens of thousands of streamers. Despite their high velocity, streamer evolution is now directly observable in experiments; a further review can be found in [2].

Streamers commonly branch in experiments if the gap and applied voltage are large enough. Recently a debate has risen about the proper physical concept for this branching. In 1939, Raether [6] proposed a mechanism for streamer propagation and Loeb and Meek [7] developed it into a branching concept that nowadays is found in many textbooks. The concept is based on a uniformly charged streamer head; ahead of it stochastic processes create secondary avalanches that subsequently develop into different branches. However, the distribution of rare electrons due to photoionization or background ionization ahead of the streamer has never been shown to agree with the conceptual pictures, and the concept has never been demonstrated to work. Furthermore, simulations in the past two decades [8–11] have shown that the fully developed streamer head is not homogeneously charged, but rather neutral and surrounded by a thin space charge layer which enhances the field ahead of it and screens it in the interior; this field enhancement allows the streamer to penetrate regions with a rather low background field. Recent simulations also show branching streamers in a fully deterministic model for charged particle densities, in a nonuniform background field [12–14] as well as in a uniform field [15–18].

Some of the present authors have proposed [15,16] a physical explanation of these numerical observations that is directly related to the formation of the thin space charge layer: the layer creates an almost equipotential streamer head that can undergo a Laplacian instability and branch in a manner similar to branching instabilities of fluid interfaces in viscous fingering. For a further discussion of the conceptual questions of streamer branching, we refer to [2]. However, the numerical codes used in [12–18] were not able to test the branching conditions on fine numerical grids. This led some researchers to question the physical nature of the instabilities [13,14,19,20] despite the arguments given in [15,16,21] and later in [22,23].

To resolve the debate from the numerical side, we have developed a code with comoving adaptive grids [24]. The algorithm enables us to run the simulations with a high spatial accuracy on large system sizes. The results presented below show that branching occurs within the deterministic model not only in overvolted gaps as in [15,16,18], but also in undervolted gaps, provided the discharge has sufficient space to develop. The branching time saturates on sufficiently fine numerical grids, giving quantitative predictions on streamer branching in a wide range of background fields. We also discuss the different branching modes as a function of the applied electric field.

MODEL AND MULTISCALE STRUCTURE OF NEGATIVE STREAMERS

We investigate a minimal continuum model for streamers, which contains the essential physics for negative streamers in a nonattaching pure gas like N_2 or Ar [8,9,15,16]. Streamers were analyzed previously with this model in [15,16], while [17,18] include photoionization and/or deal with positive streamers. The model is a two-fluid approximation for the charged particles, with a local-field-dependent impact ionization reaction coupled to the Poisson equation for electrostatic particle interactions. In dimensionless units [2], the model reads

$$\partial_\tau \sigma = \nabla \cdot (\sigma \boldsymbol{\mathcal{E}} + D \nabla \sigma) + \sigma |\boldsymbol{\mathcal{E}}| \alpha(|\boldsymbol{\mathcal{E}}|), \qquad (1)$$

$$\partial_\tau \rho = \sigma |\boldsymbol{\mathcal{E}}| \alpha(|\boldsymbol{\mathcal{E}}|), \quad \alpha(|\boldsymbol{\mathcal{E}}|) = e^{-1/|\boldsymbol{\mathcal{E}}|}, \qquad (2)$$

$$-\nabla^2 \phi = \rho - \sigma, \quad \boldsymbol{\mathcal{E}} = -\nabla \phi, \qquad (3)$$

where σ and ρ are the electron and positive ion densities, respectively. $\boldsymbol{\mathcal{E}}$ and ϕ are, respectively, the electric field and potential, D is the electron diffusion coefficient, and τ is the dimensionless time. The characteristic scales in this model depend on the neutral gas density; therefore the simulation results can be applied to high-altitude sprite discharges at low pressures as well as to high-pressure laboratory experiments.

We study the model in cylindrical symmetry in effectively two dimensions, namely, in the coordinate z between the electrodes and in the radial coordinate r. This suppresses nonsymmetric instability modes. In a strict mathematical sense, the time of branching in this cylindrical geometry is an upper bound for the branching time in a genuine three-dimensional (3D) system [21]. On the other hand, the particular instability modes will affect only the streamer configuration after branching, but not the time at which the single streamer reaches its unstable state. Therefore, we expect our predictions for the streamer branching time to be a very good approximation despite the symmetry constraint. Of course, for the evolution after branching, genuine 3D simulations should be used.

A planar cathode is placed at $z=0$ and a planar anode at $z=L_z$. The potential at the electrodes is fixed, $\phi(r,z=0,\tau)=0$, $\phi(r,z=L_z,\tau)=\phi_0>0$, generating a background electric field with strength $|\mathcal{E}_b|=\phi_0/L_z$ along the negative z direction. The streamer is initiated by an electrically neutral Gaussian ionization seed on the axis of symmetry at the cathode ($r=z=0$). There is no background ionization far from the initial seed.

We impose homogeneous Neumann conditions for the electron density at all boundaries. This results in a net inflow of electrons from the cathode if the streamer is attached to it [15,24]. In practice, the computational volume is restricted in the radial direction by a boundary L_r sufficiently far away not to disturb the solution near and in the streamer. Moreover, we choose the interelectrode distance L_z so large that the streamer is not affected by the anode proximity for the results shown.

Streamers contain a wide range of spatial scales, from the very extended nonionized medium on which the Poisson equation has to be solved through the length of the conducting channel and its width up to the inner structure of the thin space charge layer around the streamer head.

Moreover, the region just ahead of the streamer, where the field is substantially enhanced and the electron density is low, is highly unstable, in the sense that a small ionized perturbation will grow much more rapidly than in the mere background field. This unstable region ahead of the streamer tip is commonly referred to as the *leading edge* [25,26]. It requires special care when considering numerical methods [24,26]. Accurate simulations of streamers therefore pose a great computational challenge.

NUMERICAL ALGORITHM

In order to deal efficiently with the numerical challenges posed by this model, it has been implemented in a numerical code using adaptive grid refinements. We recall the essential features of this algorithm and refer to [24] for further details. The spatial discretizations are based on finite volumes, using a flux-limiting scheme to prevent spurious oscillations in the results near steep gradients. The time stepping is performed with a two-stage explicit Runge-Kutta method.

Using an explicit time-stepping method allows us to decouple the computational grids for the continuity equations (1) and (2) on the one hand from those for the Poisson equation (3) on the other hand. The particle densities are first updated on a series of nested, stepwise refined grids. The resulting electric field on that same series of grids is then computed through the Poisson equation for the electric potential, which in turn is solved on a different series of nested grids [27].

Adequate refinement criteria for the continuity and for the Poisson equation lead to a grid distribution which is especially designed to cope adequately and efficiently with the difficulties inherent to both type of equations. More specifically, the refinement criterion for the grids for the Poisson equation is based on an error estimate for the solution. The refinement criterion for grids for the continuity equations uses a curvature monitor of the solution. Moreover, it takes explicitly into account the leading edge.

The refinement criterion is computed at each time step, in such a way that the series of nested, consecutively refined grids move with the solution. Special care has been taken that the discretizations as well as the mapping of the solution from one grid to the other are charge conserving.

RESULTS

The adaptive grid refinement procedure enables us to resolve the streamer with very high accuracy, and thus to investigate the dependence of the branching process on the numerical grid. The results are obtained on increasingly finer grid sizes h_f, always taking the same coarsest mesh width h_c for both the continuity and the Poisson equations. If the branching were of numerical nature, we would expect that branching times on increasingly finer grids would not converge.

We first consider a negative streamer in a dimensionless overvolted background field of $|\mathcal{E}_b|=0.5$ corresponding to 100 kV/cm in N_2 at atmospheric pressure in a gap of $L_z=2048$ (4.6 mm) as previously in [15,16]. The initial seed is as in [16], namely, a Gaussian ionization distribution with amplitude 10^{-4} (5×10^{10} cm^{-3}) and characteristic radius of 10 (23 μm).

While [15] used a uniform grid of $h=2$ and [16] one of $h=1$, we now perform computations on a finest grid as small as $h_f=1/8$, i.e., more than a decade finer. More precisely, the coarsest mesh width is set to $h_c=2$, and the finest one to $h_f=2,1,\ldots,1/8$. Furthermore, a better numerical scheme is used: flux limiting [24] rather than third-order upwind [15,16].

Before branching, at $\tau=275$, Fig. 1 shows that there is a quantitative difference between the results on a mesh with $h_f=1$ and the finer ones. On a coarse mesh, numerical diffusion smears the space charge layer out. Numerical diffusion is aggrevated when the flux limiter switches to the diffusive first order scheme in regions with large gradients (relative to the coarse mesh). This makes the field enhancement at the

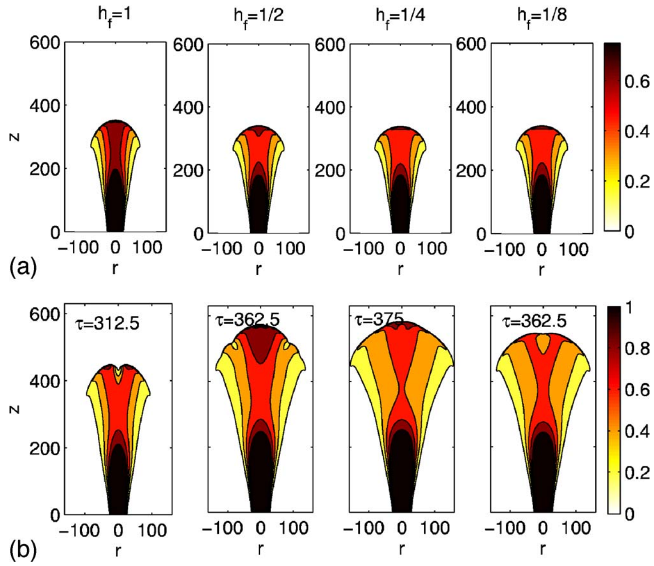

FIG. 1. (Color online) Electron density distribution computed on different finest grids, from left to right h_f=1, 1/2, 1/4, and 1/8; (a) before branching, at τ=275; (b) just after the respective branching time. In all cases the same restricted part of the total computational domain is shown.

streamer tip and the field screening in the streamer body less efficient. Consequently, the ionization reaction in the streamer body stays significant in an overvolted gap, and the electron density in the streamer body can become higher on a coarser mesh. Figure 1 shows that on meshes finer than 1/2, the results are the same during streamer propagation. It is only after branching that different states are observed on those very fine grids. However, the time of branching converges within this range of mesh widths h_f as shown in Fig. 2.

We now present results on negative streamers evolving in an undervolted field of $|\mathcal{E}_b|=0.15$, corresponding to 30 kV/cm for N_2 at atmospheric pressure. Here "undervoltage" means that an avalanche would not evolve into a streamer [28], but we show that an already existing streamer not only propagates, but can even branch. We use an electrically neutral, dense, and relatively wide Gaussian ionization seed at the cathode, with a maximum of 1/4.8 (10^{14} cm^{-3}), and a characteristic radius of 100 (230 μm). The gap length and width are set to $L_z=2L_r=2^{15}=32\,768$ (7.5 cm).

The coarsest mesh width is set to h_c=64, and the finest one to h_f=8, 4, and 2. When a finest mesh of 8 is used, the

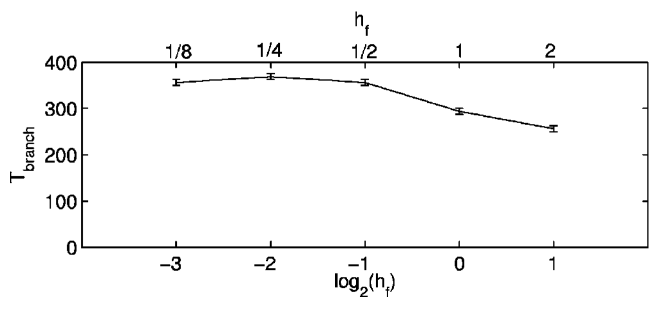

FIG. 2. Branching time in a background field $|\mathcal{E}_b|=0.5$ as function of the finest mesh size h_f=2, 1, 1/2, 1/4, 1/8.

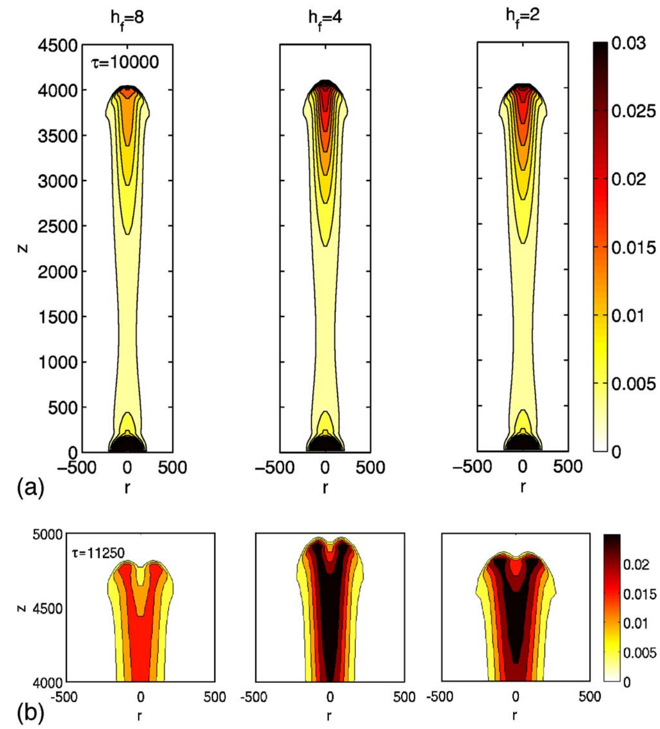

FIG. 3. (Color online) Electron density distribution before and just after streamer branching in a background field $|\mathcal{E}_b|=0.15$, computed on different finest mesh sizes h_f=8, 4, and 2 as indicated over the plots. The upper snapshots at τ=10 000 are taken before branching and the lower ones after branching, at time τ=11 250. The contours correspond to the same density levels.

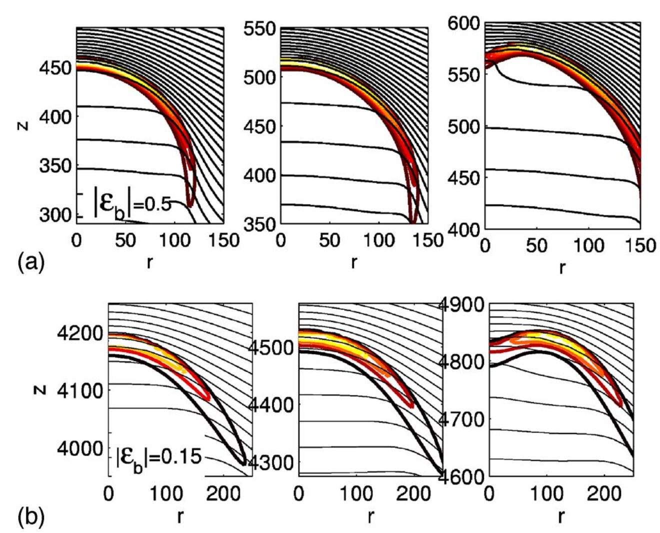

FIG. 4. (Color online) Zoom into the streamer head during branching. (a) $|\mathcal{E}_b|$=0.5 as in Fig. 2, h_f=1/8, for consecutive steps τ=325, 350, 375. (b) $|\mathcal{E}_b|$=0.15 as in Fig. 3, h_f=2, for τ=10 250, 10 750, 11 250. Contour lines (thick) of net charge density and equipotential lines (thin) are shown as a function of radius r and appropriate z. The spacing of the charge contour levels is 0.16 in the overvolted and 0.004 in the undervolted case. The spacing of equipotential lines is 5 in both cases.

electron density in the streamer is lower than on finer meshes, as can be seen in the upper row in Fig. 3—on the contrary, in the overvolted case, it was higher. As in the overvolted case, numerical diffusion decreases field enhancement and ionization rates at the streamer tip. However, the mesh-induced lower ionization rate at the tip here is not balanced by a higher one in the streamer body as the gap is undervolted.

The lower row of Fig. 3 shows that the influence of the numerical grid on the branching state decreases. Moreover, the branching time is the same in all cases. These results show that even in undervolted gaps, streamers can branch, if the gap is long enough. Branching in a marginally undervolted gap ($|\mathcal{E}_b|=0.18$) was observed once previously in [17], without checking the numerical accuracy.

DISCUSSION, CONCLUSION, AND OUTLOOK

We emphasize that the branching times converge on decreasing numerical grids in both cases. Therefore we here present quantitative numerical predictions on streamer branching. However, in contrast to the undervolted case, the lower plots in Fig. 1 show that in the overvolted case different branched modes are reached after approximately the same evolution time: in two cases, the maximal electron density and field are on the axis of symmetry, and in two other cases, they are off axis. Apparently, there are different branched states reachable at bifurcation and tiny differences determine which one will be reached. Such extreme sensitivity is well known from deterministic chaos; it is generic for nonlinear dynamics near bifurcation points. On the other hand, the unstable state is reached in a deterministic manner, and therefore the branching times converge.

But why is there once a unique branched state and once several? The answer can be found in Fig. 4 showing the two relevant spatial scales, namely, the thickness of the space charge layer and the radius of the channel. In the overvolted gap, the ratio of layer thickness to the radius is much smaller than in the undervolted gap. Moreover, the field screening and enhancement are much stronger and the equipotential lines follow the space charge layer much better. Therefore the overvolted streamer is much closer to interfacial models as discussed in [2,15,22,23,25] and can access more branching modes. This critical state in future work will be characterized by the electric charge content and electric field and potential at the streamer tip, which would then allow us to relate branching to the external electric circuit. For sketches of such ideas as well as for a discussion of photoionization effects and of continuum versus particle models, we refer to [2].

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