Strong t-perfection of bad-K_4-free graphs

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Abstract. We show that each graph not containing a bad subdivision of K_4 as a subgraph, is strongly t-perfect. Here a graph $G = (V, E)$ is *strongly t-perfect* if for each weight function $w : V \to \mathbb{Z}_+$, the maximum weight of a stable set is equal to the minimum (total) cost of a family of vertices, edges, and circuits covering any vertex v at least $w(v)$ times. By definition, the *cost* of a vertex or edge is 1, and the *cost* of a circuit C is $\lfloor \frac{1}{2}|VC| \rfloor$. A subdivision of K_4 is called *bad* if each triangle has become an odd circuit and if it is not obtained by making the edges in a 4-circuit of K_4 evenly subdivided, while the other two edges are not subdivided.

The theorem generalizes earlier results of Gerards (1989) on the strong t-perfection of odd-K_4-free graphs and of Gerards and Shepherd (1998) on the t-perfection of bad-K_4-free graphs.

1. Introduction

A graph $G = (V, E)$ is called *t-perfect* if the stable set polytope of G (= the convex hull of the incidence vectors in $\mathbb{R}^V$ of stable sets) is determined by:

$$
(1) \qquad \begin{array}{rll}
\text{(i)} & 0 \leq x_v \leq 1 & \text{for each } v \in V; \\
\text{(ii)} & x_u + x_v \leq 1 & \text{for each edge } uv \in E; \\
\text{(iii)} & x(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor & \text{for each odd circuit } C.
\end{array}
$$

Here $x(U) := \sum_{v \in U} x_v$ for any $U \subseteq V$. $V..$ and $E..$ denote the sets of vertices and edges, respectively, of .. . A circuit C is *odd* (*even*) if $|VC|$ is odd (even).

A motivation for the concept of t-perfection lies in the fact that a linear function $w^{\mathsf{T}}x$ can be maximized over (1) in strongly polynomial time (with the ellipsoid method, since the separation problem over (1) is polynomial-time solvable). Hence a maximum-weight stable set in a t-perfect graph can be found in strongly polynomial time.

G is called *strongly t-perfect* if system (1) is totally dual integral — that is, if for each weight function $w : V \to \mathbb{Z}_+$, the linear program of maximizing $w^{\mathsf{T}}x$ over (1) has an integer optimum dual solution. This implies that it also has an integer optimum primal solution. In particular, all vertices of the polytope determined by (1) are integer, and hence the polytope is the stable set polytope. So strong t-perfection implies t-perfection.

Strong t-perfection can be characterized equivalently as follows. For any $w : V \to \mathbb{Z}_+$, let $\alpha_w(G)$ denote the maximum weight of a stable set in G. Define a *w-cover* as a family of vertices, edges, and odd circuits such that each vertex v is covered at least $w(v)$ times (in a *family*, repetition is allowed). By definition, the *cost* of a vertex or edge is 1, the *cost* of a circuit C is $\lfloor \frac{1}{2}|VC| \rfloor$, and the *cost* of a w-cover is the sum of the costs of its elements (counting multiplicities). Let $\tilde{\rho}_w(G)$ denote the minimum cost of a w-cover. Then

(2) a graph G is strongly t-perfect if and only if $\alpha_w(G) = \tilde{\rho}_w(G)$ for each $w : V \to \mathbb{Z}_+$.

The classes of t-perfect and strongly t-perfect graphs are closed under taking induced subgraphs. However, no characterization is known in terms of forbidden induced subgraphs.

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If we take also noninduced subgraphs, the situation is clearer (although it does not yield a characterization). Here subdivisions of K_4 come in. A K_4-subdivision H is called *odd*, or just *an odd* K_4, if each triangle of K_4 has become an odd circuit in H. It was shown by Gerards [7] that

(3) each graph without odd K_4 is strongly t-perfect.

(By 'a graph without' odd K_4 we mean a graph not containing an odd K_4 as subgraph.) It extends an earlier result of Gerards and Schrijver [8] that such graphs are t-perfect.

There exist however odd K_4's that are t-perfect. Following Gerards and Shepherd [9], we call an odd K_4 subdivision a *bad* K_4 if it does *not* have the following property:

(4) the edges of K_4 that have become an even path, form a 4-cycle in K_4, while the two other edges of K_4 are not subdivided.

This name is motivated by the fact, shown by Barahona and Mahjoub [1], that a subdivision of K_4 is t-perfect if and only if it is not a bad K_4. Gerards and Shepherd [9] proved that

(5) each graph without bad K_4 is t-perfect.

(Gerards and Shepherd [9] also showed that graphs without bad K_4 can be recognized in polynomial time.)

In the present paper, we show more strongly that these graphs are strongly t-perfect. This generalizes (3) and (5), and implies for any graph G:

(6) each subgraph of G is t-perfect $\iff$ each subgraph of G is strongly t-perfect $\iff$ G has no bad K_4 as subgraph.

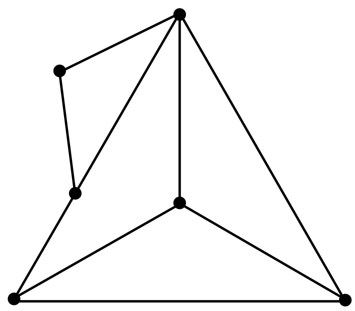

Figure 1

On the other hand, there exist strongly t-perfect graphs that contain a bad K_4 — see Figure 1.

Our proof method was inspired by a method of Geelen and Guenin [6] for proving a special case of a theorem of Seymour [14] on packing the edge sets of odd circuits in odd-K_4-free graphs.

The above results contain the strong t-perfection of series-parallel graphs, which are, as is well-known, those graphs not containing any K_4-subdivision (Boulala and Uhry [2]), and of almost bipartite graphs — graphs G having a vertex v with $G-v$ bipartite (Fonlupt and Uhry [4], Sbihi and Uhry [11]).

A related theorem was proved by Sewell and Trotter [13]. A K_4-subdivision is called a *totally odd* K_4 if it arises from K_4 by replacing each edge by an odd path. The theorem says that a graph G without totally odd K_4 satisfies $\alpha_{\mathbf{1}}(G) = \tilde{\rho}_{\mathbf{1}}(G)$, where $\mathbf{1}$ denotes the all-one weight function. This result does not follow from our methods.

The totally odd K_4's are precisely those K_4-subdivisions G with $\alpha_{\mathbf{1}}(G) < \tilde{\rho}_{\mathbf{1}}(G)$. So the theorem of Sewell and Trotter and the theorem presented in this paper suggest the question if for each graph G and each $w : VG \to \mathbb{Z}_+$ with $\alpha_w(G) < \tilde{\rho}_w(G)$, G contains a K_4-subdivision H as subgraph such that $\alpha_{w'}(H) < \tilde{\rho}_{w'}(H)$, where $w' := w|VH$. The answer is unknown.

To complete the picture, it was shown by Zang [17] and Thomassen [15] that $\chi(G) \leq 3$ for any graph G without totally odd K_4. This was conjectured by Toft [16], and was proved by Hadwiger [10] for series-parallel graphs, by Catlin [3] for odd-K_4-free graphs, and by Gerards and Shepherd [9] for bad-K_4-free graphs. (However, there exist strongly t-perfect graphs G with $\chi(G) > 3$.)

A.M.H. Gerards and P.D. Seymour proved in 1991 (personal communication) that, if G contains no odd K_4, then the stable set polytope of G has the integer decomposition property. In other words, any $w : VG \to \mathbb{Z}_+$ is the sum of the incidence vectors of k stable sets, where k is the minimum integer for which $\frac{1}{k}w$ belongs to the stable set polytope. It implies the result of Catlin mentioned above.

2. Graphs without bad K_4

In this section we prove a technical lemma on bad-K_4-free graphs. Let G be graph without bad K_4, and let C be an even circuit in G. Let $e_1, \dots, e_n$ be chords of C, such that e_i has ends s_i and s_{n+i} (say) (for $i = 1, \dots, n$), such that $s_1, \dots, s_{2n}$ are distinct and occur in this order clockwise along C, and such that for each $i = 1, \dots, 2n$, the clockwise $s_{i-1} - s_i$ path R_i along C has even length. (We take indices mod $2n$, and set $e_{n+i} := e_i$ for $i = 1, \dots, n$.) Define $D := \{e_1, \dots, e_n\}$.

Call a path B in G a *bow* if B is simple, has length at least 2, and intersects C precisely in its end vertices. We call a bow an *odd bow* if it forms with a subpath of C an odd circuit, and an *even bow* if it forms with a subpath of C an even circuit. (So an odd (even) bow need not be an odd (even) path. To avoid confusion, we therefore do not use the more familiar term 'ear'.)

We will study in particular the occurrence of odd bows. We say that a bow B *crosses* an edge $e \in D$ (and conversely), if e is disjoint from the ends a, b (say) of B and connects distinct components of the graph $C - a - b$. Then

(7) an odd bow B does not cross any edge e in D.

Otherwise C, B, and e form a bad K_4, a contradiction.

(7) implies that the ends of any odd bow belong to VR_j, for some $j = 1, \dots, 2n$. Define

(8) $J := \{j \in \{1, \dots, 2n\} |$ there exists an odd bow with ends in $VR_j\}$.

We prove:

Lemma. *There exists an* $i \in \{1, \ldots, 2n\}$ *such that* $i+1, i+2, \ldots, i+n-1 \notin J$.

Proof. Consider a counterexample with n as small as possible. Define $L := \{i | i+2, \ldots, i+n-1 \notin J\}$. Then for each i:

(9) $\quad i \in L$ or $i+n \in L$.

To see this, by symmetry it suffices to show this for $i = n$. Delete e_n. By the minimality of n, the lemma holds for the new structure. In the new structure, the paths R_n and R_{n+1} have merged to one path, and similarly the path R_{2n} and R_1 have merged to one path. If (9) does not hold for the original structure, then, for some $i \in \{2, \ldots, n-1\}$, there is no odd bow with ends in one of $VR_{i+1}, \ldots, VR_{n-1}, VR_n \cup VR_{n+1}, VR_{n+2}, \ldots, VR_{i+n-1}$ or there is no odd bow with ends in one of $VR_{i+n+1}, \ldots, VR_{2n-1}, VR_{2n} \cup VR_1, VR_2, \ldots, VR_{i-1}$. Either case implies the lemma for the original structure, a contradiction. So we have (9).

We derive from this that $n = 2$. As the lemma does not hold, we know that $i \notin L$ or $i+1 \notin L$ for each i. Hence, by (9), $i \in L$ or $i+1 \in L$ for each i. So the indices i are alternatingly in and out of L. If $n \geq 4$, then we can assume that each even i belongs to L, and hence, by the definition of L, $J = \emptyset$, a contradiction.

So $n \leq 3$. Suppose $n = 3$. We may assume $J = \{1, 3, 5\}$. For $j = 1, 3, 5$, let B_j be an odd bow with ends in VR_j. Then B_1, B_3, B_5 are pairwise disjoint, for suppose that (say) B_1 and B_3 have a vertex in common. Choose an end a of B_1 with $a \neq s_1$. Follow B_1 from a till we reach B_3. We can continue along B_3 so as to create an odd bow B (as B_3 is an odd bow). As B crosses e_1, this contradicts (7).

So B_1, B_3, B_5 are pairwise disjoint. Let R'_j be obtained from R_j by replacing part of R_j by B_j. Then $R'_1, R_2, R'_3, R_4, R'_5$ and e_1, e_2, e_3 form a bad K_4, a contradiction.

So $n = 2$. As the lemma does not hold, we know $J = \{1, 2, 3, 4\}$. For $j = 1, \ldots, 4$, let B_j be an odd bow with ends in VR_j. If the B_j are pairwise internally vertex-disjoint, we obtain a bad K_4, a contradiction. So at least two of the B_j have an internal vertex in common. Define $S := \{s_1, s_2, s_3, s_4\}$. To analyze this, we first prove:

(10) $\quad$ Let B be a bow with ends a, b and $a \in VR_1 \setminus S$ and $b \notin VR_1$. Then a and b are equal to the middle vertices of R_1 and R_3 respectively.

By (7), B is an even bow. By symmetry, we can assume that $b \in VR_2 \cup VR_3 \setminus \{s_1, s_3\}$. Let C' be the (even) circuit obtained from C by replacing the $a-b$ path P along C that traverses s_1, by B. Let e'_1 be the extension of e_1 with the $s_1 - a$ part of R_1. So e'_1 is an odd bow of C'. If $b \in VR_2$ then e_2 is a chord of C' that crosses e'_1, contradicting (7). So $b \in VR_3 \setminus S$.

Let e'_2 be the extension of e_2 with the $s_2 - b$ part of R_3. Again, e'_2 is an odd bow of C'. Then C', e'_1, e'_2 form an odd K_4-subdivision H, with trivalent vertices a, b, s_3, and s_4. As H is not bad, and as s_4 is nonadjacent (in H) to b and s_3, we know that s_4 is adjacent (in H) to a. By symmetry, a is adjacent to s_1, and b to s_2 and to s_3. This gives (10).

From this we derive:

(11) $\quad$ Let T be a tree with three end vertices a, b, c, and trivalent vertex v, such that T has only its end vertices in common with C, and such that a, b, c do not all belong to some VR_i ($i = 1, \ldots, 4$). Then for some i, $\{a, b, c\} = \{s_{i-1}, s_i, s_{i+1}\}$, s_i is adjacent to v, and the $v - s_{i-1}$ and $v - s_{i+1}$ paths along T are even.

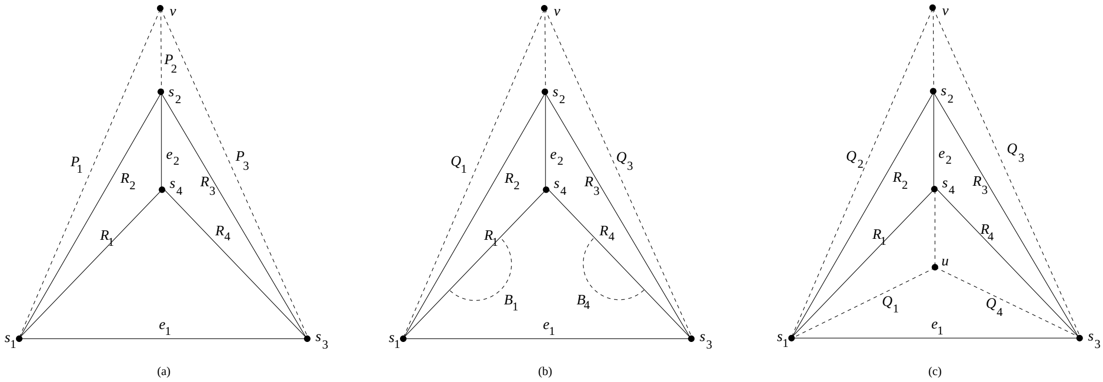

Figure 2

We first show that $a, b, c \in S$. Suppose not. Then we can assume $a \in VR_1 \setminus S$. Since a, b, c not all belong to VR_1, we can assume that $b \notin VR_1$. Then by (10), a and b are the middle vertices of R_1 and R_3 respectively. By symmetry of a and b, we can assume that $c \notin VR_1$, implying similarly that $c = b$, a contradiction. So $a, b, c \in S$.

Next we can assume that $\{a, b, c\} = \{s_1, s_2, s_3\}$. Let P_i be the $v - s_i$ path in T (for $i = 1, 2, 3$) (cf. Figure 2(a)). As P_1 and P_3 form a bow connecting s_1 and s_3, it is an even bow and we have $|EP_1| \equiv |EP_3| \pmod 2$. If moreover $|EP_1| \equiv |EP_2| \pmod 2$, then P_1, P_2, P_3, R_1, R_4, e_1 and e_2 form a bad K_4. So $|EP_1| \not\equiv |EP_2| \pmod 2$. Then P_1, P_2, P_3, R_2, R_3, and e_1 form an odd K_4. As it is not bad and as e_1 has length 1, we have $|EP_2| = 1$, implying (11).

This implies:

(12) $G - VC$ has no component K with $s_1, s_2, s_3, s_4 \in N(K)$.

Otherwise, there is a tree T intersecting VC only in its end vertices s_1, s_2, s_3, s_4. By (11), the neighbour v_i of any s_i in T has degree at least 3 (by considering a subtree with ends s_{i-1}, s_i, s_{i+1}). It also follows from (11) that $v_i \neq v_{i+1}$ for each i. So $v_1 = v_3$, contradicting (11) (by considering a subtree with ends s_1, s_2, s_3). This gives (12).

This implies that B_1 and B_3 are disjoint. Otherwise, by (11), the ends of B_1 and B_3 are s_1, s_2, s_3, s_4, contradicting (12). Similarly, B_2 and B_4 are disjoint.

So we can assume that B_2 and B_3 have a vertex in common, and hence, by (11), that there is a vertex $v \notin VC$ adjacent to s_2, and a $v - s_1$ path Q_2 and a $v - s_3$ path Q_3 such that for $i = 2, 3$, B_i is the concatenation of the edge s_2v and Q_i (cf. Figure 2(b)).

By (12), neither B_1 nor B_4 has an internal vertex in common with B_2 and B_3. If B_1 and B_4 are internally vertex-disjoint, then B_1, B_4, e_1, e_2, vs_2, Q_1, Q_2, and parts of R_1 and R_4 form a bad K_4.

So B_1 and B_4 are not internally vertex-disjoint. Hence, by (11), there is a vertex $u \notin VC$ adjacent to s_4 and a $u - s_1$ path Q_1 and a $u - s_3$ path Q_4 such that for $i = 1, 4$, B_i is the concatenation of the edge s_4u and Q_i (cf. Figure 2(c)). Then $Q_1, \ldots, Q_4$, vs_2, us_4, e_2, and e_1 form a bad K_4, a contradiction. ∎

3. Strong t-perfection of bad-K_4-free graphs

We now prove our main theorem:

Theorem. *A graph without bad K_4 is strongly t-perfect.*

Proof. Let $G = (V, E)$ be a counterexample with $|V| + |E|$ minimum. For any weight function $w : V \to \mathbb{Z}_+$, denote $\alpha_w := \alpha_w(G)$ and $\tilde{\rho}_w := \tilde{\rho}_w(G)$. For any subset U of V let χ^U be the incidence vector of U. So for an edge $e = uv$, χ^e is the $0,1$ vector in $\mathbb{R}^V$ having 1's in positions u and v.

We first show:

Claim 1. There is a $w : V \to \mathbb{Z}_+$ and an edge f such that

(13) $$\tilde{\rho}_{w+\chi^f} = \alpha_w + 1 = \tilde{\rho}_w$$

and such that

(14) $$\alpha_{w-\chi^{VC}} = \tilde{\rho}_{w-\chi^{VC}}$$

for each odd circuit C.

Proof of Claim 1. Choose a vertex u. For any $w : V \to \mathbb{Z}_+$ with $\alpha_w < \tilde{\rho}_w$ one has:

(15) $$w(u) < w(N(u))$$

(where $N(u)$ denotes the set of neighbours of u). Otherwise, by the minimality of G, setting $G' := G - u - N(u)$ and $w' := w|VG'$,

(16) $$\alpha_w(G) = w(u) + \alpha_{w'}(G') = w(u) + \tilde{\rho}_{w'}(G') \geq \tilde{\rho}_w(G),$$

since $G[\{u\} \cup N(u)]$ has a $w|N(u) \cup \{u\}$-cover of cost $w(u)$ (as $w(u) \geq w(N(u))$). (16) contradicts our assumption, which proves (15).

By (15), we can choose w such that $\alpha_w < \tilde{\rho}_w$ and such that $w(V \setminus \{u\}) - w(u)$ is as small as possible. Then:

(17) there exists a $z \in \mathbb{Z}_+^{\delta(u)}$ such that for $\tilde{w} := w + \sum_{e\in\delta(u)} z_e\chi^e$ we have $\alpha_{\tilde{w}} = \tilde{\rho}_{\tilde{w}}$.

To see this, it suffices to show:

(18) there exists a $z \in \mathbb{Z}^{\delta(u)}$ and a stable set S, such that $\tilde{w} := w + \sum_{e\in\delta(u)} z_e\chi^e$ is nonnegative and such that $\tilde{w}(S) = \tilde{\rho}_{\tilde{w}}$ and S intersects each edge incident with u.

This suffices, since if z' arises from z by replacing the negative entries by 0, and

(19) $$w' := w + \sum_{e\in\delta(u)} z'_e\chi^e,$$

then $w'(S) = \tilde{w}(S) - \sum(z_e | z_e < 0)$ and $\tilde{\rho}_{w'} \leq \tilde{\rho}_{\tilde{w}} - \sum(z_e | z_e < 0)$, as $w' = \tilde{w} - \sum(z_e\chi^e | z_e < 0)$. This implies (17).

To prove (18), first suppose that $N(u)$ is a stable set. Let G' be the graph obtained from G by contracting the edges in $\delta(u)$. Then G' contains no bad K_4. Let t be the new vertex. Let $w' : VG' \to \mathbb{Z}_+$ be defined by $w'(t) := w(N(u)) - w(u)$ and $w'(v) := w(v)$ if $v \neq t$. Since G' is smaller than G, we know $\alpha_{w'}(G') = \tilde{\rho}_{w'}(G')$.

Consider a w'-cover $\mathcal{F}'$ in G' of cost $\tilde{\rho}_{w'}(G')$. Let λ be the number of circuits in $\mathcal{F}'$ that are not circuits in G. So they traverse t, and can be made to circuits in G by adding two edges incident with u. It gives, for some $\tilde{w}$, a $\tilde{w}$-cover $\mathcal{F}$ in G of cost $\tilde{\rho}_{w'}(G') + \lambda$ such that $\tilde{w}$ coincides with w on $V \setminus (N(u) \cup \{u\})$, and such that $\tilde{w}(u) = \lambda$ and $\tilde{w}(N(u)) = w'(t) + \lambda$. Hence the cost is $\tilde{\rho}_{w'}(G') + \tilde{w}(u)$ and $\tilde{w}(N(u)) - \tilde{w}(u) = w(N(u)) - w(u)$. This last implies that $\tilde{w} = w + \sum_{e\in\delta(u)} z_e\chi^e$ for some $z \in \mathbb{Z}^{\delta(v)}$.

Now let S' be a stable set in G' with $w'(S') = \alpha_{w'}(G')$. If $t \in S'$, define $S := (S' \setminus \{t\}) \cup N(u)$, and if $t \notin S'$, define $S := S' \cup \{u\}$. So S is a stable set in G. Then $w(S) = w'(S') + w(u)$ and S intersects each edge incident with u. So

$$\tilde{w}(S) = w'(S') + \tilde{w}(u) = \tilde{\rho}_{w'}(G') + \tilde{w}(u) \geq \tilde{\rho}_{\tilde{w}}(G). \tag{20}$$

This gives (18) in case $N(u)$ is a stable set.

If $N(u)$ is not a stable set, let $G' := G - u - N(u)$ and $w' := w|VG'$. By the minimality of G, $\alpha_{w'}(G') = \tilde{\rho}_{w'}(G')$. Let $\mathcal{F}'$ be a w'-cover in G' of cost $\tilde{\rho}_{w'}(G')$. By adding to $\mathcal{F}'$ a number of times a triangle incident with u we obtain a $\tilde{w}$-cover $\mathcal{F}$ in G for some $\tilde{w} : V \to \mathbb{Z}_+$, where $\tilde{w}$ coincides with w on $V \setminus (\{u\} \cup N(u))$, where $\tilde{w}(N(u)) - \tilde{w}(u) = w(N(u)) - w(u)$, and where $\mathcal{F}$ has cost $\tilde{\rho}_{w'}(G') + \tilde{w}(u)$.

Now let S' be a stable set in G' with $w'(S') = \alpha_{w'}(G')$. Define $S := S' \cup \{u\}$. So S is a stable set in G. Then $w(S) = w'(S') + w(u)$ and S intersects each edge incident with u. Moreover, $\tilde{w}(S) = w'(S') + \tilde{w}(u) = \tilde{\rho}_{w'}(G') + \tilde{w}(u) \geq \tilde{\rho}_{\tilde{w}}(G)$. So we have (18), and hence (17).

Choose z in (17) with $z(\delta(u))$ as small as possible. Choose $f \in \delta(u)$ with $z_f \geq 1$. We can assume that $z_f = 1$ and $z_e = 0$ for all other edges e, as we can reset $w := \tilde{w} - \chi^f$. (This resetting does not change the value of $w(V \setminus \{u\}) - w(u)$.) Then (14) follows from the minimality of $w(V \setminus \{u\}) - w(u)$.

We finally show (13). By the definition of z, $\tilde{\rho}_{w+\chi^f} = \alpha_{w+\chi^f}$. Also we have $\alpha_{w+\chi^f} \leq \alpha_w + 1$, since any stable set S satisfies $(w + \chi^f)(S) \leq w(S) + 1$. As $\tilde{\rho}_w \leq \tilde{\rho}_{w+\chi^f}$, this implies (13). *End of Proof of Claim 1.*

As of now we assume that w and f satisfy (13) and (14). Let f connect vertices u and u'. Since by the minimality of G, G has no isolated vertices, there exists a minimum-cost $w + \chi^f$-cover consisting only of edges and odd circuits, say, $e_1, \ldots, e_t$, $C_1, \ldots, C_k$. We choose f and $e_1, \ldots, e_t$, $C_1, \ldots, C_k$ such that

$$|VC_1| + \cdots + |VC_k| \tag{21}$$

is as small as possible. Then:

(22) at least two of the C_i traverse f.

To see this, let $G' := G - f$. If $\alpha_w(G') = \alpha_w(G)$, then by induction G' has a w-cover of cost α_w. As this is a w-cover in G as well, this would imply $\alpha_w = \tilde{\rho}_w$, a contradiction.

So $\alpha_w(G') > \alpha_w(G)$. That is, there exists a stable set S in G' with $w(S) > \alpha_w$. Necessarily, S contains both u and u'. Then for any circuit C traversing f:

$$|VC \cap S| \leq \lfloor \tfrac{1}{2}|VC| \rfloor + 1. \tag{23}$$

Also, f is not among $e_1,\ldots,e_t$, since otherwise $\tilde{\rho}_w \le \tilde{\rho}_{w+\chi^f} - 1$, contradicting (13). Setting l to the number of C_i traversing f, we obtain:

(24)
$$\tilde{\rho}_{w+\chi^f} \le \alpha_w + 1 \le w(S) = (w+\chi^f)(S) - 2 \le -2 + \sum_{j=1}^{t} |e_j \cap S| + \sum_{i=1}^{k} |VC_i \cap S|$$
$$\le -2 + t + \sum_{i=1}^{k} \lfloor \tfrac{1}{2}|VC_i| \rfloor + l = \tilde{\rho}_{w+\chi^f} + l - 2.$$

So $l \ge 2$, which is (22).

By (22) we can assume that C_1 and C_2 traverse f. It is convenient to assume that $EC_1 \setminus \{f\}$ and $EC_2 \setminus \{f\}$ are disjoint; this can be achieved by adding parallel edges. So $EC_1 \cap EC_2 = \{f\}$.

Then:

(25) if C is an odd circuit with $EC \subseteq EC_1 \cup EC_2$, then $f \in EC$ and $EC_1 \triangle EC_2 \triangle EC$ is again an odd circuit.

To see this, define $C_1' := C$. As $EC_1 \triangle EC_2 \triangle EC$ is an odd cycle (a *cycle* is an edge-disjoint union of circuits), it can be decomposed into circuits $C_2',\ldots,C_p'$, with $C_2',\ldots,C_q'$ odd and $C_{q+1}',\ldots,C_p'$ even ($q \ge 2$). Choose for each $i = q+1,\ldots,p$ a perfect matching M_i in C_i'. Let $e_1',\ldots,e_r'$ be the edges in the matchings M_i and in $\{f\} \setminus EC$. Then

(26)
$$\chi^{VC_1} + \chi^{VC_2} = \sum_{i=1}^{q} \chi^{VC_i'} + \sum_{j=1}^{r} \chi^{e_j'}$$

and

(27)
$$\lfloor \tfrac{1}{2}|VC_1| \rfloor + \lfloor \tfrac{1}{2}|VC_2| \rfloor = \tfrac{1}{2}|EC_1| + \tfrac{1}{2}|EC_2| - 1 = r - 1 + \tfrac{1}{2}\sum_{i=1}^{q} |EC_i'|$$
$$\ge r + \sum_{i=1}^{q} \lfloor \tfrac{1}{2}|VC_i'| \rfloor.$$

So replacing C_1, C_2 by $C_1',\ldots,C_q'$ and adding $e_1',\ldots,e_r'$ to $e_1,\ldots,e_t$, gives again a $w+\chi^f$-cover of cost at most $\tilde{\rho}_{w+\chi^f}$.

If $f \notin EC$, then f is among $e_1',\ldots,e_r'$. Hence deleting f gives a w-cover of cost at most $\tilde{\rho}_{w+\chi^f} - 1 \le \alpha_w$, contradicting (13). So $f \in EC$. As this is true for any odd circuit in $EC_1 \cup EC_2$ we know that $f \in EC_i'$ for $i = 1,\ldots,q$ and that $q = 2$.

If $p \ge 3$ or $r \ge 1$, then $|EC_1'| + |EC_2'| < |EC_1| + |EC_2|$, contradicting the minimality of (21). This proves (25).

First, it implies

(28) a circuit in $EC_1 \cup EC_2$ is odd if and only if it contains f.

A second consequence is as follows. Let P_i be the $u-u'$ path $C_i \setminus \{f\}$. Orient the edges occurring in the path $P_i := C_i \setminus \{f\}$ in the direction from u to u', for $i = 1, 2$. Then

(29) the orientation is acyclic.

For suppose there exists a directed circuit C. Then $(EC_1 \cup EC_2) \setminus EC$ contains a directed $u-u'$ path, and hence an odd circuit C'. Hence by (25), $EC_1 \triangle EC_2 \triangle EC'$ is an odd circuit, however containing the even circuit EC, a contradiction.

Let A and B be the colour classes of the bipartite graph $(VP_1 \cup VP_2, EP_1 \cup EP_2)$, such that $u, u' \in A$. So

(30) $$\begin{aligned} A &:= \{v \in VP_1 \cup VP_2 | \text{there exists an even-length directed } u-v \text{ path}\}, \\ B &:= \{v \in VP_1 \cup VP_2 | \text{there exists an odd-length directed } u-v \text{ path}\}. \end{aligned}$$

Define

(31) $$\begin{aligned} &X := VP_1 \cap VP_2 \text{ and} \\ &U := \{v \in V | w(v) = \sum_{j=1}^{t} |e_j \cap \{v\}| + \sum_{j=1}^{k} |VC_j \cap \{v\}|\}. \end{aligned}$$

We next show the following technical, but straightforward to prove, claim:

Claim 2.Let $z \in A$, let Q be an even length directed $u-z$ path, and let S be a stable set in G. Then

(32) $$(w - \chi^{VQ})(S) \geq \alpha_w - \lfloor \tfrac{1}{2}|VQ| \rfloor + 1$$

if and only if

(33)
(i) $|e_j \cap S| = 1$ *for each* $j = 1, \ldots, t$,
(ii) $|VC_j \cap S| = \lfloor \frac{1}{2}|VC_j| \rfloor$ *for* $j = 3, \ldots, k$,
(iii) $S \subseteq U$,
(iv) S *contains* $B \setminus VQ$ *and is disjoint from* $A \setminus VQ$,
(v) S *contains* $B \cap X$ *and is disjoint from* $A \cap X$.

Proof of Claim 2. We can assume that $EQ \subseteq EC_1$. Set $W := VC_1 \setminus VQ$. So $|W|$ is even. Consider the following sequence of (in)equalities:

(34) $$\begin{aligned} &(w - \chi^{VQ})(S) = w(S) - |VQ \cap S| \leq (w + \chi^f)(S) - |VQ \cap S| \\ &\leq \sum_{j=1}^{t} |e_j \cap S| + \sum_{j=1}^{k} |VC_j \cap S| - |VQ \cap S| = \sum_{j=1}^{t} |e_j \cap S| + \sum_{j=2}^{k} |VC_j \cap S| + |W \cap S| \\ &\leq t + \sum_{j=2}^{k} \lfloor \tfrac{1}{2}|VC_j| \rfloor + |W \cap S| = \tilde{\rho}_{w+\chi^f} - \lfloor \tfrac{1}{2}|VC_1| \rfloor + |W \cap S| \\ &\leq \tilde{\rho}_{w+\chi^f} - \lfloor \tfrac{1}{2}|VC_1| \rfloor + \tfrac{1}{2}|W| = \alpha_w + 1 - \lfloor \tfrac{1}{2}|VQ| \rfloor. \end{aligned}$$

Hence (32) holds if and only if equality holds throughout in (34), which is equivalent to (33). *End of Proof of Claim 2.*

By (29), we can order the vertices in X as $v_0 = u, v_1, \ldots, v_s = u'$ such that both P_1 and P_2 traverse them in this order. For $j = 0, \ldots, s$, let $\mathcal{P}_j$ be the collection of directed $u-x$ paths, where $x = v_j$ if $v_j \in A$ and x is an in-neighbour of v_j if $v_j \in B$. So $x \in A$.

Let j be the largest index for which there exists a path $Q \in \mathcal{P}_j$ with

$$(35) \qquad \alpha_{w-\chi^{VQ}} \leq \alpha_w - \lfloor \tfrac{1}{2}|VQ| \rfloor.$$

Such a j exists, since (35) holds for the trivial directed $u-u$ path, as $\alpha_{w-\chi^u} \leq \alpha_w$. Also, $j < s$, since otherwise $VQ = VC$ for some odd circuit C, and hence, with (14) we have

$$(36) \qquad \tilde{\rho}_w \leq \tilde{\rho}_{w-\chi^{VC}} + \lfloor \tfrac{1}{2}|VC| \rfloor = \alpha_{w-\chi^{VC}} + \lfloor \tfrac{1}{2}|VC| \rfloor \leq \alpha_w,$$

contradicting (13).

Let Q_1 and Q_2 be the two paths in $\mathcal{P}_{j+1}$ that extend Q. By the maximality of j, we know

$$(37) \qquad \alpha_{w-\chi^{VQ_i}} \geq \alpha_w - \lfloor \tfrac{1}{2}|VQ_i| \rfloor + 1.$$

Hence there exist stable sets S_1 and S_2 with

$$(38) \qquad (w - \chi^{VQ_i})(S_i) \geq \alpha_w - \lfloor \tfrac{1}{2}|VQ_i| \rfloor + 1$$

for $i = 1, 2$. So for $i = 1, 2$, (33) holds for Q_i, S_i. By (33)(iv), S_1 and S_2 coincide on $VP_1 \cup VP_2$ except on $VQ_1 \cup VQ_2$. In other words:

$$(39) \qquad (S_1 \triangle S_2) \cap (VP_1 \cup VP_2) \subseteq VQ_1 \cup VQ_2.$$

By (33)(v), S_1 and S_2 moreover coincide on X.

Let H be the subgraph of G induced by $S_1 \triangle S_2$. So H is a bipartite graph, with colour classes $S_1 \setminus S_2$ and $S_2 \setminus S_1$. Define

$$(40) \qquad Y_i := VQ_i \setminus VQ$$

for $i = 1, 2$. Then

(41) $\quad H$ contains a path connecting Y_1 and Y_2.

For suppose not. Let K be the union of the components of H that intersect Y_1. So K is disjoint from Y_2. Define $S := S_1 \triangle K$. Then $S \cap Y_1 = S_2 \cap Y_1$ and $S \cap Y_2 = S_1 \cap Y_2$. This implies that Q, S satisfy (33). Hence (32) holds, contradicting (35). This proves (41).

Let C be the (even) circuit formed by the two directed $v_j - v_{j+1}$ paths. So Y_1 and Y_2 are subsets of VC. Let R be a shortest path in H that connects Y_1 and Y_2; say it connects $y_1 \in Y_1$ and $y_2 \in Y_2$.

Since $y_1, y_2 \in S_1 \triangle S_2$, we know by (33)(v) that $y_1, y_2 \notin X$. By (33)(iv), if $y_1 \in S_1 \setminus S_2$ then $y_1 \in A$ and if $y_1 \in S_2 \setminus S_1$ then $y_1 \in B$. Similarly, if $y_2 \in S_2 \setminus S_1$ then $y_2 \in A$ and if $y_2 \in S_1 \setminus S_2$ then $y_2 \in B$.

So if R is even then y_1 and y_2 belong to different sets A, B, and if R is odd then y_1 and y_2 belong to the same set among A, B. Hence R forms with part of C an odd circuit.

By (39) and as $(S_1 \triangle S_2) \cap X = \emptyset$, there exist a directed $u - v_j$ path N' and a directed $v_{j+1} - u'$ path N'' that are (vertex-)disjoint from $S_1 \triangle S_2$. N', N'', and f make a $v_{j+1} - v_j$ path N. Then N, R, and C make an odd K_4, with 3-valent vertices v_j, v_{j+1}, y_1, y_2.

By assumption, it is not a bad K_4; that is, it satisfies (4). Suppose first that R has even length. Then by (4) also N has even length. Hence v_j and v_{j+1} belong to different sets

A, B. Then by (4) and the symmetry of y_1 and y_2, we may assume that y_1 is adjacent to v_j and that y_2 is adjacent to v_{j+1}. Hence, as $y_1, y_2 \in S_1 \cup S_2$, v_j and v_{j+1} do not belong to $S_1 \cap S_2$, and so $v_j, v_{j+1} \notin B$ (by (33)(v)), a contradiction.

So R has length 1. Hence N has length 1 as well, and v_j, v_{j+1}, y_1, y_2 lie in the same colour class of the bipartition A, B of C. So we know:

(42) $\quad v_j = u$, $v_{j+1} = u'$, $y_1, y_2 \in A$, and R has length 1.

Let D be the set of edges of G connecting two vertices in A. So $f \in D$ and $y_1 y_2 \in D$. Hence $|D| \geq 2$. We consider the edges in D as chords of the circuit C with $EC = EP_1 \cup EP_2$.

Now any edge d in D can play the same role as f, since, if C_1' and C_2' denote the two odd circuits in $EC \cup \{d\}$, then:

(43) $\quad C_1', C_2', C_3, \ldots, C_k, e_1, \ldots, e_t$ form a $w + \chi^d$-cover of cost $\tilde{\rho}_{w+\chi^d} = \tilde{\rho}_{w+\chi^f}$.

Indeed, as $\chi^{C_1'} + \chi^{C_2'} = \chi^d + \chi^{C_1} + \chi^{C_2} - \chi^f$, the collection $C_1', C_2', C_3, \ldots, C_k, e_1, \ldots, e_t$ is a $w + \chi^d$-cover of cost $\tilde{\rho}_{w+\chi^f}$ with $|VC_1'| + |VC_2'| + |VC_3| + \cdots + |VC_k|$ at most (21). Hence (43) follows from the choice of f.

So each $d \in D$ has all the properties derived for f so far and it would lead to the same circuit C and to the same bipartition A, B of C.

This is used to prove:

(44) $\quad$ any edge in D crosses any chord of C.

Indeed, we only need to prove this for f. However, by the minimality of (21) all circuits among $C_1, \ldots, C_k$ are chordless, so each chord of C crosses f.

Let $n := |D|$, and let $s_1, s_2, \ldots, s_{2n}$ be the ends of the edges in D, in cyclic order. Let $f_1, \ldots, f_{2n}$ be the edges in D incident with $s_1, \ldots, s_{2n}$, respectively. So $f_{n+j} = f_j$ for all j (taking indices mod $2n$). For $j = 1, \ldots, 2n$, let R_j be the $s_{j-1} - s_j$ path along C that does not contain any other of the vertices s_i.

By the Lemma, we can assume that $2, \ldots, n \notin J$, where J is as defined in (8). Let Q_1 be the path of the form $Q = R_{j+1} R_{j+2} \cdots R_n$ with $0 \leq j \leq n$ such that

(45) $\quad \alpha_{w-\chi^{VQ}} \geq \alpha_w - \lfloor \frac{1}{2} |VQ| \rfloor + 1,$

and such that j is maximal. This path exists, since for $j = 0$ we have (45), as otherwise (36) would again yield a contradiction.

Trivially, $j < n$, since the empty path does not satisfy (45). Let $Q_2 := R_{j+2} R_{j+3} \cdots R_{j+1+n}$. Since also Q_2 satisfies (45) (as again, (36) would yield a contradiction otherwise), there exist stable sets S_1 and S_2 with

(46) $\quad (w - \chi^{VQ_i})(S_i) \geq \alpha_w - \lfloor \frac{1}{2} |VQ_i| \rfloor + 1$

for $i = 1, 2$. So for $i = 1, 2$, (33) holds for Q_i, S_i where we can take for f any edge not incident with an internal vertex of Q_i. By (33)(iv),

(47) $\quad (S_1 \triangle S_2) \cap VC \subseteq VQ_1 \cup VQ_2.$

We (re)define H as the subgraph of G induced by $S_1 \triangle S_2$. Define

(48) $\quad Y_1 := VR_{j+1}$ and $Y_2 := VR_{n+1} \cup VR_{n+2} \cup \cdots \cup VR_{n+j+1}.$

Then

(49) H contains a path connecting Y_1 and Y_2.

For suppose not. Let K be the union of the components of H that intersect Y_1. So K is disjoint from Y_2. Define $S := S_1 \triangle K$. Then $S \cap Y_1 = S_2 \cap Y_1$ and $S \cap Y_2 = S_1 \cap Y_2$. This implies that $Q := R_{j+2}R_{j+3} \cdots R_n$ and S satisfy (33), taking $f := f_n$. Hence (32) holds for Q, contradicting the maximality of j. This proves (49).

Let R be a shortest path in H that connects Y_1 and Y_2; say it connects $y_1 \in Y_1$ and $y_2 \in Y_2$. By (47), any internal vertex of R that is on C, is an internal vertex of $R_{j+2}R_{j+3} \cdots R_n$. If $y_1 \in S_1 \setminus S_2$, as y_1 is not an internal vertex of Q_2, we know $y_1 \in A$. Similarly, if $y_1 \in S_2 \setminus S_1$, then $y_1 \in B$. Similarly, if $y_2 \in S_2 \setminus S_1$, then $y_2 \in A$, and if $y_2 \in S_1 \setminus S_2$, then $y_2 \in B$. So R together with the $y_1 - y_2$ part of $R_{j+1}R_{j+2} \cdots R_{n+j+1}$ forms an odd cycle. Hence it contains an odd circuit, and so R contains an odd bow. By (7), this bow connects two vertices in some $R_{j+2}, \ldots, R_n$. This contradicts the fact that $j+2, \ldots, n \notin J$. ∎

Figure 1 gives a strongly t-perfect graph that contains a bad K_4. So the implication in the Theorem cannot be reversed. However one has:

Corollary. *For any graph G, the following are equivalent:*

(50) (i) *G contains no bad K_4;*

(ii) *each subgraph of G is t-perfect;*

(iii) *each subgraph of G is strongly t-perfect.*

Proof. The implication (i)⇒(iii) follows from the Theorem, while the implication (iii)⇒(ii) follows by the observations made in Section 1.

The implication (ii)⇒(i) was proved by Barahona and Mahjoub [1]. It suffices to show that a bad K_4 is not t-perfect. Choose a smallest counterexample G. As G is t-perfect, $G \neq K_4$. If (4) does not hold then G has a vertex v such that contracting the edges in $\delta(v)$ gives an odd K_4-subdivision G' that again does not satisfy (4). As G' is again a t-perfect odd K_4 (as one easily checks), this contradicts the minimality of G. ∎

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