Counting 1-factors in regular bipartite graphs

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Abstract. We show that any k-regular bipartite graph with $2n$ vertices has at least

$$\left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n$$

perfect matchings (1-factors). Equivalently, this is a lower bound on the permanent of any nonnegative integer $n \times n$ matrix with each row and column sum equal to k.

For any k, the base $\frac{(k-1)^{k-1}}{k^{k-2}}$ is largest possible.

1. Introduction

In this paper we show that any k-regular bipartite graph with $2n$ vertices has at least

$$(1) \qquad \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n$$

perfect matchings. (A *perfect matching* or 1-*factor* is a set of disjoint edges covering all vertices.) This generalizes a result of Voorhoeve [11] for the case $k = 3$, stating that any 3-regular bipartite graph with $2n$ vertices has at least $(\frac{4}{3})^n$ perfect matchings.

The base in (1) is best possible for any k: let α_k be the largest real number such that any k-regular bipartite graph with $2n$ vertices has at least $(\alpha_k)^n$ perfect matchings; then

$$(2) \qquad \alpha_k = \frac{(k-1)^{k-1}}{k^{k-2}}.$$

Here, the inequality $\leq$ was shown in [10], where moreover equality was conjectured for all k. That this conjecture is true is thus the result of the present paper. For completeness, we sketch the argument showing (2) in Section 3 below.

The result can be equivalently stated in terms of permanents (for the definition of permanent, see Section 4 below): the permanent of any nonnegative integer $n \times n$ matrix with each row and column sum equal to k, is at least (1).

The result of Voorhoeve [11] for the case $k = 3$ answered a question posed by Erdős and Rényi [3]: is there an $\varepsilon > 0$ such that the permanent of any nonnegative integer $n \times n$ matrix with all row and column sums equal to 3 is at least $(1 + \varepsilon)^n$? So Voorhoeve's result shows that one can take $\varepsilon = \frac{1}{3}$.

Voorhoeve's result was obtained before Van der Waerden's permanent conjecture was resolved, in 1981. This conjecture states that the permanent of any doubly stochastic $n \times n$ matrix is at least $\frac{n!}{n^n}$. (A matrix is *doubly stochastic* if it is nonnegative and each row and column sum is equal to 1.) Van der Waerden's conjecture was proved by Falikman [4] and a sharper version by Egorychev [2].

Van der Waerden's bound implies that for any k, n, the permanent of any nonnegative integer $n \times n$ matrix A with all row and column sums equal to k, is at least

$$(3) \qquad \frac{k^n n!}{n^n},$$

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since the matrix $\frac{1}{k}A$ is doubly stochastic. Bound (3) is at least $(\frac{k}{\mathrm{e}})^n$. Since $\frac{3}{\mathrm{e}} > 1$, it implies the Erdős-Rényi conjecture. Also Bang [1] and Friedland [5] showed the Erdős-Rényi conjecture by proving that any doubly stochastic $n \times n$ matrix has permanent at least e^{-n}. Since

$$(4) \qquad \frac{(k-1)^{k-1}}{k^{k-2}} \geq \frac{k}{\mathrm{e}}$$

for each k, also the bound (1) implies that the permanent of any doubly stochastic $n \times n$ matrix is at least e^{-n}.

The proof of Voorhoeve [11] for the case $k = 3$ is very elegant and simple (see, for instance, Lovász and Plummer [6] pp.313–314). Compared to the simplicity of Voorhoeve's method and of the general statement, our method is quite complicated. Yet, the method forms a generalization of Voorhoeve's method. In fact, it generalizes bound (1) to weighted bipartite graphs, so as to enable induction. Although it leads to slightly complicated formulas, they all are quite natural and precise for our purposes. Nevertheless, the question remains if a simpler proof could be given.

Another question is whether there is a common generalization of the Van der Waerden bound and the bound given in this paper. For any k, n, let $p(k, n)$ be the minimum number of perfect matchings in any k-regular bipartite graph with $2n$ vertices. Then the Van der Waerden bound states that for each n one has

$$(5) \qquad \inf_{k \in \mathbb{N}} \frac{p(k,n)}{k^n} = \frac{n!}{n^n},$$

while our bound states that for each k one has

$$(6) \qquad \inf_{n \in \mathbb{N}} p(k, n)^{1/n} = \frac{(k-1)^{k-1}}{k^{k-2}}.$$

So both bounds are best possible, in different asymptotic directions. It might be possible to derive a sharper lower bound for $p(k, n)$ with the methods of the present paper.

We give our main theorem and its proof in Section 2, after which we derive bound (1) in Section 3. The theorem also implies a bound on the permanent of certain matrices derived from doubly stochastic matrices, which we discuss in Section 4. Finally, in Section 5 we observe that our bound also gives tight bounds for the number of 1-factorizations (edge-colourings) of regular bipartite graphs, conjectured in [9].

In this paper, a bipartite graph $G = (V, E)$ can have multiple edges. For any vertex v, the set of edges incident with v is denoted by $\delta(v)$. For any function $w : E \to \mathbb{Z}_+$, we generally put w_e for $w(e)$ $(e \in E)$, and

$$(7) \qquad w(F) = \sum_{e \in F} w_e$$

for any $F \subseteq E$. For any $e \in E$, χ^e denotes the function $\chi^e : E \to \{0, 1\}$ with $\chi^e(f) = 1$ if and only if $f = e$.

2. The main theorem and proof

We now formulate and prove a theorem that implies bound (1). In this section we fix k. Let $G = (V, E)$ be a bipartite graph, and let $w : E \longrightarrow \mathbb{Z}_+$. For any perfect matching M in G define

$$(8) \qquad \phi(w, M) := \prod_{e \in M} w_e(k - w_e).$$

Next let

$$\tau(w) := \sum_M \phi(w, M), \tag{9}$$

where the summation extends over all perfect matchings M in G. So $\tau(w)$ is equal to the number of perfect matchings in the graph obtained from G by replacing each edge e by $w_e(k - w_e)$ parallel edges (assuming $w_e \leq k$).

Call w *k-regular* if $w(\delta(v)) = k$ for each $v \in V$.

Theorem 1. *For any bipartite graph $G = (V, E)$ and any k-regular $w : E \to \mathbb{Z}_+$,*

$$\tau(w) \geq k^{|V|-|E|} \prod_{e \in E} (k - w_e). \tag{10}$$

Proof. We prove a generalization. Call a function $w : E \to \mathbb{Z}_+$ a 1-*weighting* if either w is k-regular or there exist two vertices t and u such that $w(\delta(t)) = k-1$, $w(\delta(u)) = k+1$, and $w(\delta(v)) = k$ for all $v \neq t, u$. (Necessarily, t and u belong to the same colour class of G.)

Call $w : E \to \mathbb{Z}_+$ a -1-*weighting* if there exist two vertices t and u such that $w(\delta(t)) = w(\delta(u)) = k - 1$, and $w(\delta(v)) = k$ for all $v \neq t, u$. (Necessarily, t and u belong to different colour classes of G.)

Note that any α-weighting can be obtained as follows from a k-regular $w : E \to \mathbb{Z}_+$. Choose a simple path P in G, with edges $e_1, \ldots, e_t$, in this order (possibly $t = 0$), such that $w_e > 0$ if $e = e_i$ for odd $i \leq t$. Now reset $w_e := w_e - 1$ if $e = e_i$ for some odd $i \leq t$ and $w_e := w_e + 1$ if $e = e_i$ for some even $i \leq t$. Then the resulting w is an α-weighting with $\alpha := (-1)^t$.

Let $\alpha \in \{+1, -1\}$. For any α-weighting w define

$$\beta(w) := \tfrac{k+\alpha}{k+1} k^{|V|-|E|} \prod_{e \in E} (k - w_e). \tag{11}$$

We show that for any bipartite graph $G = (V, E)$, any $\alpha \in \{+1, -1\}$, and any α-weighting $w : E \to \mathbb{Z}_+$, one has

$$\tau(w) \geq \beta(w). \tag{12}$$

This implies the theorem.

Suppose (12) does not hold. Choose a graph $G = (V, E)$ for which there exist α, w violating (12), with $|E|$ minimal. Then G is connected, since otherwise a component of G will give a smaller counterexample.

Having G, we choose α, w violating (12) so that the quotient

$$\frac{\tau(w)}{\beta(w)} \tag{13}$$

is minimized (this is possible, as $\beta(w) > 0$). We call any w attaining this minimum *minimizing*.

If possible, we choose w, among all minimizing w, such that w is k-regular; otherwise, we choose w such that the two vertices v with $w(\delta(v)) \neq k$ have minimum distance in G. (So the path P described above is minimized.)

Since we can delete edges e with $w_e = 0$, we know that $w_e \geq 1$. Since $\beta(w) = 0$ if $w_e = k$ for some edge e, we know that $w_e \leq k-1$. So it follows that $k \geq 2$.

For any edge e let

$$\tau(w,e) := \sum_{M \ni e} \phi(w,M). \tag{14}$$

So for each vertex v one has

$$\sum_{e \in \delta(v)} \tau(w,e) = \tau(w). \tag{15}$$

Let u be a vertex satisfying $w(\delta(u)) = k + \alpha$, if it exists, and let u be any vertex otherwise. (So $\alpha = 1$ and $w(\delta(u)) = k$ in the latter case.) Then $w - \alpha\chi^e$ is a $-\alpha$-weighting for any edge $e \in \delta(u)$.

Claim 1. For each edge $e \in \delta(u)$,

$$\alpha(k - 2w_e + \alpha)\tau(w,e) \leq \alpha(k - 2w_e + \alpha)\tfrac{w_e}{k+\alpha}\tau(w). \tag{16}$$

If equality holds, then $w - \alpha\chi^e$ is minimizing.

Proof. Since $w - \alpha\chi^e$ is a $-\alpha$-weigting and since w is minimizing we have

$$\tau(w - \alpha\chi^e) \geq \frac{\beta(w - \alpha\chi^e)}{\beta(w)}\tau(w) = \tfrac{k-\alpha}{k+\alpha} \cdot \tfrac{k-w_e+\alpha}{k-w_e}\tau(w). \tag{17}$$

Moreover, we can express $\tau(w - \alpha\chi^e)$ in terms of $\tau(w)$ and $\tau(w,e)$:

$$\tau(w - \alpha\chi^e) = \tau(w) + (\tfrac{(w_e-\alpha)(k-w_e+\alpha)}{w_e(k-w_e)} - 1)\tau(w,e) = \tau(w) - \alpha\tfrac{k-2w_e+\alpha}{w_e(k-w_e)}\tau(w,e). \tag{18}$$

Combining (17) and (18) gives:

$$\begin{aligned} &\alpha(k - 2w_e + \alpha)\tau(w,e) = w_e(k - w_e)(\tau(w) - \tau(w - \alpha\chi^e)) \leq \\ &w_e(k - w_e)\tau(w)(1 - \tfrac{k-\alpha}{k+\alpha}\tfrac{k-w_e+\alpha}{k-w_e}) = \\ &w_e(k - w_e)\tau(w)\tfrac{(k^2+\alpha k-kw_e-\alpha w_e)-(k^2-\alpha^2-kw_e+\alpha w_e)}{(k+\alpha)(k-w_e)} = \\ &w_e(k - w_e)\tau(w)\tfrac{\alpha(k-2w_e+\alpha)}{(k+\alpha)(k-w_e)} = \alpha(k - 2w_e + \alpha)\tfrac{w_e}{k+\alpha}\tau(w). \end{aligned} \tag{19}$$

As equality in (19) implies equality in (17), this shows Claim 1. □

From this we derive:

Claim 2. There exists an edge $e \in \delta(u)$ satisfying

$$w_e \geq \tfrac{1}{2}(k + \alpha) \text{ and } \alpha \cdot \tau(w,e) > \alpha\tfrac{w_e}{k+\alpha}\tau(w). \tag{20}$$

Proof. Suppose not. Then by Claim 1,

$$\alpha \cdot \tau(w,e) \leq \alpha\tfrac{w_e}{k+\alpha}\tau(w) \tag{21}$$

for each $e \in \delta(u)$ (since if $w_e < \frac{1}{2}(k+\alpha)$, (16) amounts to (21)). Hence

$$\alpha \cdot \tau(w) = \alpha \sum_{e\in\delta(u)} \tau(w,e) \leq \alpha \sum_{e\in\delta(u)} \tfrac{w_e}{k+\alpha}\tau(w) = \alpha\tfrac{w(\delta(u))}{k+\alpha}\tau(w) \leq \alpha\cdot\tau(w), \tag{22}$$

since $\alpha\cdot w(\delta(u)) \leq \alpha(k+\alpha)$. So equality holds throughout in (22), implying $w(\delta(u)) = k+\alpha$ and implying equality in (16) for each $e \in \delta(u)$. So by Claim 1, $w - \alpha\chi^e$ is minimizing for each $e \in \delta(u)$.

Now let e be the first edge of the shortest path connecting u with the vertex $v \neq u$ satisfying $w(\delta(v)) = k-1$. (w is not k-regular since $w(\delta(u)) = k+\alpha$.) Then replacing w by $w - \alpha\chi^e$ we obtain a minimizing $-\alpha$-weighting which is either k-regular or has a shorter distance between the vertices v with $w(\delta(v)) \neq k$, contradicting our assumption. □

Let us fix edge e as in Claim 2, and let e connect vertex u with vertex v (thus fixing v from here on in this proof). Let F be the set of edges $f \neq e$ incident with v. Then

Claim 3. $w_f < k - w_e$ *for each* $f \in F$.

Proof. For if not, then $w_f + w_e \geq k$, implying that $w(\delta(v)) = k$ (since $w(\delta(v)) \leq k$), and that e and f are the only edges incident with v. So $w_f = k - w_e$.

If e and f are not parallel, we can contract them and obtain a graph G' with a smaller number of edges, and an α-weighting w'. Then by the minimality of G we know $\tau(w') \geq \beta(w')$, and hence

$$\tau(w) = w_e w_f \tau(w') \geq w_e w_f \beta(w') = \beta(w), \tag{23}$$

again contradicting the fact that w gives a counterexample.

If e and f are parallel and form the whole graph, then $\alpha = 1$ and $\tau(w) = w_e(k-w_e) + w_f(k-w_f) = 2w_e w_f$. So

$$\tau(w) = 2w_e w_f \geq w_e w_f = \beta(w), \tag{24}$$

contradicting the fact that we have a counterexample.

If e and f parallel and do not form the whole graph, then $w(\delta(u)) = k+1$ for the vertex u adjacent to v. Hence $\alpha = 1$. Then deleting u and v, and the edges incident with u and v, we obtain a graph G' with -1-weighting w'. Since G is a counterexample with $|E|$ smallest, we know that $\tau(w') \geq \beta(w')$. However, $\tau(w) = 2w_e w_f \tau(w')$ and hence

$$\tau(w) = 2w_e w_f \tau(w') \geq 2w_e w_f \beta(w') \geq \tfrac{k+1}{k-1}\tfrac{k-1}{k} w_e w_f \beta(w') = \beta(w), \tag{25}$$

contradicting the fact that w gives a counterexample. □

Since $w_e \geq \frac{1}{2}(k+\alpha)$, Claim 3 implies

$$w_f < \tfrac{1}{2}(k-\alpha);\ \text{equivalently, } k - 2w_f - \alpha > 0. \tag{26}$$

So we can define for any $f \in F$,

$$\lambda_f := -\tfrac{k-2w_e+\alpha}{w_e(k-w_e)} \cdot \tfrac{w_f(k-w_f)}{k-2w_f-\alpha}. \tag{27}$$

By (26) and (20) we have that $\lambda_f \geq 0$ for each $f \in F$. Moreover, one has:

Claim 4. $\sum_{f \in F} \lambda_f < 1.$

Proof. Since this is trivial if $k - 2w_e + \alpha = 0$ (in which case $\lambda_f = 0$ for all $f \in F$), we can assume $k - 2w_e + \alpha < 0$. So we must prove

$$\sum_{f \in F} \tfrac{w_f(k-w_f)}{k-2w_f-\alpha} < -\tfrac{w_e(k-w_e)}{k-2w_e+\alpha}. \tag{28}$$

To prove this, first observe that the function $h(x) = \frac{x(k-x)}{k-2x-\alpha}$ satisfies $h(0) = 0$ and is strictly increasing and strictly convex for $x < \frac{1}{2}(k-\alpha)$, since

$$\begin{aligned} h'(x) &= \tfrac{(k-2x)(k-2x-\alpha)+2x(k-x)}{(k-2x-\alpha)^2} = \tfrac{(k^2-4kx+4x^2-\alpha k+2\alpha x)+(2kx-2x^2)}{(k-2x-\alpha)^2} = \\ &\tfrac{k^2-2kx+2x^2-\alpha k+2\alpha x}{(k-2x-\alpha)^2} = \tfrac{\frac{1}{2}(k-2x-\alpha)^2+\frac{1}{2}(k^2-\alpha^2)}{(k-2x-\alpha)^2} = \tfrac{1}{2} + \tfrac{\frac{1}{2}(k^2-\alpha^2)}{(k-2x-\alpha)^2}, \end{aligned} \tag{29}$$

and therefore $h'(x)$ is positive and strictly increasing for $x < \frac{1}{2}(k-\alpha)$.

Since $\sum_{f \in F} w_f = w(\delta(v)) - w_e \le k - w_e$, the strict monotonicity and strict convexity of h imply that

$$\sum_{f \in F} \tfrac{w_f(k-w_f)}{k-2w_f-\alpha} < \tfrac{(k-w_e)w_e}{k-2(k-w_e)-\alpha} = \tfrac{w_e(k-w_e)}{-k+2w_e-\alpha} \tag{30}$$

(the inequality is strict because of Claim 3), which is (28). □

We now finish the proof by deriving a contradiction. For each $f \in F$, $w - \alpha\chi^e + \alpha\chi^f$ is an α-weighting. Hence, since w is minimizing, we have:

$$\tau(w - \alpha\chi^e + \alpha\chi^f) \ge \frac{\beta(w - \alpha\chi^e + \alpha\chi^f)}{\beta(w)}\tau(w) = \tfrac{(k-w_e+\alpha)(k-w_f-\alpha)}{(k-w_e)(k-w_f)}\tau(w). \tag{31}$$

Moreover, we can express $\tau(w - \alpha\chi^e + \alpha\chi^f)$ in terms of $\tau(w)$, $\tau(w,e)$, and $\tau(w,f)$:

$$\begin{aligned} &\tau(w - \alpha\chi^e + \alpha\chi^f) = \\ &\tau(w) + (\tfrac{(w_e-\alpha)(k-w_e+\alpha)}{w_e(k-w_e)} - 1)\tau(w,e) + (\tfrac{(w_f+\alpha)(k-w_f-\alpha)}{w_f(k-w_f)} - 1)\tau(w,f) = \\ &\tau(w) + \tfrac{-\alpha(k-2w_e+\alpha)}{w_e(k-w_e)}\tau(w,e) + \tfrac{\alpha(k-2w_f-\alpha)}{w_f(k-w_f)}\tau(w,f). \end{aligned} \tag{32}$$

Combining (31) and (32) then gives a bound for $\tau(w,f)$ in terms of $\tau(w)$ and $\tau(w,e)$:

$$\begin{aligned} &\alpha \cdot \tau(w,f) = \tfrac{w_f(k-w_f)}{k-2w_f-\alpha}(\tau(w - \alpha\chi^e + \alpha\chi^f) - \tau(w)) - \alpha\lambda_f\tau(w,e) \ge \\ &\tfrac{w_f(k-w_f)}{k-2w_f-\alpha}(\tfrac{(k-w_e+\alpha)(k-w_f-\alpha)}{(k-w_e)(k-w_f)} - 1)\tau(w) - \alpha\lambda_f\tau(w,e) = \\ &\tfrac{w_f(k-w_f)}{k-2w_f-\alpha}\tfrac{\alpha(w_e-w_f-\alpha)}{(k-w_e)(k-w_f)}\tau(w) - \alpha\lambda_f\tau(w,e) = \alpha\tfrac{w_f(w_e-w_f-\alpha)}{(k-2w_f-\alpha)(k-w_e)}\tau(w) - \alpha\lambda_f\tau(w,e). \end{aligned} \tag{33}$$

Hence, using (15), (20) and Claim 4, we obtain the following contradiction:

$$\begin{aligned} &\alpha \cdot \tau(w) = \alpha \cdot \tau(w,e) + \sum_{f \in F} \alpha \cdot \tau(w,f) \ge \\ &\alpha \cdot \tau(w,e) + \sum_{f \in F} \alpha \cdot (\tfrac{w_f(w_e-w_f-\alpha)}{(k-2w_f-\alpha)(k-w_e)}\tau(w) - \lambda_f\tau(w,e)) = \\ &\alpha \sum_{f \in F} \tfrac{w_f(w_e-w_f-\alpha)}{(k-w_e)(k-2w_f-\alpha)}\tau(w) + \alpha(1 - \sum_{f \in F}\lambda_f)\tau(w,e) > \\ &\alpha \sum_{f \in F} \tfrac{w_f(w_e-w_f-\alpha)}{(k-w_e)(k-2w_f-\alpha)}\tau(w) + \alpha(1 - \sum_{f \in F}\lambda_f)\tfrac{w_e}{k+\alpha}\tau(w) \ge \alpha \cdot \tau(w). \end{aligned} \tag{34}$$

The last inequality can be seen as follows. First we have for each $f \in F$,

$$(35) \qquad \frac{w_f(w_e-w_f-\alpha)}{(k-w_e)(k-2w_f-\alpha)} - \lambda_f \frac{w_e}{k+\alpha} = w_f \frac{k-w_e+\alpha}{(k-w_e)(k+\alpha)},$$

as follows directly from definition (27) of λ_f. (Indeed,

$$(36) \qquad \frac{w_f(w_e-w_f-\alpha)}{(k-w_e)(k-2w_f-\alpha)} - \lambda_f \frac{w_e}{k+\alpha} = \frac{w_f(w_e-w_f-\alpha)}{(k-w_e)(k-2w_f-\alpha)} + \frac{w_f(k-2w_e+\alpha)(k-w_f)}{(k-w_e)(k-2w_f-\alpha)(k+\alpha)} =$$
$$w_f \frac{(w_e-w_f-\alpha)(k+\alpha)+(k-2w_e+\alpha)(k-w_f)}{(k-w_e)(k-2w_f-\alpha)(k+\alpha)} =$$
$$w_f \frac{(kw_e-kw_f-\alpha k+\alpha w_e-\alpha w_f-\alpha^2)+(k^2-2kw_e+\alpha k-kw_f+2w_ew_f-\alpha w_f)}{(k-w_e)(k-2w_f-\alpha)(k+\alpha)} =$$
$$w_f \frac{k^2-kw_e-2kw_f+2w_ew_f-2\alpha w_f+\alpha w_e-\alpha^2}{(k-w_e)(k-2w_f-\alpha)(k+\alpha)} = w_f \frac{(k-2w_f-\alpha)(k-w_e+\alpha)}{(k-w_e)(k-2w_f-\alpha)(k+\alpha)} =$$
$$w_f \frac{k-w_e+\alpha}{(k-w_e)(k+\alpha)}.)$$

Now (35) gives, using the inequality $\alpha w(\delta(v)) \geq \alpha k$,

$$(37) \qquad \alpha \sum_{f\in F} \left(\frac{w_f(w_e-w_f-\alpha)}{(k-w_e)(k-2w_f-\alpha)} - \lambda_f \frac{w_e}{k+\alpha}\right) = \alpha \frac{k-w_e+\alpha}{(k-w_e)(k+\alpha)} \sum_{f\in F} w_f =$$
$$\alpha \frac{(k-w_e+\alpha)(w(\delta(v))-w_e)}{(k-w_e)(k+\alpha)} \geq \alpha \frac{(k-w_e+\alpha)(k-w_e)}{(k-w_e)(k+\alpha)} = \alpha\left(1-\frac{w_e}{k+\alpha}\right),$$

implying the last inequality in (34). As (34) is a contradiction, there is no counterexample to (12). ∎

3. Derivation of bound (1)

Corollary 1a. *Any k-regular bipartite graph $G=(V,E)$ with $2n$ vertices has at least*

$$(38) \qquad \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n$$

perfect matchings.

Proof. Define $w: E \longrightarrow \mathbb{Z}_+$ by $w_e = 1$ for each $e \in E$. So w is k-regular in the sense of Section 2. Now $\tau(w)$ is equal to $(k-1)^n$ times the number of perfect matchings in G (since $w_e(k-w_e) = k-1$ for each edge e). Moreover,

$$(39) \qquad k^{|V|-|E|} \prod_{e\in E} (k-w_e) = \left(\frac{(k-1)^k}{k^{k-2}}\right)^n.$$

So Theorem 1 implies the corollary. ∎

We sketch a proof that the base in (38) is best possible; that is, we show (2). Fix k and n. Let Π be the set of permutations of $\{1,\ldots,kn\}$. For any $\pi \in \Pi$, let G_π be the bipartite graph with vertices $u_1,\ldots,u_n,v_1,\ldots,v_n$ and edges $e_1,\ldots,e_{kn}$, where

$$(40) \qquad e_i \text{ connects } u_{\lceil \frac{i}{k} \rceil} \text{ and } v_{\lceil \frac{\pi(i)}{k} \rceil}$$

for $i = 1, \ldots, kn$. (Here $\lceil x \rceil$ denotes the upper integer part of x.) So G_π is a k-regular bipartite graph with $2n$ vertices. Hence, by definition of α_k,

$$(41) \qquad \sigma(G_\pi) \geq (\alpha_k)^n,$$

where $\sigma(G_\pi)$ denotes the number of perfect matchings in G_π.

On the other hand,

$$(42) \qquad \sum_{\pi \in \Pi} \sigma(G_\pi) = k^n k^n n!((k-1)n)!.$$

This can be seen as follows. The left hand side is equal to the number of pairs (π, I), where π is a permutation of $\{1 \ldots, kn\}$ and where I is a subset of $\{1, \ldots, kn\}$ such that $\{e_i | i \in I\}$ forms a perfect matching in G_π; that is, such that

$$(43) \qquad \begin{array}{ll} \text{(i)} & |I \cap \{jk - k + 1, \ldots, jk\}| = 1 \text{ for each } j = 1, \ldots, n, \\ \text{(ii)} & |\pi(I) \cap \{jk - k + 1, \ldots, jk\}| = 1 \text{ for each } j = 1, \ldots, n. \end{array}$$

Now by first choosing I satisfying (43)(i) (which can be done in k^n ways), and next choosing a permutation π of $\{1, \ldots, kn\}$ satisfying (43)(ii) (which can be done in $k^n n!((k-1)n)!$ ways), we obtain (42).

Since $|\Pi| = (kn)!$, (41) and (42) imply

$$(44) \qquad \alpha_k \leq \left(\frac{k^{2n} n!((k-1)n)!}{(kn)!}\right)^{1/n}$$

yielding (2), with Stirling's formula.

4. Consequences on permanents

Our result can also be expressed in terms of permanents. Recall that for any $n \times n$ matrix $A = (a_{i,j})$, the *permanent* per$A$ is defined as

$$(45) \qquad \text{per}A := \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)},$$

where the summation extends over all permutations π of $\{1, \ldots, n\}$. (For background on permanents, see Minc [7], [8].)

Then we have:

Corollary 1b. *Let $A = (a_{i,j})$ be a nonnegative integer $n \times n$ matrix with each row and column sum equal to k. Then*

$$(46) \qquad \text{per}A \geq \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n.$$

Proof. Make a bipartite graph G with vertex set $u_1, \ldots, u_n, v_1, \ldots, v_n$, where u_i and v_j are connected by $a_{i,j}$ edges (parallel if $a_{i,j} \geq 2$). Then per(A) is equal to the number of perfect matchings in G, and hence Corollary 1a implies the present corollary. ∎

Our more general Theorem 1 implies another theorem on permanents. For any real number a let $\tilde{a} := a(1-a)$, and for any matrix $A = (a_{i,j})$ let

$$\tilde{A} := (\tilde{a}_{i,j}). \tag{47}$$

Corollary 1c. *For any doubly stochastic $n \times n$ matrix $A = (a_{i,j})$,*

$$\mathrm{per}\tilde{A} \geq \prod_{i=1}^{n}\prod_{j=1}^{n}(1-a_{i,j}). \tag{48}$$

Proof. By continuity, we can assume that A is rational. Hence there exists a natural number k such that kA is an integer matrix, with all row and column sums equal to k.

Let $G = (V, E)$ be the complete bipartite graph with colour classes $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$. Let $w : E \to \mathbb{Z}_+$ be defined by $w_e := ka_{i,j}$ for the edge e connecting u_i and v_j $(i, j = 1, \ldots, n)$. So w is k-regular, and hence by Theorem 1,

$$k^{2n}\mathrm{per}\tilde{A} = \tau(w) \geq k^{|V|-|E|}\prod_{e\in E}(k-w_e) = k^{2n}\prod_{i=1}^{n}\prod_{j=1}^{n}(1-a_{i,j}), \tag{49}$$

implying the corollary. ∎

In fact, this corollary can be seen to be equivalent to Theorem 1. We have tried to find a direct proof of it, based on continuity and differentiability, but we did not succeed.

5. 1-factorizations

Our bound also implies a tight bound on the number of 1-factorizations of regular bipartite graphs. Let $G = (V, E)$ be a bipartite graph. A 1-*factorization* of G is a partition of E into perfect matchings $M_1, \ldots, M_k$ ('factors'). A 1-factorization can also be considered as an edge colouring.

The following was conjectured in [9] (and proved for all k of the form 2^a3^b):

Corollary 1d. *The number of 1-factorizations of a k-regular bipartite graph with $2n$ vertices is at least*

$$(\tfrac{k!^2}{k^k})^n. \tag{50}$$

Proof. By Corollary 1a, the first factor M_1 can be chosen in at least

$$(\tfrac{(k-1)^{k-1}}{k^{k-2}})^n \tag{51}$$

ways. Deleting the edges in M_1 we obtain a $(k-1)$-regular bipartite graph, having (by induction) at least

$$(\tfrac{(k-1)!^2}{(k-1)^{k-1}})^n \tag{52}$$

1-factorizations. Multiplying (51) and (52) we obtain (50). ∎

Again, by an argument similar to that of Section 3, one shows that the base in (50) is best possible (cf. [9]).

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