Minimum circulation of railway stock

Alexander Schrijver[1]

Abstract. We describe some research in progress performed for Nederlandse Spoorwegen (Dutch Railways) to determine a minimum circulation of train-units needed to execute a given timetable with given bounds on demands and capacities.

1. The problem

Nederlandse Spoorwegen (Dutch Railways) runs an hourly train service on its route Amsterdam-Schiphol Airport-Leyden-The Hague-Rotterdam-Dordrecht-Roosendaal-Middelburg-Vlissingen *vice versa*, with the following timetable, for each day from Monday till Friday:

train number		2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam	V		6.48	7.55	8.56	9.56	10.56	11.56	12.56	13.56
Rotterdam	A		7.55	8.58	9.58	10.58	11.58	12.58	13.58	14.58
Rotterdam	V	7.00	8.01	9.02	10.03	11.02	12.03	13.02	14.02	15.02
Roosendaal	A	7.40	8.41	9.41	10.43	11.41	12.41	13.41	14.41	15.41
Roosendaal	V	7.43	8.43	9.43	10.45	11.43	12.43	13.43	14.43	15.43
Vlissingen	A	8.38	9.38	10.38	11.38	12.38	13.38	14.38	15.38	16.38
train number		2159	2163	2167	2171	2175	2179	2183	2187	2191
Amsterdam	V	14.56	15.56	16.56	17.56	18.56	19.56	20.56	21.56	22.56
Rotterdam	A	15.58	16.58	17.58	18.58	19.58	20.58	21.58	22.58	23.58
Rotterdam	V	16.00	17.01	18.01	19.02	20.02	21.02	22.02	23.02	
Roosendaal	A	16.43	17.43	18.42	19.41	20.41	21.41	22.41	23.54	
Roosendaal	V	16.45	17.45	18.44	19.43	20.43	21.43			
Vlissingen	A	17.40	18.40	19.39	20.38	21.38	22.38			

train number		2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen	V			5.30	6.54	7.56	8.56	9.56	10.56	11.56
Roosendaal	A			6.35	7.48	8.50	9.50	10.50	11.50	12.50
Roosendaal	V		5.29	6.43	7.52	8.53	9.53	10.53	11.53	12.53
Rotterdam	A		6.28	7.26	8.32	9.32	10.32	11.32	12.32	13.32
Rotterdam	V	5.31	6.29	7.32	8.35	9.34	10.34	11.34	12.34	13.35
Amsterdam	A	6.39	7.38	8.38	9.40	10.38	11.38	12.38	13.38	14.38
train number		2144	2148	2152	2156	2160	2164	2168	2172	2176
Vlissingen	V	12.56	13.56	14.56	15.56	16.56	17.56	18.56	19.55	
Roosendaal	A	13.50	14.50	15.50	16.50	17.50	18.50	19.50	20.49	
Roosendaal	V	13.53	14.53	15.53	16.53	17.53	18.53	19.53	20.52	21.53
Rotterdam	A	14.32	15.32	16.32	17.33	18.32	19.32	20.32	21.30	22.32
Rotterdam	V	14.35	15.34	16.34	17.35	18.34	19.34	20.35	21.32	22.34
Amsterdam	A	15.38	16.40	17.38	18.38	19.38	20.38	21.38	22.38	23.38

Table 1. Timetable Amsterdam-Vlissingen vice versa

[1]CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands, and Department of Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

The trains have more stops, but for our purposes only those given in the table are of interest.

For each of the stages of any scheduled train, Nederlandse Spoorwegen has determined an expected number of passengers, divided into first class and second class, given in the following table:

train number	2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam-Rotterdam		*47* 340	*100* 616	*61* 407	*41* 336	*31* 282	*46* 287	*42* 297	*33* 292
Rotterdam-Roosendaal	*4* 58	*35* 272	*52* 396	*41* 364	*26* 240	*25* 221	*27* 252	*27* 267	*28* 287
Roosendaal-Vlissingen	*14* 328	*19* 181	*27* 270	*26* 237	*24* 208	*32* 188	*15* 180	*21* 195	*23* 290

train number	2159	2163	2167	2171	2175	2179	2183	2187	2191
Amsterdam-Rotterdam	*39* 378	*84* 527	*109* 616	*78* 563	*44* 320	*28* 184	*21* 161	*28* 190	*10* 123
Rotterdam-Roosendaal	*52* 497	*113* 749	*98* 594	*51* 395	*29* 254	*22* 165	*13* 130	*8* 77	
Roosendaal-Vlissingen	*41* 388	*76* 504	*67* 381	*43* 276	*20* 187	*15* 136			

train number	2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen-Roosendaal			*28* 138	*100* 448	*48* 449	*57* 436	*24* 224	*19* 177	*19* 184
Roosendaal-Rotterdam		*16* 167	*88* 449	*134* 628	*57* 397	*71* 521	*34* 281	*26* 214	*22* 218
Rotterdam-Amsterdam	*7* 61	*26* 230	*106* 586	*105* 545	*56* 427	*75* 512	*47* 344	*36* 303	*32* 283

train number	2144	2148	2152	2156	2160	2164	2168	2172	2176
Vlissingen-Roosendaal	*17* 181	*19* 165	*22* 225	*39* 332	*30* 309	*19* 164	*15* 142	*11* 121	
Roosendaal-Rotterdam	*21* 174	*25* 206	*35* 298	*51* 422	*32* 313	*20* 156	*14* 155	*14* 130	*7* 64
Rotterdam-Amsterdam	*34* 330	*39* 338	*67* 518	*74* 606	*37* 327	*23* 169	*18* 157	*17* 154	*11* 143

Table 2. Numbers of required first class (in italics) and second class seats

The problem to be solved is: What is the minimum amount of train stock necessary to perform the service in such a way that at each stage there are enough seats?

In order to answer this question, one should know a number of further characteristics and constraints. In a first variant of the problem considered, the train stock consists of one type of two-way train-units, each consisting of three carriages. The number of seats in any unit is:

first class	*38*
second class	163

Table 3. Number of seats

Each unit has at both ends an engineer's cabin, and units can be coupled together, up to a certain maximum number of units. This maximum is trajectory-dependent, and depends on lengths of station platforms, curvature of bends, required acceleration speed and braking distance, etc. On each of the trajectories of the line Amsterdam-Vlissingen this maximum number is 15 carriages, meaning 5 train-units.

The train length can be changed, by coupling or decoupling units, at the terminal stations of the line, that is at Amsterdam and Vlissingen, and *en route* at two intermediate stations: Rotterdam and Roosendaal. Any train-unit decoupled from a train arriving at place X at time t can be linked up to any other train departing from X at any time later than t. (The Amsterdam-Vlissingen schedule is such that in practice this gives enough time to make the necessary switchings.)

A last condition put is that for each place $X \in$ {Amsterdam, Rotterdam, Roosendaal, Vlissingen}, the number of train-units staying overnight at X should be constant during the week (but may vary for different places). This requirement is made to facilitate surveying the stock, and to equalize at any place the load of overnight cleaning and maintenance throughout the week. It is not required that the same train-unit, after a night in Roosendaal, say, should return to Roosendaal at the end of the day. Only the number of units is of importance.

Given these problem data and characteristics, one may ask for the minimum number of train-units that should be available to perform the daily cycle of train rides required.

It is assumed that if there is sufficient stock for Monday till Friday, then this should also be enough for the weekend services, since in the weekend a few early trains are cancelled, and on the remaing trains there is a smaller expected number of passengers. Moreover, it is not taken into consideration that stock can be exchanged during the day with other lines of the network. In practice this will happen, but initially this possibility is ignored. (We will return below to this issue.)

Another point left out of consideration is the regular maintenance and repair of stock and the amount of reserve stock that should be maintained, as this generally amounts to just a fixed percentual addition on top of the net minimum.

2. A network model

If only one type of railway stock is used, a classical method can be applied to solve the problem, based on min-cost circulations in networks (see [2], cf. also [3], [6], [7], [14], [15], [16]).

To this end, a directed graph $D = (V, A)$ is constructed as follows. For each place $X \in$ {Amsterdam, Rotterdam, Roosendaal, Vlissingen} and for each time t at which any train leaves or arrives at X, we make a vertex (X, t). So the vertices of D correspond to all 198 time entries in the timetable (Table 1).

For any stage of any train ride, leaving place X at time t and arriving at place Y at time t', we make a directed arc from (X, t) to (Y, t'). For instance, there is an arc from (Roosendaal, 7.43) to (Vlissingen, 8.38).

Moreover, for any place X and any two successive times t, t' at which any time leaves or arrives at X, we make an arc from (X, t) to (X, t'). Thus in our example there will be arcs, e.g., from (Rotterdam, 8.01) to (Rotterdam, 8.32), from (Rotterdam, 8.32) to (Rotterdam, 8.35), from (Vlissingen, 8.38) to (Vlissingen, 8.56), and from (Vlissingen, 8.56) to (Vlissingen, 9.38).

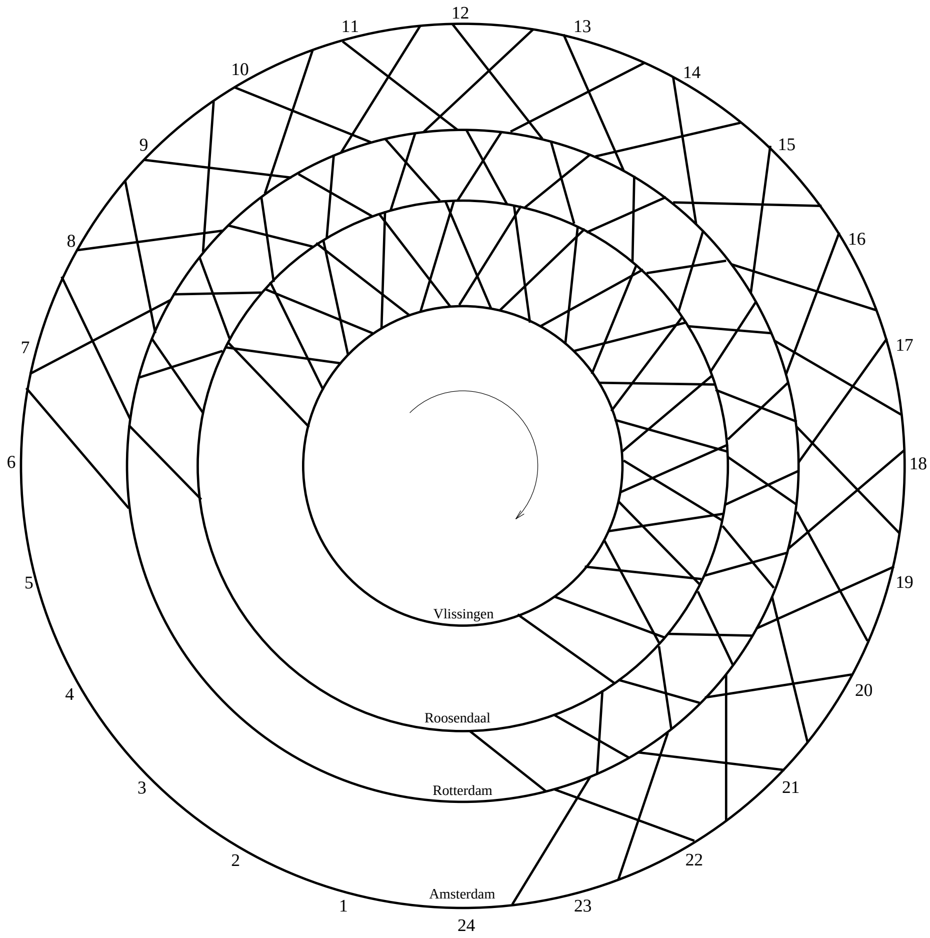

Figure 1. The graph D. All arcs are oriented clockwise

Finally, for each place X there will be an arc from (X, t) to (X, t'), where t is the last time of the day at which any train leaves or arrives at X and where t' is the first time of the day at which any train leaves or arrives at X. So there is an arc from (Roosendaal, 23.54) to (Roosendaal, 5.29).

We can now describe any possible routing of train stock as a function $f : A \longrightarrow \mathbb{Z}_+$, where $f(a)$ denotes the following. If a corresponds to a ride stage, then $f(a)$ is the number of units deployed for that stage. If a corresponds to an arc from (X, t) to (X, t'), then $f(a)$ is equal to the number of units present at place X in the time period t–t' (possibly overnight).

First of all, this function is a *circulation*. That is, at any vertex v of D one should have:

(1) $$\sum_{a \in \delta^+(v)} f(a) = \sum_{a \in \delta^-(v)} f(a),$$

the *flow conservation law*. Here $\delta^+(v)$ denotes the set of arcs of D that are entering vertex v and $\delta^-(v)$ denotes the set of arcs of D that are leaving v.

Moreover, in order to satisfy the demand and capacity constraints, f should satisfy the following condition for each arc a corresponding to a stage:

$$d(a) \leq f(a) \leq c(a). \tag{2}$$

Here $c(a)$ gives the 'capacity' for the stage, in our example $c(a) = 15$ throughout. Furthermore, $d(a)$ denotes the 'demand' for that stage, that is, the lower bound on the number of units required by the expected number of passengers as given in Table 2. That is, with Table 3 we obtain the following lower bounds on the numbers of train-units:

train number	2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam-Rotterdam		3	4	3	3	2	2	2	2
Rotterdam-Roosendaal	1	2	3	3	2	2	2	2	2
Roosendaal-Vlissingen	3	2	2	2	2	2	2	2	2

train number	2159	2163	2167	2171	2175	2179	2183	2187	2191
Amsterdam-Rotterdam	3	4	4	4	2	2	1	2	1
Rotterdam-Roosendaal	4	5	4	3	2	2	1	1	
Roosendaal-Vlissingen	3	4	3	2	2	1			

train number	2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen-Roosendaal			1	3	3	3	2	2	2
Roosendaal-Rotterdam		2	3	4	3	4	2	2	2
Rotterdam-Amsterdam	1	2	4	4	3	4	3	2	2

train number	2144	2148	2152	2156	2160	2164	2168	2172	2176
Vlissingen-Roosendaal	2	2	2	3	2	2	1	1	
Roosendaal-Rotterdam	2	2	2	3	2	1	1	1	1
Rotterdam-Amsterdam	3	3	4	4	3	2	1	1	1

Table 4. Lower bounds on the number of train-units

Note that, by the flow conservation law, at any section of the graph in Figure 1, the total flow on the arcs crossing the section is independent of the choice of the section. It gives the number of train-units that are used. This number is also equal to the total flow on the 'overnight' arcs. So if we wish to minimize the total number of units deployed, we could restrict ourselves to:

$$\text{Minimize} \sum_{a \in A^\circ} f(a). \tag{3}$$

Here A° denotes the set of overnight arcs. So $|A^\circ| = 4$ in the example.

It is easy to see that this fully models the problem. Hence determining the minimum number of train-units amounts to solving a minimum-cost circulation problem, where the cost function is quite trivial: we have $\text{cost}(a) = 1$ if a is an overnight arc, and $\text{cost}(a) = 0$ for all other arcs.

Having this model, we can apply standard min-cost circulation algorithms, based on min-cost augmenting paths and cycles ([12], [11], [4], [5]) or on 'out-of-kilter' ([10], [13]).

Implementation gives solutions of the problem (for the above data) in about 0.05 CPUseconds on an SGI R4400. (See also the classical standard reference [9] and the recent encyclopedic treatment [1].)

Alternatively, the problem can be solved easily with any linear programming package, since by the integrality of the input data and by the total unimodularity of the underlying matrix the optimum basic solution will have integer values only. With the fast linear programming package CPLEX (version 2.1) the following optimum solution was obtained in 0.05 CPUseconds (on an SGI R4400):

train number	2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam-Rotterdam		3	4	3	3	2	2	2	2
Rotterdam-Roosendaal	1	2	3	3	2	2	2	2	2
Roosendaal-Vlissingen	3	2	2	2	2	2	2	2	2
train number	2159	2163	2167	2171	2175	2179	2183	2187	2191
Amsterdam-Rotterdam	5	5	4	4	2	2	1	2	1
Rotterdam-Roosendaal	4	5	4	3	2	2	1	1	
Roosendaal-Vlissingen	3	4	3	2	2	1			

train number	2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen-Roosendaal			1	3	3	3	2	2	2
Roosendaal-Rotterdam		2	4	4	3	4	2	2	2
Rotterdam-Amsterdam	1	2	4	4	3	4	3	2	2
train number	2144	2148	2152	2156	2160	2164	2168	2172	2176
Vlissingen-Roosendaal	2	2	2	3	2	2	1	4	
Roosendaal-Rotterdam	2	3	2	4	3	1	1	1	1
Rotterdam-Amsterdam	3	3	4	4	3	2	1	1	1

Table 5. Minimum circulation with one type of stock

Required are 22 units, divided during the night over the four couple-stations as follows:

	number of units	number of carriages
Amsterdam	4	12
Rotterdam	2	6
Roosendaal	8	24
Vlissingen	8	24
Total	22	66

Table 6. Required stock (one type)

It is quite direct to modify and extend the model so as to contain several other problems. Instead of minimizing the number of train-units one can minimize the amount of carriage-kilometers that should be made every day, or any linear combination of both quantities. In addition, one can put an upper bound on the number of units that can be stored at any of the stations.

Instead of considering one line only, one can more generally consider *networks* of lines that share the same stock of railway material, including trains that are scheduled to be split or combined. (Nederlandse Spoorwegen has trains from The Hague and Rotterdam to Leeuwarden and Groningen that are combined to one train on the common trajectory between Utrecht and Zwolle.)

If only one type of unit is employed for that part of the network, each unit having the same capacity, the problem can be solved fast even for large networks.

3. More types of trains

The problem becomes harder if there are several types of trains that can be deployed for the train service. Clearly, if for each scheduled train we would prescribe which type of unit should be deployed, the problem could be decomposed into separate problems of the type above. But if we do not make such a prescription, and if some of the types can be coupled together to form a train of mixed composition, we should extend the model to a 'multi-commodity circulation' model.

Let us restrict ourselves to the case Amsterdam-Vlissingen again, where now we can deploy two types of two-way train-units, that can be coupled together. The two types are type III, each unit of which consists of 3 carriages, and type IV, each unit of which consists of 4 carriages. The capacities are given in the following table:

type	III	IV
first class	*38*	*65*
second class	163	218

Table 7. Number of seats

Again, the demands of the train stages are given in Table 2. The maximum number of carriages that can be in any train is again 15. This means that if a train consists of x units of type III and y units of type IV then $3x + 4y \leq 15$ should hold.

It is quite easy to extend the model above to the present case. Again we consider the directed graph $D = (V, A)$ as above. At each arc a let $f(a)$ be the number of units of type III on the stage corresponding to a and let $g(a)$ similarly represent type IV. So both $f : A \longrightarrow \mathbb{Z}_+$ and $g : A \longrightarrow \mathbb{Z}_+$ are circulations, that is, satisfy the flow circulation law:

(4)
$$\sum_{a \in \delta^-(v)} f(a) = \sum_{a \in \delta^+(v)} f(a),$$
$$\sum_{a \in \delta^-(v)} g(a) = \sum_{a \in \delta^+(v)} g(a),$$

for each vertex v. The capacity constraint now is:

(5)
$$3f(a) + 4g(a) \leq 15$$

for each arc a representing a stage.

The demand constraint can be formulated as follows:

(6)
$$38f(a) + 65g(a) \geq p_1(a),$$
$$163f(a) + 218g(a) \geq p_2(a),$$

for each arc a representing a stage, where $p_1(a)$ and $p_2(a)$ denote the number of first class and second class seats required (Table 2). Note that in contrary to the case of one type of unit, now we cannot speak of a minimum number of units required: there are now two dimensions, so that minimum train compositions need not be unique.

Let cost_{III} and cost_{IV} represent the cost of purchasing one unit of type III and of type IV, respectively. Although train-units of type IV are more expensive than those of type III, they are cheaper per carriage; that is:

$$\text{cost}_{\text{III}} < \text{cost}_{\text{IV}} < \tfrac{4}{3}\text{cost}_{\text{III}}. \tag{7}$$

This is due to the fact that engineer's cabins are relatively expensive.

One variant of the problem is to find f and g so as to

$$\text{Minimize} \sum_{a \in A^\circ} (\text{cost}_{\text{III}} f(a) + \text{cost}_{\text{IV}} g(a)). \tag{8}$$

However, the classical min-cost circulation algorithms do not apply now. One could implement variants of augmenting paths and cycles techniques, but they generally lead to *fractional* circulations, that is, with certain values being non-integer.

Similarly, when solving the problem as a linear programming problem, we loose the pleasant phenomenon observed above that we automatically would obtain an optimum solution $f, g : A \longrightarrow \mathbb{R}$ with *integer* values only. (Also Ford and Fulkerson's column generation technique [8] yields fractional solutions.)

So the problem is an integer linear programming problem, with 198 integer variables. Solving the problem in this form with the integer programming package CPLEX (version 2.1) would give (for the Amsterdam-Vlissingen example) a running time of several hours, which is too long if one wishes to compare several problem data. This long running time is caused by the fact that, despite a fractional optimum solution is found quickly, a large number of possibilities should be checked in a branching tree (corresponding to rounding fractional values up or down) before one has found an integer-valued optimum solution.

However, there are ways of speeding up the process, by sharpening the constraints and by exploiting more facilities offered by CPLEX.

The conditions (5) and (6) can be sharpened in the following way. For each arc a representing a stage, the two-dimensional vector $(f(a), g(a))$ should be an integer vector in the polygon

$$P_a := \{(x,y) | x \geq 0, y \geq 0, 3x + 4y \leq 15, 38x + 65y \geq p_1(a), 163x + 218y \geq p_2(a)\}. \tag{9}$$

For instance, the trajectory Rotterdam-Amsterdam of train 2132 gives the polygon

$$P_a = \{(x,y) | x \geq 0, y \geq 0, 3x + 4y \leq 15, 38x + 65y \geq 47, 163x + 218y \geq 344\}. \tag{10}$$

In a picture:

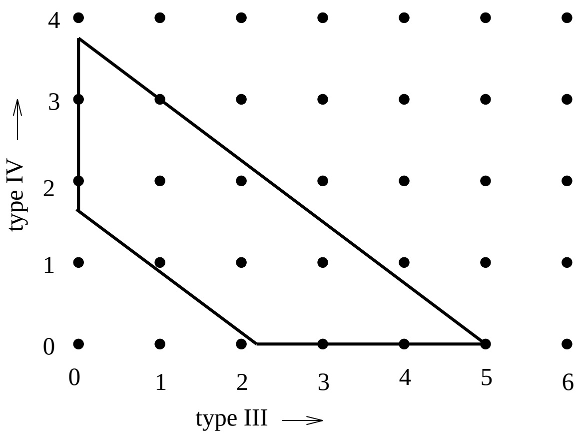

Figure 2. The polygon P_a

In a sense, the inequalities are too wide. The constraints given in (10) could be tightened so as to describe exactly the convex hull of the integer vectors in the polygon P_a (the 'integer hull'), as in:

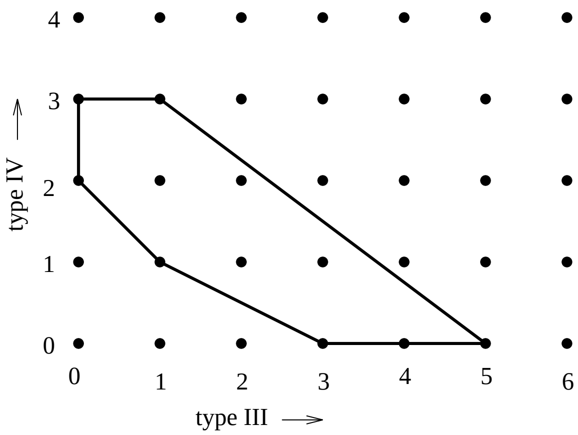

Figure 3. The integer hull of P_a

Thus for segment Rotterdam-Amsterdam of train 2132 the constraints in (10) can be sharpened to:

(11) $$x \geq 0, y \geq 0, x + y \geq 2, x + 2y \geq 3, y \leq 3, 3x + 4y \leq 15.$$

Doing this for each of the 99 polygons representing a stage gives a sharper set of inequalities, which helps to obtain more easily an integer optimum solution from a fractional solution. (This is a weak form of application of the technique of *polyhedral combinatorics*.) Finding all these inequalities can be done in a pre-processing phase, and takes about 0.04 CPUseconds.

Another ingredient that improves the performance of CPLEX when applied to this problem is to give it an order in which the branch-and-bound procedure should select variables. In particular, one can give higher priority to variables that correspond to peak hours (as one may expect that they form the bottleneck in obtaining a minimum circulation), and lower priority to those corresponding to off-peak periods.

Implementation of these techniques makes that CPLEX gives a solution to the Amsterdam-Vlissingen problem in 1.58 CPUseconds (taking $\mathrm{cost}_{\mathrm{III}} = 4$ and $\mathrm{cost}_{\mathrm{IV}} = 5$).

train number	2123	2127	2131	2135	2139	2143	2147	2151	2155
Amsterdam-Rotterdam		0+2	0+3	4+0	0+2	0+2	1+2	0+2	1+1
Rotterdam-Roosendaal	0+1	0+2	0+2	4+0	0+2	0+2	1+3	0+3	1+1
Roosendaal-Vlissingen	0+2	0+2	0+2	2+0	0+1	0+1	0+2	0+2	2+0
train number	2159	2163	2167	2171	2175	2179	2183	2187	2191
Amsterdam-Rotterdam	0+3	2+1	0+3	1+2	0+2	0+1	1+2	0+1	0+1
Rotterdam-Roosendaal	0+3	2+2	0+3	0+2	1+1	2+0	1+3	1+0	
Roosendaal-Vlissingen	0+2	2+1	0+2	0+2	2+0	0+1			

train number	2108	2112	2116	2120	2124	2128	2132	2136	2140
Vlissingen-Roosendaal			1+0	0+3	1+2	0+2	0+2	0+1	1+1
Roosendaal-Rotterdam		1+2	3+0	0+3	0+2	1+2	0+2	2+1	1+3
Rotterdam-Amsterdam	0+1	0+2	4+0	0+3	0+3	1+2	0+2	2+0	0+2
train number	2144	2148	2152	2156	2160	2164	2168	2172	2176
Vlissingen-Roosendaal	1+1	0+1	0+2	0+2	2+0	0+2	2+0	0+1	
Roosendaal-Rotterdam	0+1	0+3	1+3	0+3	1+1	0+1	2+2	0+1	1+0
Rotterdam-Amsterdam	1+1	0+3	1+2	0+3	1+1	0+1	0+2	0+1	0+1

Table 8. Minimum circulation with two types of stock.
$x + y$ means: x units of type III and y units of type IV

In total, one needs 7 units of type III and 12 units of type IV, divided during the night as follows:

	number of units of type III	number of units of type IV	total number of units	total number of carriages
Amsterdam	0	2	2	8
Rotterdam	0	2	2	8
Roosendaal	3	3	6	21
Vlissingen	2	5	7	26
Total	5	12	17	63

Table 9. Required stock (two types)

So comparing this solution with the solution for one type only (Table 6), the possibility of having two types gives both a decrease in the number of train-units and in the number of carriages needed.

Interestingly, it turns out that 17 is the minimum number of units needed and 63 is the minimum number of carriages needed. (This can be shown by finding a minimum circulation first for $\mathrm{cost}_{\mathrm{III}} = \mathrm{cost}_{\mathrm{IV}} = 1$ and next for $\mathrm{cost}_{\mathrm{III}} = 3, \mathrm{cost}_{\mathrm{IV}} = 4$.)

So any feasible circulation with stock of Types III and IV requires at least 17 train-units and at least 63 carriages. In other words, the circulation is optimum for any cost function satisfying (7). We observed a similar phenomenon when checking other input data (although there is no mathematical reason for this fact and it is not difficult to construct examples where it does not show up).

Again variants as described at the end of Section 2 also apply to this more extended model. One can include minimizing the number of carriage-kilometers as an objective, or the option that in some of the trains a buffet section is scheduled (where some of the types contain a buffet). Moreover, one can consider networks of lines.

Our research for NS in fact has focused on more extended problems that require more complicated models and techniques. One requirement is that in any train ride Amsterdam-Vlissingen there should be at least one unit that makes the whole trip. Moreover, it is required that, at any of the four stations given (Amsterdam, Rotterdam, Roosendaal, Vlissingen) one may either couple units to or decouple units from a train, but not both simultaneously. Moreover, one may couple fresh units only to the front of the train, and decouple laid off units only from the rear. (One may check that these conditions are not met by all trains in the solution given in Table 8.)

This all causes that the order of the different units in a train does matter, and that conditions have a more global impact: the order of the units in a certain morning train can still influence the order of some evening train. This does not fit directly in the circulation model described above, and requires an extension. The method we have developed for NS so far, based on introducing extra variables, extending the graph described above and utilizing some heuristic arguments, yields a running time (with CPLEX) of about 30 CPUseconds for the Amsterdam-Vlissingen problem.

Acknowledgement. I thank Adri Steenbeek for efficiently implementing the min-cost circulation and convex hull algorithms.

References

[1] R.K. Ahuja, T.L. Magnanti, and J.B. Orlin, *Network Flows — Theory, Algorithms, and Applications*, Prentice Hall, Englewood Cliffs, New Jersey, 1993.

[2] T.E. Bartlett, An algorithm for the minimum number of transport units to maintain a fixed schedule, *Naval Research Logistics Quarterly* 4 (1957) 139–149.

[3] A.W. Boldyreff, Determination of the maximal steady state flow of traffic through a railroad network, *Journal of the Operations Research Society of America* 3 (1955) 443–465.

[4] R.G. Busacker and P.J. Gowen, *A procedure for determining minimal-cost network flow patterns*, ORO Technical Report 15, Operational Research Office, Johns Hopkins University, Baltimore, Maryland, 1961.

[5] J. Edmonds and R.M. Karp, Theoretical improvements in algorithmic efficiency for network flow problems, *Journal of the Association for Computing Machinery* 19 (1972) 248–264.

[6] G.J. Feeney, The empty boxcar distribution problem, *Proceedings of the First International Conference on Operational Research (Oxford 1957)*, M. Davies, R.T. Eddison, T. Page, eds., Operations Research Society of America, Baltimore, Maryland, 1957, pp. 250–265.

[7] A.R. Ferguson and G.B. Dantzig, The problem of routing aircraft, *Aeronautical Engineering Review* 14 (1955) 51–55.

[8] L.R. Ford, Jr and D.R. Fulkerson, A suggested computation for maximal multi-commodity network flows, *Management Science* 5 (1958-59) 97–101.

[9] L.R. Ford, Jr and D.R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton, New Jersey, 1962.

[10] D.R. Fulkerson, An out-of-kilter method for minimal cost flow problems, *Journal of the Society for Industrial and Applied Mathematics* 9 (1961) 18–27.

[11] M. Iri, A new method of solving transportation-network problems, *Journal of the Operations Research Society of Japan* 3 (1960) 27–87.

[12] W.S. Jewell, *Optimal flows through networks*, Interim Technical Report 8, Operations Research Center, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1958.

[13] G.J. Minty, Monotone networks, *Proceedings of the Royal Society of London* 257A (1960) 194–212.

[14] A.R.D. Norman and M.J. Dowling, *Railroad Car Inventory: Empty Woodrack Cars on the Louisville and Nashville Railroad,* Technical Report 320-2926, IBM New York Scientific Center, New York, 1968.

[15] J.W.H.M.T.S.J. van Rees, Een studie omtrent de circulatie van materieel, *Spoor- en Tramwegen* 38 (1965) 363–367.

[16] W.W. White and A.M. Bomberault, A network algorithm for empty freight car allocation, *IBM Systems Journal* 8 (1969) 147–169.