A short proof of Mader's $\mathcal{S}$-paths theorem

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Abstract. For an undirected graph $G = (V, E)$ and a collection $\mathcal{S}$ of disjoint subsets of V, an *$\mathcal{S}$-path* is a path connecting different sets in $\mathcal{S}$. We give a short proof of Mader's min-max theorem for the maximum number of disjoint $\mathcal{S}$-paths.

Let $G = (V, E)$ be an undirected graph and let $\mathcal{S}$ be a collection of disjoint subsets of V. An *$\mathcal{S}$-path* is a path connecting two different sets in $\mathcal{S}$. Mader [4] gave the following min-max relation for the maximum number of (vertex-)disjoint $\mathcal{S}$-paths, where $S := \bigcup \mathcal{S}$.

Mader's $\mathcal{S}$-paths theorem. *The maximum number of disjoint $\mathcal{S}$-paths is equal to the minimum value of*

$$|U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|B_i| \rfloor, \tag{1}$$

taken over all partitions $U_0, \ldots, U_n$ of V such that each $\mathcal{S}$-path disjoint from U_0, traverses some edge spanned by some U_i. Here B_i denotes the set of vertices in U_i that belong to S or have a neighbour in $V \setminus (U_0 \cup U_i)$.

Lovász [3] gave an alternative proof, by deriving it from his matroid matching theorem. Here we give a short proof of Mader's theorem.

Let μ be the minimum value obtained in (1). Trivially, the maximum number of disjoint $\mathcal{S}$-paths is at most μ, since any $\mathcal{S}$-path disjoint from U_0 and traversing an edge spanned by U_i, traverses at least two vertices in B_i.

I. First, the case where $|T| = 1$ for each $T \in \mathcal{S}$ was shown by Gallai [2], by reduction to matching theory as follows. Let the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ arise from G by adding a disjoint copy G' of $G - S$, and making the copy v' of each $v \in V \setminus S$ adjacent to v and to all neighbours of v in G. We claim that $\tilde{G}$ has a matching of size $\mu + |V \setminus S|$. Indeed, by the Tutte-Berge formula ([5],[1]), it suffices to prove that for any $\tilde{U}_0 \subseteq \tilde{V}$:

$$|\tilde{U}_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|\tilde{U}_i| \rfloor \geq \mu + |V \setminus S|, \tag{2}$$

where $\tilde{U}_1, \ldots, \tilde{U}_n$ are the components of $\tilde{G} - \tilde{U}_0$. Now if for some $v \in V \setminus S$ exactly one of v, v' belongs to $\tilde{U}_0$, then we can delete it from $\tilde{U}_0$, thereby not increasing the left hand side of (2). So we can assume that for each $v \in V \setminus S$, either $v, v' \in \tilde{U}_0$ or $v, v' \notin \tilde{U}_0$. Let $U_i := \tilde{U}_i \cap V$ for $i = 0, \ldots, n$. Then $U_1, \ldots, U_n$ are the components of $G - U_0$, and we have:

$$|\tilde{U}_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|\tilde{U}_i| \rfloor = |U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|U_i \cap S| \rfloor + |V \setminus S| \geq \mu + |V \setminus S| \tag{3}$$

(since in this case $B_i = U_i \cap S$ for $i = 1, \ldots, n$), showing (2).

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So $\tilde{G}$ has a matching M of size $\mu + |V \setminus S|$. Let N be the matching $\{vv'|v \in V \setminus S\}$ in $\tilde{G}$. As $|M| = \mu + |V \setminus S| = \mu + |N|$, the union $M \cup N$ has at least μ components with more edges in M than in N. Each such component is a path connecting two vertices in S. Then contracting the edges in N yields μ disjoint $\mathcal{S}$-paths in G.

II. We now consider the general case. Fixing V, choose a counterexample E, $\mathcal{S}$ minimizing

$$|E| - |\{\{t,u\}|t,u \in V, \exists T,U \in \mathcal{S} : t \in T, u \in U, T \neq U\}|. \tag{4}$$

By part I, there exists a $T \in \mathcal{S}$ with $|T| \geq 2$. Then T is independent in G, since any edge e spanned by T can be deleted without changing the maximum and minimum value in Mader's theorem (as any $\mathcal{S}$-path traversing e contains an $\mathcal{S}$-path not containing e, and as deleting e does not change any set B_i), while decreasing (4).

Choose $s \in T$. Replacing $\mathcal{S}$ by $\mathcal{S}' := (\mathcal{S} \setminus \{T\}) \cup \{T \setminus \{s\}, \{s\}\}$ decreases (4), but not the minimum in Mader's theorem (as each $\mathcal{S}$-path is an $\mathcal{S}'$-path and as $\bigcup \mathcal{S}' = S$). So there exists a collection $\mathcal{P}$ of μ disjoint $\mathcal{S}'$-paths. We can assume that no path in $\mathcal{P}$ has any internal vertex in S.

Necessarily, there is a path $P_0 \in \mathcal{P}$ connecting s with another vertex in T, all other paths in $\mathcal{P}$ being $\mathcal{S}$-paths. Let u be an internal vertex of P_0. Replacing $\mathcal{S}$ by $\mathcal{S}'' := (\mathcal{S} \setminus \{T\}) \cup \{T \cup \{u\}\}$ decreases (4), but not the minimum in Mader's theorem (as each $\mathcal{S}$-path is an $\mathcal{S}''$-path and as $\bigcup \mathcal{S}'' \supset S$). So there exists a collection $\mathcal{Q}$ of μ disjoint $\mathcal{S}''$-paths. Choose $\mathcal{Q}$ such that no internal vertex of any path in $\mathcal{Q}$ belongs to $S \cup \{u\}$, and such that $\mathcal{Q}$ uses a minimal number of edges not used by $\mathcal{P}$.

Necessarily, u is an end of some path $Q_0 \in \mathcal{Q}$, all other paths in $\mathcal{Q}$ being $\mathcal{S}$-paths. As $|\mathcal{P}| = |\mathcal{Q}|$ and as u is not an end of any path in $\mathcal{P}$, there exists an end v of some path $P \in \mathcal{P}$ that is not an end of any path in $\mathcal{Q}$. Now P intersects at least one path in $\mathcal{Q}$ (since otherwise $P \neq P_0$, and $(\mathcal{Q} \setminus \{Q_0\}) \cup \{P\}$ would consist of μ disjoint $\mathcal{S}$-paths). So when following P starting at v, there is a first vertex w that is on some path in $\mathcal{Q}$, say on $Q \in \mathcal{Q}$.

For any end x of Q let Q^x be the $x - w$ part of Q, let P^v be the $v - w$ part of P, and let U be the set in $\mathcal{S}''$ containing v. Then for any end x of Q we have that Q^x is part of P or the other end of Q belongs to U, since otherwise by rerouting part Q^x of Q along P^v, Q remains an $\mathcal{S}''$-path disjoint from the other paths in $\mathcal{Q}$, while we decrease the number of edges used by $\mathcal{Q}$ and not by $\mathcal{P}$, contradicting the minimality assumption.

Let y, z be the ends of Q. We can assume that $y \notin U$. Then Q^z is part of P, hence Q^y is not part of P (as Q is not part of P, as otherwise $Q = P$, and hence v is an end of Q), so $z \in U$. As z is on P and as also v belongs to U and is on P, we have $P = P_0$. So $U = T \cup \{u\}$ and $Q = Q_0$ (since Q^z is part of P, so $z = u$). But then rerouting part Q^z of Q along P^v gives μ disjoint $\mathcal{S}$-paths, contradicting our assumption.

References

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