Graph parameters and reflection positivity

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July 27, 2005

We characterize which real-valued (undirected) graph parameters are of the following type, where H is a graph and $\alpha : VH \to \mathbb{R}_+$ and $\beta : EH \to \mathbb{R}$:

$$f_{H,\alpha,\beta}(G) := \sum_{\substack{\phi:VG\to VH \\ \phi \text{ homomorphism}}} \Big(\prod_{v\in VG} \alpha_{\phi(v)}\Big)\Big(\prod_{uv\in EG} \beta_{\phi(u)\phi(v)}\Big). \qquad (1)$$

Here $\phi : VG \to VH$ is a *homomorphism* if $\phi(u)\phi(v) \in EH$ for all $uv \in EG$. (So if $\phi(u) = \phi(v)$, then H has a loop at $\phi(u)$.) To reduce technicalities, it has turned out to be convenient to assume that G has no loops but may have multiple edges, while H has no multiple edges but may have loops.

Several graph parameters are indeed of this type. A first example of such a parameter is $f(G)$:= the number of k-vertex-colourings of G (for some fixed k). Then we can take $H = K_k$ (the complete loopless graph on k vertices), and α and β the all-one functions on VG and EG respectively. More generally, by taking any graph H and $\alpha \equiv 1$ and $\beta \equiv 1$, $f(G)$ counts the number of homomorphism of G into H. By taking H to be a two-vertex graph with one edge connecting the two vertices and a loop at one of the two vertices, $f(G)$ then counts the number of stable sets of G.

Other examples are given by the partition functions of several models in statistical mechanics. Then H can be taken to be a complete graph with all loops attached, and VH is interpreted as the set of states certain elements of a system G can adopt. The function $\beta : EH \to \mathbb{R}$ describes the energy of the interaction of two neighbouring states, while $\alpha : VH \to \mathbb{R}_+$ can be the external energy of the different states, or, alternatively, if $\sum_{v\in VH}\alpha_v = 1$, α_v may be the probability that an element is in state v. Then any function $\phi : VG \to VH$ is a *configuration* of system G, and $f_{H,\alpha,\beta}(G)$ is the total or average energy of the system. (A different interpretation of this model is in economics, where β gives the profit or cost of certain interactions, and $f_{H,\alpha,\beta}$ gives the expected profit or cost.)

It will follow from our theorem (but also a direct construction based on characters can be made) that also the following graph parameters are of the type above. Let Γ be a finite abelian group and let S be a subset of Γ with $-S = S$ (i.e., $-s \in S$ if $s \in S$). For any graph G, fix an arbitrary orientation. Call a function $x : EG \to \Gamma$ an *S-flow* if all values of x are in S and x satisfies the flow conservation law at each vertex v of G: the inflow is equal to the outflow. Let $f(G)$ be the number of S-flows. (Since $-S = S$, this number is independent of the orientation chosen.) A well-known example is when Γ is the cyclic group with k elements and $S = \Gamma \setminus \{0\}$. Then an S-flow corresponds to a nowhere-zero k-flow, and Tutte's nowhere-zero 5-flow conjecture says that $f(G) > 0$ if $k = 5$ and G has no bridges. (It can be shown that for the case of nowhere-zero k-flows, we can take for H the complete graph on k vertices with all loops attached, and set $\alpha(v) = 1/k$ for each $v \in VH$, $\beta(e) = -1$ for each nonloop edge e of H, and $\beta(e) = k-1$ for each loop e of H.)

The question of characterizing the graph parameters of form (1) is motivated, among others, by the question of the physical realizability of certain graph parameters. It turns out that two conditions on certain matrices derived from the graph parameter are necessary and sufficient: restricted (namely exponential) growth of the ranks and positive semidefiniteness — a condition that corresponds to the well-known *reflection positivity* in statistical mechanics.

These matrices are described as follows. For any integer $k \geq 0$, let $\mathcal{G}_k$ be the set of graphs in which k of the vertices are labeled $1, \ldots, k$, while the remaining vertices are unlabeled. For $G, G' \in \mathcal{G}_k$, let GG' denote the graph obtained by first taking the disjoint sum of G and G', and next identifying equally labeled vertices. (So GG' has $|VG| + |VG'| - k$ vertices.) For any graph parameter f, let $M_{f,k}$ be the (infinite) $\mathcal{G}_k \times \mathcal{G}_k$ matrix whose entry in position G, G' is equal to $f(GG')$.

Then for any graph parameter f (where K_0 is the graph with no vertices and edges):

Theorem. *There exist H, $\alpha : VH \to \mathbb{R}_+$ and $\beta : EH \to \mathbb{R}$ such that $f = f_{H,\alpha,\beta}$ if and only if $f(K_0) = 1$ and there exists a c such that each $M_{f,k}$*

is positive semidefinite and has degree at most c^k*.*

Necessity can be shown rather straightforwardly. The method for proving sufficiency is based on considering each $\mathcal{G}_k$ as a semigroup (taking GG' above as multiplication), making the semigroup algebra over $\mathcal{G}_k$, and taking the quotient algebra over the null-space of $M_{f,k}$, thus obtaining a finite-dimensional Banach algebra, which has a basis of idempotents. The interaction of the idempotents between these algebras for different values of k gives us the combinatorics to find H and the functions α and β.

Extension of this method gives similar results for directed graph and hypergraph parameters, and more generally for any parameter for systems that have a certain semigroup structure.

Reference

[1] M.H. Freedman, L. Lovász, A. Schrijver, Reflection positivity, rank connectivity, and homomorphisms of graphs, preprint, 2003.