Survey of Problems, Questions, and Conjectures

We here collect unsolved problems, questions, and conjectures mentioned in this book. For terminology and background, we refer to the pages indicated.

1 (page 41). Is NP≠P?

2 (page 42). Is P=NP∩co-NP?

3 (page 65). The *Hirsch conjecture*: A d-dimensional polytope with m facets has diameter at most $m - d$.

4 (page 161). Is there an $O(nm)$-time algorithm for finding a maximum flow?

5 (page 232). Berge [1982b] posed the following conjecture generalizing the Gallai-Milgram theorem. Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then for each path collection $\mathcal{P}$ partitioning V and minimizing

$$(1) \qquad \sum_{P \in \mathcal{P}} \min\{|VP|, k\},$$

there exist disjoint stable sets $C_1, \ldots, C_k$ in D such that each $P \in \mathcal{P}$ intersects $\min\{|VP|, k\}$ of them. This was proved by Saks [1986] for acyclic graphs.

6 (page 403). The following open problem was mentioned by Fulkerson [1971b]: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a set S and let $w \in \mathbb{Z}_+^S$. What is the maximum number k of common transversals $T_1, \ldots, T_k$ of $\mathcal{A}$ and $\mathcal{B}$ such that

$$(2) \qquad \chi^{T_1} + \cdots + \chi^{T_k} \leq w?$$

7 (page 459). Can the weighted matching problem be formulated as a linear programming problem of size bounded by a polynomial in the size of the graph, by extending the set of variables? That is, is the matching polytope of a graph $G = (V, E)$ equal to the projection of some polytope $\{x \mid Ax \leq b\}$ with A and b having size bounded by a polynomial in $|V| + |E|$?

8 (pages 472,646). The 5-*flow conjecture* of Tutte [1954a]:

(3) (?) each bridgeless graph has a nowhere-zero 5-flow. (?)

(A *nowhere-zero k-flow* is a flow over $\mathbb{Z}_k$ in some orientation of the graph, taking value 0 nowhere.)

9 (pages 472,498,645,1426). The 4-*flow conjecture* of Tutte [1966]:

(4) (?) each bridgeless graph without Petersen graph minor has a nowhere-zero 4-flow. (?)

This implies the four-colour theorem. For cubic graphs, (4) was proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

Seymour [1981c] showed that the 4-flow conjecture is equivalent to the following more general conjecture, also due to Tutte [1966]:

(5) (?) each bridgeless matroid without F_7^*, $M^*(K_5)$, or $M(\mathbf{P}_{10})$ minor has a nowhere-zero flow over GF(4). (?)

Here $\mathbf{P}_{10}$ denotes the Petersen graph.

10 (page 472). The 3-*flow conjecture* (W.T. Tutte, 1972 (cf. Bondy and Murty [1976], Unsolved problem 48)):

(6) (?) each 4-edge-connected graph has a nowhere-zero 3-flow. (?)

11 (page 473). The *weak* 3-*flow conjecture* of Jaeger [1988]:

(7) (?) there exists a number k such that each k-edge-connected graph has a nowhere-zero 3-flow. (?)

12 (page 473). The following *circular flow conjecture* of Jaeger [1984] generalizes both the 3-flow and the 5-flow conjecture:

(8) (?) for each $k \geq 1$, any $4k$-connected graph has an orientation such that in each vertex, the indegree and the outdegree differ by an integer multiple of $2k+1$. (?)

13 (pages 475,645). The *generalized Fulkerson conjecture* of Seymour [1979a]:

(9) (?) $\lceil \chi'^*(G) \rceil = \lceil \frac{1}{2}\chi'(G_2) \rceil$ (?)

for each graph G. (Here $\chi'^*(G)$ denotes the fractional edge-colouring number of G, and G_2 the graph obtained from G by replacing each edge by two parallel edges.) This is equivalent to the conjecture that

(10) (?) for each k-graph G there exists a family of $2k$ perfect matchings, covering each edge precisely twice. (?)

(A *k-graph* is a k-regular graph $G = (V, E)$ with $|\delta(U)| \geq k$ for each odd-size subset U of V.)

14 (pages 476,645). Fulkerson [1971a] asked if in each bridgeless cubic graph there exist 6 perfect matchings, covering each edge precisely twice (the *Fulkerson conjecture*). It is a special case of Seymour's generalized Fulkerson conjecture.

15 (page 476). Berge [1979a] conjectures that the edges of any bridgeless cubic graph can be covered by 5 perfect matchings. (This would follow from the Fulkerson conjecture.)

16 (page 476). Gol'dberg [1973] and Seymour [1979a] conjecture that for each (not necessarily simple) graph G one has

(11) (?) $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \chi'^*(G) \rceil\}$. (?)

An equivalent conjecture was stated by Andersen [1977].

17 (page 476). Seymour [1981c] conjectures the following generalization of the four-colour theorem:

(12) (?) each planar k-graph is k-edge-colourable. (?)

For $k = 3$, this is equivalent to the four-colour theorem. For $k = 4$ and $k = 5$, it was derived from the case $k = 3$ by Guenin [2002b].

18 (pages 476,644). Lovász [1987] conjectures more generally:

(13) (?) each k-graph without Petersen graph minor is k-edge-colourable. (?)

This is equivalent to stating that the incidence vectors of perfect matchings in a graph without Petersen graph minor, form a Hilbert base.

19 (page 481). The following question was asked by Vizing [1968]: Is there a simple planar graph of maximum degree 6 and with edge-colouring number 7?

20 (page 481). Vizing [1965a] asked if a minimum edge-colouring of a graph can be obtained from an arbitrary edge-colouring by iteratively swapping colours on a colour-alternating path or circuit and deleting empty colours.

21 (page 482). Vizing [1976] conjectures that the list-edge-colouring number of any graph is equal to its edge-colouring number.

(The *list-edge-colouring number* $\chi^l(G)$ of a graph $G = (V, E)$ is the minimum number k such that for each choice of sets L_e for $e \in E$ with $|L_e| = k$, one can select $l_e \in L_e$ for $e \in E$ such that for any two incident edges e, f one has $l_e \neq l_f$.)

22 (page 482). Behzad [1965] and Vizing [1968] conjecture that the total colouring number of a simple graph G is at most $\Delta(G)+2$. (The *total colouring* *number* of a graph $G = (V, E)$ is a colouring of $V \cup E$ such that each colour consists of a stable set and a matching, vertex-disjoint.)

23 (page 482). More generally, Vizing [1968] conjectures that the total colouring number of a graph G is at most $\Delta(G) + \mu(G) + 1$, where $\mu(G)$ is the maximum edge multiplicity of G.

24 (pages 497,645). Seymour [1979b] conjectures that each even integer vector in the circuit cone of a graph is a nonnegative integer combination of incidence vectors of circuits.

25 (pages 497,645,1427). A special case of this is the *circuit double cover conjecture* (asked by Szekeres [1973] and conjectured by Seymour [1979b]): each bridgeless graph has circuits such that each edge is covered by precisely two of them.

Jamshy and Tarsi [1989] proved that the circuit double cover conjecture is equivalent to a generalization to matroids:

(14) (?) each bridgeless binary matroid without F_7^* minor has a circuit double cover. (?)

26 (page 509). Is the system of T-join constraints totally dual quarter-integral?

27 (page 517). L. Lovász asked for the complexity of the following problem: given a graph $G = (V, E)$, vertices $s, t \in V$, and a length function $l : E \to \mathbb{Q}$ such that each circuit has nonnegative length, find a shortest odd $s - t$ path.

28 (page 545). What is the complexity of deciding if a given graph has a 2-factor without circuits of length at most 4?

29 (page 545). What is the complexity of finding a maximum-weight 2-factor without circuits of length at most 3?

30 (page 646). Tarsi [1986] mentioned the following strengthening of the circuit double cover conjecture:

(15) (?) in each bridgeless graph there exists a family of at most 5 cycles covering each edge precisely twice. (?)

31 (page 657). Is the dual of any algebraic matroid again algebraic?

32 (page 892). A special case of a question asked by A. Frank (cf. Schrijver [1979b], Frank [1995]) amounts to the following:

(16) (?) Let $G = (V, E)$ be an undirected graph and let $s \in V$. Suppose that for each vertex $t \neq s$, there exist k internally vertex-disjoint $s-t$ paths. Then G has k spanning trees such that for each vertex

$t \neq s$, the $s-t$ paths in these trees are internally vertex-disjoint. (?)

(The spanning trees need not be edge-disjoint — otherwise $G = K_3$ would form a counterexample.) For $k = 2$, (16) was proved by Itai and Rodeh [1984, 1988], and for $k = 3$ by Cheriyan and Maheshwari [1988] and Zehavi and Itai [1989].

33 (page 962). Can a maximum number of disjoint directed cut covers in a directed graph be found in polynomial time?

34 (page 962). Woodall [1978a,1978b] conjectures (*Woodall's conjecture*):

(17) (?) In a digraph, the minimum size of a directed cut is equal to the maximum number of disjoint directed cut covers. (?)

35 (page 985). Let $G = (V, E)$ be a complete undirected graph, and consider the system

(18)
$$\begin{aligned} &0 \leq x_e \leq 1 \quad \text{for each edge } e,\\ &x(\delta(v)) = 2 \quad \text{for each vertex } v,\\ &x(\delta(U)) \geq 2 \text{ for each } U \subseteq V \text{ with } \emptyset \neq U \neq V. \end{aligned}$$

Let $l : E \to \mathbb{R}_+$ be a length function. Is the minimum length of a Hamiltonian circuit at most $\frac{4}{3}$ times the minimum value of $l^{\mathsf{T}}x$ over (18)?

36 (page 990). Padberg and Grötschel [1985] conjecture that the diameter of the symmetric traveling salesman polytope of a complete graph is at most 2.

37 (page 1076). Frank [1994a] conjectures:

(19) (?) Let $D = (V, A)$ be a simple acyclic directed graph. Then the minimum size of a k-vertex-connector for D is equal to the maximum of $\sum_{v \in V} \max\{0, k - \deg^{\text{in}}(v)\}$ and $\sum_{v \in V} \max\{0, k - \deg^{\text{out}}(v)\}$. (?)

(A *k-vertex-connector* for D is a set of (new) arcs whose addition to D makes it k-vertex-connected.)

38 (page 1087). *Hadwiger's conjecture* (Hadwiger [1943]): If $\chi(G) \geq k$, then G contains K_k as a minor.

Hadwiger's conjecture is trivial for $k = 1, 2, 3$, was shown by Hadwiger [1943] for $k = 4$ (also by Dirac [1952]), is equivalent to the four-colour theorem for $k = 5$ (by a theorem of Wagner [1937a]), and was derived from the four-colour theorem for $k = 6$ by Robertson, Seymour, and Thomas [1993]. For $k \geq 7$, the conjecture is unsettled.

39 (page 1099). Chvátal [1973a] asked if for each fixed t, the stable set problem for graphs for which the stable set polytope arises from $P(G)$ by at most

t rounds of cutting planes, is polynomial-time solvable. Here $P(G)$ is the polytope determined by the nonnegativity and clique inequalities.

40 (page 1099). Chvátal [1975b] conjectures that there is no polynomial $p(n)$ such that for each graph G with n vertices we can obtain the inequality $x(V) \leq \alpha(G)$ from the system defining $Q(G)$ by adding at most $p(n)$ cutting planes. Here $Q(G)$ is the polytope determined by the nonnegativity and edge inequalities. (This conjecture would be implied by NP$\neq$co-NP.)

41 (page 1105). Gyárfás [1987] conjectures that there exists a function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\chi(G) \leq g(\omega(G))$ for each graph G without odd holes.

42 (page 1107). Can perfection of a graph be tested in polynomial time?

43 (page 1131). Berge [1982a] conjectures the following. A directed graph $D = (V, A)$ is called *α-diperfect* if for every induced subgraph $D' = (V', A')$ and each maximum-size stable set S in D' there is a partition of V' into directed paths each intersecting S in exactly one vertex. Then for each directed graph D:

(20) (?) D is α-diperfect if and only if D has no induced subgraph C whose underlying undirected graph is a chordless odd circuit of length ≥ 5, say with vertices $v_1, \ldots, v_{2k+1}$ (in order) such that each of $v_1, v_2, v_3, v_4, v_6, v_8, \ldots, v_{2k}$ is a source or a sink. (?)

44 (page 1170). Is $\vartheta(C_n) = \Theta(C_n)$ for each odd n?

45 (page 1170). Can Haemers' bound $\eta(G)$ on the Shannon capacity of a graph G be computed in polynomial time?

46 (page 1187). Is every t-perfect graph strongly t-perfect?

Here a graph is *t-perfect* if its stable set polytope is determined by the nonnegativity, edge, and odd circuit constraints. It is *strongly t-perfect* if this system is totally dual integral.

47 (page 1195). T-perfection is closed under taking induced subgraphs and under contracting all edges in $\delta(v)$ where v is a vertex not contained in a triangle. What are the minimally non-t-perfect graphs under this operation?

48 (page 1242). For any k, let $f(k)$ be the smallest number such that in any $f(k)$-connected undirected graph, for any choice of distinct vertices $s_1, t_1, \ldots, s_k, t_k$ there exist vertex-disjoint $s_1 - t_1, \ldots, s_k - t_k$ paths. Thomassen [1980] conjectures that $f(k) = 2k + 2$ for $k \geq 2$.

49 (page 1242). For any k, let $g(k)$ be the smallest number such that in any $g(k)$-edge-connected undirected graph, for any choice of vertices $s_1, t_1, \ldots, s_k, t_k$ there exist edge-disjoint $s_1 - t_1, \ldots, s_k - t_k$ paths. Thomassen [1980] conjectures that $g(k) = k$ if k is odd and $g(k) = k + 1$ if k is even.

50 (page 1243). What is the complexity of the k arc-disjoint paths problem in directed planar graphs, for any fixed $k \geq 2$? This is even unknown for $k = 2$, also if we restrict ourselves to two opposite nets.

51 (page 1274). Karzanov [1991] conjectures that if the nets in a multiflow problem form two disjoint triangles and if the capacities and demands are integer and satisfy the Euler condition, then the existence of a fractional multiflow implies the existence of a half-integer multiflow.

52 (page 1274). The previous conjecture implies that for each graph $H = (T, R)$ without three disjoint edges, there is an integer k such that for each graph $G = (V, E)$ with $V \supseteq T$ and any $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, if there is a feasible multiflow, then there exists a $\frac{1}{k}$-integer multiflow.

53 (page 1276). Okamura [1998] conjectures the following. Let $G = (V, E)$ be an l-edge-connected graph (for some l). Let $H = (T, R)$ be a 'demand' graph, with $T \subseteq V$, such that $d_R(U) \leq l$ for each $U \subseteq V$. Then the edge-disjoint paths problem has a half-integer solution.

54 (page 1293). Is each Mader matroid a gammoid?

55 (page 1294). Is each Mader matroid linear?

56 (page 1299). Is the undirected edge-disjoint paths problem for planar graphs polynomial-time solvable if all terminals are on the outer boundary? Is it NP-complete?

57 (page 1310). Is the integer multiflow problem polynomial-time solvable if the graph and the nets form a planar graph such that the nets are spanned by a fixed number of faces?

58 (page 1310). Pfeiffer [1990] raised the question if the edge-disjoint paths problem has a half-integer solution if the graph $G + H$ (the union of the supply graph and the demand graph) is embeddable in the torus and there exists a quarter-integer solution.

59 (page 1320). Let $G = (V, E)$ be a planar bipartite graph and let q be a vertex on the outer boundary. Do there exist disjoint cuts $C_1, \ldots, C_p$ such that any pair s, t of vertices with s and t on the outer boundary, or with $s = q$, is separated by $\mathrm{dist}_G(s, t)$ cuts?

60 (page 1345). Fu and Goddyn [1999] asked: Is the class of graphs for which the incidence vectors of cuts form a Hilbert base, closed under taking minors?

61 (page 1382). Füredi, Kahn, and Seymour [1993] conjecture that for each hypergraph $H = (V, \mathcal{E})$ and each $w : \mathcal{E} \to \mathbb{R}_+$, there exists a matching $\mathcal{M} \subseteq \mathcal{E}$ such that

$$(21) \qquad \sum_{F\in\mathcal{M}} \big(|F| - 1 + \frac{1}{|F|}\big) w(F) \geq \nu_w^*(H),$$

where $\nu_w^*(H)$ is the maximum weight $w^{\mathsf{T}}y$ of a fractional matching $y : \mathcal{E} \to \mathbb{R}_+$.

62 (pages 1387,1408). Seymour [1981a] conjectures:

(22) (?) a binary hypergraph is ideal if and only if it has no $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, or F_7 minor. (?)

63 (page 1392). Seymour [1990b] asked the following. Suppose that $H = (V, \mathcal{E})$ is a hypergraph without J_n minor ($n \geq 3$). Let $l, w : V \to \mathbb{Z}_+$ be such that

$$(23) \qquad \tau(H^w) \cdot \tau(b(H)^l) > l^{\mathsf{T}} w.$$

Is there a minor H' of H and $l', w' : VH' \to \{0, 1\}$ such that

$$(24) \qquad \tau((H')^{w'}) \cdot \tau(b(H')^{l'}) > l'^{\mathsf{T}} w'$$

and such that $\tau((H')^{w'}) \leq \tau(H^w)$ and $\tau(b(H')^{l'}) \leq \tau(b(H)^l)$?

Here, for each $n \geq 3$: $J_n :=$ the hypergraph with vertex set $\{1, \ldots, n\}$ and edges $\{2, \ldots, n\}$, $\{1, 2\}, \ldots, \{1, n\}$.

64 (page 1392). Seymour [1990b] also asked the following. Let $H = (V, \mathcal{E})$ be a nonideal hypergraph. Is the minimum of $\tau(H')$ over all parallelizations and minors H' of H with $\tau^*(H') < \tau(H')$ attained by a minor of H?

65 (page 1395). Cornuéjols and Novick [1994] conjecture that there are only finitely many minimally nonideal hypergraphs H with $r_{\min}(H) > 2$ and $\tau(H) > 2$.

66 (page 1396). Ding [1993] asked whether there exists a number t such that each minimally nonideal hypergraph H satisfies $r_{\min}(H) \leq t$ or $\tau(H) \leq t$.

(The above conjecture of Cornuéjols and Novick [1994] implies a positive answer to this question.)

67 (page 1396). Ding [1993] conjectures that for each fixed $k \geq 2$, each minor-minimal hypergraph H with $\tau_k(H) < k \cdot \tau(H)$, contains some J_n minor ($n \geq 3$) or satisfies the regularity conditions of Lehman's theorems (Theorem 78.4 and 78.5).

68 (page 1401). Conforti and Cornuéjols [1993] conjecture:

(25) (?) a hypergraph is Mengerian if and only if it is packing. (?)

69 (page 1401). Cornuéjols, Guenin, and Margot [1998,2000] conjecture:

(26) (?) each minimally nonideal hypergraph H with $r_{\min}(H)\tau(H) = |VH| + 1$ is minimally nonpacking. (?)

70 (page 1401). Cornuéjols, Guenin, and Margot [1998,2000] conjecture that $\tau(H) = 2$ for each ideal minimally nonpacking hypergraph H.

71 (page 1404). Seymour [1981a] conjectures that T_{30} is the unique minor-minimal binary ideal hypergraph H with the property $\nu_2(H) < 2\tau(H)$.

Here the hypergraph T_{30} arises as follows. Replace each edge of the Petersen graph by a path of length 2, making the graph G. Let $T := VG \setminus \{v\}$, where v is an arbitrary vertex of v of degree 3. Let $\mathcal{E}$ be the collection of T-joins. Then $T_{30} := (EG, \mathcal{E})$.

72 (page 1405). P.D. Seymour (personal communication 1975) conjectures that for each ideal hypergraph H there exists an integer k such that $\nu_k(H) = k \cdot \tau(H)$ and such that $k = 2^i$ for some i. He also asks if $k = 4$ would do in all cases.

73 (page 1405). Seymour [1979a] conjectures that for each ideal hypergraph H, the g.c.d. of those k with $\nu_k(H) = k \cdot \tau(H)$ is equal to 1 or 2.

74 (page 1409). Is the following true for binary hypergraphs H:

(27) (?) $\nu(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+$ with $w(B)$ even for all $B \in b(H) \iff \frac{1}{2}\nu_2(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+ \iff$ H has no $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, F_7, or T_{15} minor. (?)

Here T_{15} is the hypergraph of $V\mathbf{P}_{10}$-joins in the Petersen graph $\mathbf{P}_{10}$.

75 (page 1421). Seymour [1981a] conjectures that for any binary matroid M:

(28) (?) M is 1-cycling $\iff$ M is 1-flowing $\iff$ M has no AG(3,2), T_{11}, or T_{11}^* minor. (?)

Here T_{11} is the binary matroid represented by the 11 vectors in $\{0,1\}^5$ with precisely 3 or 5 ones. Moreover, AG(3,2) is the matroid with 8 elements obtained from the 3-dimensional affine geometry over GF(2).