Reduction of symmetric semidefinite programs using the regular $*$-representation

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Abstract. We consider semidefinite programming problems on which a permutation group is acting. We describe a general technique to reduce the size of such problems, exploiting the symmetry. The technique is based on a low-order matrix $*$-representation of the commutant (centralizer ring) of the matrix algebra generated by the permutation matrices.

We apply it to extending a method of de Klerk et al. that gives a semidefinite programming lower bound to the crossing number of complete bipartite graphs. It implies that $\mathrm{cr}(K_{8,n}) \geq 2.9299n^2 - 6n$, $\mathrm{cr}(K_{9,n}) \geq 3.8676n^2 - 8n$, and (for any $m \geq 9$)

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8594 \frac{m}{m-1},$$

where $Z(m,n)$ is the *Zarankiewicz number* $\lfloor \frac{1}{4}(m-1)^2 \rfloor \lfloor \frac{1}{4}(n-1)^2 \rfloor$, which is the conjectured value of $\mathrm{cr}(K_{m,n})$. Here the best factor previously known was 0.8303 instead of 0.8594.

1. Introduction

This paper is inspired by papers of Kanno, Ohsaki, Murota, and Katoh [5] and Gatermann and Parrilo [4], that study semidefinite programming problems whose underlying matrices have symmetries that enable us to reduce the size of the problems, and it extends results of de Klerk, Maharry, Pasechnik, Richter, and Salazar [7] on the crossing number of complete bipartite graphs.

The new contribution of the present paper is a general but explicit method to reduce the order of the matrices in a semidefinite programming problem if the problem is invariant under a group acting on its variables. The method is based on constructing a 'regular $*$-representation' of a matrix $*$-algebra. A *matrix $*$-algebra* is a collection of matrices closed under addition, scalar and matrix multiplication, and transposition. In this paper, all matrices are real, and all positive semidefinite matrices are symmetric.

The results in this paper relate to representation theory (cf. [3]), C$*$-algebra (cf. [10]), and the theory of association schemes (cf. [2]) — however, this paper is mainly self-contained.

The method applies to problems of the form

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$$(1) \qquad \min\{\mathrm{tr}(CX) \mid X \text{ positive semidefinite},\ X \geq 0,\ \mathrm{tr}(A_j X) = b_j \text{ for } j = 1, \ldots, m\},$$

where C and $A_1, \ldots, A_m$ are given real symmetric matrices (all of the same order), and $b_1, \ldots, b_m$ are given real numbers. (This is a generic form of a semidefinite programming problem.)

The method is in particular effective when the order of the matrices C and A_j is large, whereas there is a relatively large multiplicative group G of permutation matrices that commute with each of $C, A_1, \ldots, A_m$. In that case, we can assume without loss of generality that also X commutes with all matrices in G. As we will show below, this makes it possible to reduce the order of the matrices involved to the *dimension* of the algebra of matrices commuting with all matrices in G. This often is much smaller than the order of the original input matrices, which allows to solve the semidefinite programming problem much more efficiently, or to solve it at all.

As application of the method we extend the bounds on the crossing number of complete bipartite graphs $K_{m,n}$ found by de Klerk et al. [7], as we will describe in Section 3.

2. The regular $*$-representation

Let G be a finite group acting on a finite set Z. That is, we have a homomorphism $h : G \to S_Z$, where S_Z is the group of all permutations of Z. So for each $\pi \in G$, h_π is a bijection $Z \to Z$ with $h_{\pi\pi'} = h_\pi h_{\pi'}$ and $h_{\pi^{-1}} = h_\pi^{-1}$ for all π, $\pi' \in G$.

For each $\pi \in G$, let M_π be the $Z \times Z$ matrix with

$$(2) \qquad (M_\pi)_{x,y} := \begin{cases} 1 & \text{if } h_\pi(x) = y, \\ 0 & \text{otherwise,} \end{cases}$$

for $x, y \in Z$. So M_π is the $Z \times Z$ permutation matrix corresponding to the permutation h_π of Z. Hence $\pi \mapsto M_\pi$ defines an orthogonal representation of G, i.e., it satisfies

$$(3) \qquad M_{\pi\pi'} = M_\pi M_{\pi'} \text{ and } M_{\pi^{-1}} = M_\pi^T$$

for all $\pi, \pi' \in G$.

Let $\mathcal{A}$ be the matrix $*$-algebra

$$(4) \qquad \mathcal{A} := \{\sum_\pi \lambda_\pi M_\pi \mid \lambda_\pi \in \mathbb{R} \quad (\pi \in G)\}.$$

The *invariant matrices* are the $Z \times Z$ matrices X satisfying

$$(5) \qquad XM_\pi = M_\pi X$$

for all $\pi \in G$. In other words, $M_\pi X M_{\pi^{-1}} = X$.

So the collection of invariant matrices is precisely the *commutant* $\mathcal{A}'$ of $\mathcal{A}$:

$$\mathcal{A}' := \{X \in \mathbb{R}^{Z\times Z} \mid XM = MX \text{ for all } M \in \mathcal{A}\}. \tag{6}$$

(The commutant is also known as the centralizer ring.) The commutant is again a matrix $*$-algebra.

The matrix $*$-algebra $\mathcal{A}'$ has a basis of $\{0,1\}$-matrices $E_1, \ldots, E_d$ such that

$$E_1 + \cdots + E_d = J, \tag{7}$$

where J is the all-one $Z \times Z$ matrix. They correspond to the orbits of the action of G on $Z \times Z$. (This is the action $(x, y) \mapsto (h_\pi(x), h_\pi(y))$ for $x, y \in Z$ and $\pi \in G$.)

Computationally, we do not need to work with these matrices, but we should be able to identify them and to calculate their multiplication parameters, as will be specified below.

Observe that for each i there is an i^* with

$$E_{i^*} = (E_i)^T \tag{8}$$

(possibly $i^* = i$).

We normalize the E_i to

$$B_i := \mathrm{tr}(E_i^T E_i)^{-1/2} E_i \tag{9}$$

for $i = 1, \ldots, d$. Then

$$\mathrm{tr}(B_i^T B_j) = \delta_{i,j}, \tag{10}$$

where $\delta_{i,j}$ is the Kronecker delta.

The *multiplication parameters* $\lambda_{i,j}^k$ are defined by

$$B_i B_j = \sum_k \lambda_{i,j}^k B_k \tag{11}$$

for $i, j = 1, \ldots, d$.

Define the $d \times d$ matrices $L_1, \ldots, L_d$ by

$$(L_k)_{i,j} := \lambda_{k,j}^i \tag{12}$$

for $k, i, j = 1, \ldots, d$. Let $\mathcal{L}$ be the linear space

$$\mathcal{L} := \{\sum_{k=1}^d x_k L_k \mid x_1, \ldots, x_k \in \mathbb{R}\}. \tag{13}$$

Let ϕ be the linear function $\mathcal{A}' \to \mathcal{L}$ determined by $\phi(B_k) = L_k$ for $k = 1, \ldots, d$. We will show that ϕ is a $*$-*isomorphism*; that is, it is a bijection and satisfies $\phi(YZ) = \phi(Y)\phi(Z)$ and $\phi(Y^T) = \phi(Y)^T$ for all $X, Y \in \mathcal{A}'$.

Theorem 1. *ϕ is a $*$-isomorphism.*

Proof. Consider any $k = 1, \ldots, d$. Let the linear function $\Phi_k : \mathcal{A}' \to \mathcal{A}'$ be defined by: $\Phi_k(X) := B_k X$ for each $X \in \mathcal{A}'$. Then L_k is the matrix corresponding to Φ_k, using the basis $B_1, \ldots, B_d$ of $\mathcal{A}'$. Indeed,

$$\Phi_k(B_j) = B_k B_j = \sum_i \lambda^i_{k,j} B_i = \sum_i (L_k)_{i,j} B_i. \tag{14}$$

So $\phi(B_k)$ is the matrix corresponding to the linear operator $X \mapsto B_k X$ on $\mathcal{A}'$ (since $L_k = \phi(B_k)$). Hence, as the B_k span $\mathcal{A}'$, it follows that, for each $Y \in \mathcal{A}'$, $\phi(Y)$ is the matrix corresponding to the linear operator $X \mapsto YX$ on $\mathcal{A}'$. This implies that $\phi(YZ) = \phi(Y)\phi(Z)$ for all $Y, Z \in \mathcal{A}'$ (since $(YZ)X = Y(ZX)$). Moreover, ϕ is one-to-one, since if $\phi(Y) = 0$, then $YX = 0$ for all $X \in \mathcal{A}'$, hence $YY^T = 0$, and so $Y = 0$.

Finally, $\phi(Y^T) = \phi(Y)^T$. Indeed, we have for each i,

$$YB_j = \sum_t \phi(Y)_{t,j} B_t. \tag{15}$$

Hence

$$\mathrm{tr}(B_i^T Y B_j) = \sum_t \phi(Y)_{t,j} \mathrm{tr}(B_i^T B_t) = \sum_t \phi(Y)_{t,j} \delta_{i,t} = \phi(Y)_{i,j}. \tag{16}$$

Since similarly $\phi(Y^T)_{j,i} = \mathrm{tr}(B_j^T Y^T B_i)$ and since $\mathrm{tr}(B_j^T Y^T B_i) = \mathrm{tr}(B_i^T Y B_j)$, we have $\phi(Y^T)_{j,i} = \phi(Y)_{i,j}$. So $\phi(Y^T) = \phi(Y)^T$. ∎

Those familiar with representation theory will see that ϕ is the regular $*$-representation of $\mathcal{A}'$ associated with the orthonormal basis $B_1, \ldots, B_d$ of $\mathcal{A}'$.

An important consequence of Theorem 1 is that, for any $x_1, \ldots, x_d \in \mathbb{R}$,

$$\sum_{i=1}^d x_i B_i \text{ is positive semidefinite} \iff \sum_{i=1}^d x_i L_i \text{ is positive semidefinite.} \tag{17}$$

(Recall that each positive semidefinite matrix is symmetric, and that $\sum_i x_i B_i$ and $\sum_i x_i L_i$ are symmetric if and only if $x_{i^*} = x_i$ for each i.) (17) is a well-known and easy fact from C$*$-algebra. It can be seen as follows. Trivially, as ϕ is a $*$-isomorphism, ϕ maintains symmetry of matrices. Now let $M \in \mathcal{A}'$ be symmetric and let $p(x)$ be the minimal polynomial of M. Then p is also the minimal polynomial of $\phi(M)$, as ϕ is an algebra $*$-isomorphism (since $X = 0 \iff \phi(X) = 0$). Then M is positive semidefinite $\iff$ all roots of p are nonnegative $\iff$ $\phi(M)$ is positive semidefinite.

Since the order d of the matrices L_i is equal to the number of matrices B_i (that is, to the number of orbits of the action of G on $Z \times Z$), this may give a considerable reduction of the size of the matrices to which we want to apply semidefinite programming.

To be more precise, let the matrices C and A_j in (1) be $Z \times Z$ matrices commuting with

M_π for each π in some finite group acting on Z. Then there is an optimum solution X that commutes with each of the M_π, since we can replace any optimum solution X by

$$(18) \qquad X' := |G|^{-1} \sum_{\pi \in G} M_\pi X M_\pi^T,$$

as X' is feasible again and $\mathrm{tr}(CX') = \mathrm{tr}(CX)$. Hence we can require $X = \sum_i x_i B_i$ for some x_i. Then by (17)

$$(19) \qquad \begin{aligned} &\min\{\mathrm{tr}(CX) \mid X \text{ positive semidefinite, } X \geq 0, \mathrm{tr}(A_j X) = b_j \text{ for } j = 1, \ldots, m\} = \\ &\min\{\sum_{i=1}^{d} \mathrm{tr}(CB_i) x_i \mid \sum_{i=1}^{d} x_i B_i \text{ positive semidefinite, } x_i \geq 0 \text{ for } i = 1, \ldots, d, \\ &\sum_{i=1}^{d} \mathrm{tr}(A_j B_i) x_i = b_j \text{ for } j = 1, \ldots, m\} = \\ &\min\{\sum_{i=1}^{d} \mathrm{tr}(CB_i) x_i \mid \sum_{i=1}^{d} x_i L_i \text{ positive semidefinite, } x_i \geq 0 \text{ for } i = 1, \ldots, d, \\ &\sum_{i=1}^{d} \mathrm{tr}(A_j B_i) x_i = b_j \text{ for } j = 1, \ldots, m\}. \end{aligned}$$

Assuming that we can compute the values of $\mathrm{tr}(CB_i)$ and $\mathrm{tr}(A_j B_i)$, this gives a smaller semidefinite programming problem.

Since the matrix $\sum_i x_i L_i$ is symmetric if and only if $x_i = x_{i^*}$ for each i, the number of variables in (19) can be reduced to the *reduced dimension* d_{reduced}, which is the number of distinct pairs $\{i, i^*\}$. In other words, it is the dimension of the subspace of $\mathcal{A}'$ of symmetric matrices.

Finally, we mention the following equality, that may be useful in determining the matrices L_k:

$$(20) \qquad \lambda_{i,j}^k = \mathrm{tr}(D_{k^*} D_i D_j)$$

(which can be derived from (16)). It implies $\lambda_{i,j}^k = \lambda_{k^*,i}^{j^*} = \lambda_{j,k^*}^{i^*} = \lambda_{j^*,i^*}^{k^*} = \lambda_{i^*,k}^{j} = \lambda_{k,j^*}^{i}$.

3. Crossing numbers

As application we give an extension of a method of de Klerk, Maharry, Pasechnik, Richter, and Salazar [7] to lower bound the crossing number $\mathrm{cr}(K_{m,n})$ of a complete bipartite graph $K_{m,n}$. (The *crossing number* of a graph G is the minimum number of intersections of edges when G is drawn in the plane such that all vertices are distinct.) This is based on finding, for some fixed m, a lower bound for $\mathrm{cr}(K_{m,n})$ using semidefinite programming.

The bound relates to the problem raised by the paper of Zarankiewicz [12], asking if

$$(21) \qquad \mathrm{cr}(K_{m,n}) \stackrel{?}{=} Z(m,n) := \lfloor \tfrac{1}{4}(m-1)^2 \rfloor \lfloor \tfrac{1}{4}(n-1)^2 \rfloor.$$

(In fact, Zarankiewicz claimed to have a proof, which however was shown to be incorrect.) Here $\leq$ follows from a direct construction. This equality was proved by Kleitman [6] if $\min\{m,n\} \leq 6$ and by Woodall [11] if $m \in \{7,8\}$ and $n \in \{7,8,9,10\}$.

Consider any m,n. Let $K_{m,n}$ have colour classes $\{1,\ldots,m\}$ and $\{u_1,\ldots,u_n\}$. (This notation will be convenient for our purposes.) Let Z_m be the set of cyclic permutations of $\{1,\ldots,m\}$ (that is, the permutations with precisely one orbit). For any drawing of $K_{m,n}$ in the plane and for any u_i, let $\gamma(u_i)$ be the cyclic permutation $(1,i_2,\ldots,i_m)$ such that the edges leaving u_i in clockwise order, go to $1,i_2,\ldots,i_m$ respectively.

For $\sigma,\tau \in Z_m$, let $C_{\sigma,\tau}$ be equal to the minimum number of crossings when we draw $K_{m,2}$ in the plane such that $\gamma(u_1) = \sigma$ and $\gamma(u_2) = \tau$. De Klerk et al. applied a direct algorithm to compute $C_{\sigma,\tau}$, due to Kleitman [6] and described in detail by Woodall [11]. One may show that for any $\sigma \in Z_m$:

$$(22) \qquad C_{\sigma,\sigma^{-1}} = 0 \text{ and } C_{\sigma,\sigma} = \lfloor \tfrac{1}{4}(m-1)^2 \rfloor.$$

The $C_{\sigma,\tau}$ define a matrix $C = (C_{\sigma,\tau})$ in $\mathbb{R}^{Z_m \times Z_m}$ (see [7] for more details about this matrix). Then define the number α_m by:

$$(23) \qquad \alpha_m := \min\{\operatorname{tr}(CX) \mid X \in \mathbb{R}_+^{Z_m \times Z_m}, X \text{ positive semidefinite}, \operatorname{tr}(JX) = 1\},$$

where J is the all-one matrix in $\mathbb{R}^{Z_m \times Z_m}$.

De Klerk et al. [7] showed:

Theorem 2. $\operatorname{cr}(K_{m,n}) \geq \frac{1}{2}n^2\alpha_m - \frac{1}{2}n\lfloor \frac{1}{4}(m-1)^2 \rfloor$ *for all* m,n.

Proof. Consider a drawing of $K_{m,n}$ in the plane with $\operatorname{cr}(K_{m,n})$ crossings. For each cyclic permutation σ, let d_σ be the number of vertices u_i with $\gamma(u_i) = \sigma$. Consider d as column vector in $\mathbb{R}^{Z_m}$, and define

$$(24) \qquad X := n^{-2}dd^T.$$

Then X satisfies the constraints in (23), hence $\alpha_m \leq \operatorname{tr}(CX)$. For $i,j = 1,\ldots,n$, let $\beta_{i,j}$ denote the number of crossings of the edges leaving u_i with the edges leaving u_j. So if $i \neq j$, then $\beta_{i,j} \geq C_{\gamma(u_i),\gamma(u_j)}$. Hence

$$(25) \qquad n^2\operatorname{tr}(CX) = \operatorname{tr}(Cdd^T) = d^TCd = \sum_{i,j=1}^{n} C_{\gamma(u_i),\gamma(u_j)} \leq \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \beta_{i,j} + \sum_{i=1}^{n} C_{\gamma(u_i),\gamma(u_i)} = 2\operatorname{cr}(K_{m,n}) + n\lfloor \tfrac{1}{4}(m-1)^2 \rfloor.$$

Therefore,

$$(26) \qquad \operatorname{cr}(K_{m,n}) \geq \tfrac{1}{2}n^2\operatorname{tr}(CX) - \tfrac{1}{2}n\lfloor \tfrac{1}{4}(m-1)^2 \rfloor \geq \tfrac{1}{2}\alpha_m n^2 - \tfrac{1}{2}n\lfloor \tfrac{1}{4}(m-1)^2 \rfloor. \qquad \blacksquare$$

This implies:

Corollary 2a. $\mathrm{cr}(K_{m,n}) \geq \dfrac{m(m-1)}{k(k-1)}(\frac{1}{2}n^2\alpha_k - \frac{1}{2}n\lfloor\frac{1}{4}(k-1)^2\rfloor)$ *for all* n *and* $k \leq m$.

Proof. Consider a drawing of $K_{m,n}$ in the plane with $\mathrm{cr}(K_{m,n})$ crossings. Let $\mathcal{G}$ be the collection of all subgraphs of $K_{m,n}$ isomorphic to $K_{k,n}$, obtained by selecting k vertices from $1,\ldots,m$. Then $|\mathcal{G}| = \binom{m}{k}$. Moreover, any two disjoint edges in $K_{m,n}$ occur in $\binom{m-2}{k-2}$ of the graphs in $\mathcal{G}$. So each crossing of $K_{m,n}$ occurs in $\binom{m-2}{k-2}$ of the graphs in $\mathcal{G}$. Therefore,

$$\mathrm{cr}(K_{m,n}) \geq \frac{\binom{m}{k}}{\binom{m-2}{k-2}}\mathrm{cr}(K_{k,n}) = \frac{m(m-1)}{k(k-1)}\mathrm{cr}(K_{k,n}). \tag{27}$$ ∎

This in turn implies:

Corollary 2b. $\lim\limits_{n\to\infty} \dfrac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq \dfrac{8\alpha_k}{k(k-1)}\dfrac{m}{m-1}$ *for all* $k \leq m$.

Proof. Using Corollary 2a:

$$\begin{aligned} &\lim_{n\to\infty}\frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq \lim_{n\to\infty}\frac{m(m-1)(\frac{1}{2}n^2\alpha_k - \frac{1}{2}n\lfloor\frac{1}{4}(k-1)^2\rfloor)}{k(k-1)Z(m,n)} = \\ &\lim_{n\to\infty}\frac{m(m-1)(\frac{1}{2}n^2\alpha_k - \frac{1}{2}n\lfloor\frac{1}{4}(k-1)^2\rfloor)}{k(k-1)\lfloor\frac{1}{4}(m-1)^2\rfloor\lfloor\frac{1}{4}(n-1)^2\rfloor} = \frac{2\alpha_k}{k(k-1)}\frac{m(m-1)}{\lfloor\frac{1}{4}(m-1)^2\rfloor} \geq \\ &\frac{8\alpha_k}{k(k-1)}\frac{m}{m-1}. \end{aligned} \tag{28}$$ ∎

The parameter α_m is defined by the conceptually very simple semidefinite programming problem (23), but the order $(m-1)!$ of the matrices increases fast with m. For $m \geq 7$, it is too large for present-day semidefinite programming software.

However, using the symmetry of C, de Klerk et al. [7] computed $\alpha_7 = 4.3593154965\ldots$, which implies

$$\mathrm{cr}(K_{7,n}) \geq 2.1796n^2 - 4.5n, \tag{29}$$

and also, for each $m \geq 7$ and n:

$$\mathrm{cr}(K_{m,n}) \geq 0.0518m(m-1)n^2 - \tfrac{3}{28}m(m-1)n \tag{30}$$

and for each $m \geq 7$:

$$\lim_{n\to\infty}\frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8303\frac{m}{m-1}. \tag{31}$$

We describe the approach to exploiting the symmetry further, and apply the method described in Section 2. Fix $m \in \mathbb{N}$. Let $G := S_m \times \{-1,+1\}$, and define $h : G \to S_{Z_m}$ by

$$h_{\pi,i}(\sigma) := \pi\sigma^i\pi^{-1} \tag{32}$$

for $\pi \in S_m$, $i \in \{-1,+1\}$, $\sigma \in Z_m$. So G acts on Z_m. Moreover, the cost matrix C satisfies $M_\pi C M_\pi^T = C$ for each $\pi \in G$ (cf. [7]), and also $M_\pi J M_\pi^T = J$ for each $\pi \in G$. Hence the method of Section 2 applies, and we can reduce (23) as in (19). Let the algebra $\mathcal{A}$ (as defined in (4)) in this case be denoted by $\mathcal{C}_m$. So its commutant is $\mathcal{C}_m'$.

Applying this method requires that we are able to identify the matrices E_i and the multiplication parameters $\lambda_{i,j}^k$. This indeed is possible for this application, where we have to identify the equivalence classes of pairs $(\sigma,\tau) \in Z_m \times Z_m$ under the equivalence relation

$$(33)\qquad (\sigma,\tau) \cong (\sigma',\tau') \iff \exists(\pi,i) \in G : h_{\pi,i}(\sigma) = \sigma', h_{\pi,i}(\tau) = \tau'.$$

Since each equivalence class contains a pair (ι,τ), where ι is the permutation $\iota := (1,\ldots,m)$, this can be done for instance by enumerating all $(m-1)!$ pairs (ι,τ) and check their equivalences. (We note here that $(9-1)! = 40320$ is still computationally feasible in this respect, whereas 40320×40320 matrices are too large for present-day semidefinite programming software.) Also the multiplication parameters $\lambda_{i,j}^k$ can be computed (for $m = 9$) within reasonable time.

With this method we were able to compute α_8 and α_9. It turns out that $\alpha_8 = 5.8599856444\ldots$, implying

$$(34)\qquad \mathrm{cr}(K_{8,n}) \geq 2.9299n^2 - 6n,$$

and also, for each $m \geq 8$ and n:

$$(35)\qquad \mathrm{cr}(K_{m,n}) \geq 0.0523m(m-1)n^2 - \tfrac{3}{28}m(m-1)n$$

and for each $m \geq 8$:

$$(36)\qquad \lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8371\frac{m}{m-1}.$$

Moreover, $\alpha_9 = 7.7352126\ldots$, implying

$$(37)\qquad \mathrm{cr}(K_{9,n}) \geq 3.8676063n^2 - 8n,$$

and also, for each $m \geq 9$ and n:

$$(38)\qquad \mathrm{cr}(K_{m,n}) \geq 0.0537m(m-1)n^2 - \tfrac{1}{9}m(m-1)n,$$

and for each $m \geq 9$:

$$(39)\qquad \lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8594\frac{m}{m-1}.$$

The dimension d of $\mathcal{C}_m'$ and the reduced dimension d_{reduced} (cf. the end of Section 2) are given in the following table:

m	d	d_{reduced}
1	1	1
2	1	1
3	2	2
4	3	3
5	8	7
6	20	17
7	78	56
8	380	239
9	2438	1366
10	18744	9848

Table 1: Table of dimension d and reduced dimension d_{reduced}

Computations for this paper were done on an SGI Altrix cluster running 64-bit Linux on 32 Itanium II processors, and with 128 GB of shared memory. We used the interior point implementation CSDP by Borchers [1] that relies upon BLAS/LAPACK matrix library routines (for the latter we used the parallel implementation by SGI).

For $m = 9$, the SDP problem to compute α_9 had more than 44 million nonzero data entries. This is larger than any SDP benchmark problem known to the authors. Its solution on the SGI Altrix cluster required more than seven days of wall clock time and used 1.47GB of memory.

It is therefore safe to say that the computation of α_{10} is out of reach of present-day computing power, at least when general-purpose interior point SDP solvers are used, even if we would be able to find the most economical representation of the problem (i.e., a block-diagonalization), simply because the number of variables remains too large. Any interior point method has to form and solve dense linear systems of order $d_{\text{reduced}} = 9848$ at each iteration when computing α_{10} (cf. Table 1). This is regardless of whether a block-diagonalization is known for the regular representation of $\mathcal{C}'_m$.

Moreover, an interior point algorithm will have to compute Choleski and/or singular value decompositions of matrices of order $d \times d$ at each iteration (or of order the largest block if a block-diagonalization is used).

Figure 1 shows the lower bounds obtained on the ratio $\lim_{m,n\to\infty} \operatorname{cr}(K_{m,n})/Z(m,n)$ by computing α_k for $k = 2, \ldots, 9$ (cf. Corollary 2b). So far, odd values of k gave relatively large improvements compared to the even values. This is reminiscent of the fact that, if the Zarankiewicz conjecture holds for $K_{2m-1,n}$, it also holds for $K_{2m,n}$.

We finally note that, for $m \geq 6$, the number of orbits of $Z_m \times Z_m$ under the actions of $G = S_m \times \{-1, +1\}$ is strictly smaller than if we restrict the actions to $S_m \times \{1\}$. In fact, G is precisely the full automorphism group of the matrix C.

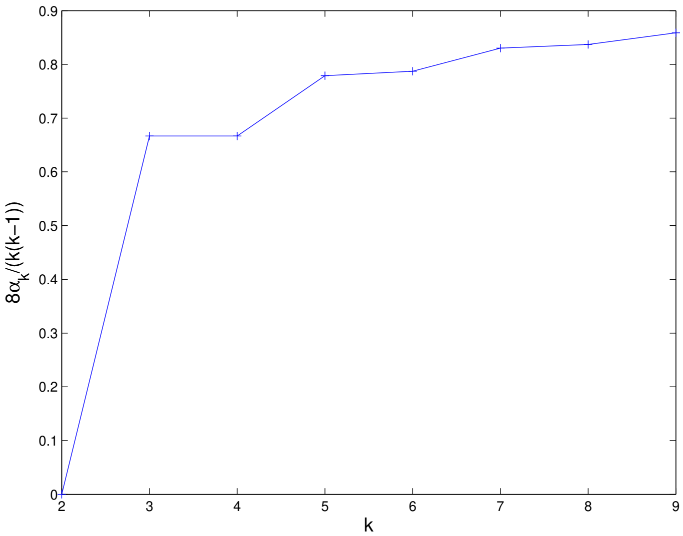

Figure 1: Each computed value α_k gives the lower bound $\lim_{n,m\to\infty} \frac{\mathrm{cr}(K_{n,m})}{Z(n,m)} \geq \frac{8\alpha_k}{k(k-1)}$.

4. Concluding remarks

We discuss what is new in this paper compared with [4] and [5]. Gatermann and Parrilo [4] only consider the situation where the canonical representation of the commutant is known. In the example we consider, this is not the case. In the paper of Kanno, Ohsaki, Murota, and Katoh [5], it is shown that the central path in semidefinite programming converges to a group symmetrical optimal solution (i.e. a solution in the commutant). Our approach restricts the optimization process to the commutant (in fact to a more economical representation of it). Thus the desirable feature of a symmetric optimal solution is retained, but with the additional advantage of a reduction in the size of the optimization problem.

Our method may also be applied to compute upper bounds on the size of error-correcting codes. For instance, it may reduce the Terwilliger algebra of the Hamming scheme H_n (cf. [9]), whose matrices have order $2^n \times 2^n$, to an algebra of matrices of order $\binom{n+3}{3} \times \binom{n+3}{3}$. This makes the corresponding bounds computable in time bounded by a polynomial in n (rather than in 2^n). However, for this application the block-diagonalization has been found ([9]), which allows a more efficient computation of the bounds. Laurent [8] showed that with the method of the present paper a hierarchy of further, polynomial-time computable sharpenings can be obtained for the coding problem.

Related to the coding application is computing the Lovász's ϑ bound of graphs G (and its variant ϑ') when the commutant of the automorphism group of G has low dimension (or when the algebra generated by the adjacency matrix and the all-one matrix has low dimension). Another potential application would be the truss topology design problem described in Kanno, Ohsaki, Murota, and Katoh [5] for trusses with suitable symmetry.

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