COMBINATORICA
Bolyai Society – Springer-Verlag

TOTAL DUAL INTEGRALITY OF MATCHING FOREST CONSTRAINTS

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Received August 31, 1998

Let $G=(V,E,A)$ be a mixed graph. That is, (V,E) is an undirected graph and (V,A) is a directed graph.

A *matching forest* (introduced by R. Giles) is a subset F of $E\cup A$ such that F contains no circuit (in the underlying undirected graph) and such that for each $v\in V$ there is at most one $e\in F$ such that v is head of e. (For an undirected edge e, both ends of e are called head of e.)

Giles gave a polynomial-time algorithm to find a maximum-weight matching forest, yielding as a by-product a characterization of the inequalities determining the convex hull of the incidence vectors of the matching forests.

We prove that these inequalities form a totally dual integral system. It is equivalent to an "all-integer" min-max relation for the maximum weight of a matching forest. Our proof is based on an exchange property for matching forests, and implies Giles' characterization.

1. Introduction

In an interesting trilogy, Rick Giles [9], [10], [11] introduced the concept of a matching forest in a mixed graph — a common generalization of matchings in undirected graphs and branchings in directed graphs — and he gave a polynomial-time algorithm to find a maximum-weight matching forest, yielding as a by-product a characterization of the matching forest polytope (the convex hull of the incidence vectors of matching forests).

Giles' results generalize the polynomial-time solvability and the polyhedral characterizations for matchings (Edmonds [4]) and for branchings (Chu and Liu [2], Edmonds [5], Bock [1]).

Mathematics Subject Classification (1991):

In this paper we prove that the system given by Giles is *totally dual integral* (cf. [15]). This means that the linear program of maximizing an integer objective function over the constraints has integer primal and dual solutions. It generalizes the total dual integrality of the matching constraints in an undirected graph, proved by Cunningham and Marsh [3] (which generalizes the Tutte-Berge formula for the maximum size of a matching (cf. [13])), and the total dual integrality of the branching constraints in a directed graph, proved implicitly by Edmonds [5], Bock [1], and Fulkerson [8].

We give some definitions and background, and formulate Giles' inequalities for the matching forest polytope.

A *mixed graph* is a triple (V,E,A), where (V,E) is an undirected graph and (V,A) is a directed graph. (So E and A are disjoint.) In this paper, a graph can have multiple edges, but no loops. If an edge e is directed from u to v, then u is called the *tail* and v the *head* of e. If e is undirected and connects u and v, then both u and v are called *head* of e. The *underlying* undirected graph of a mixed graph is the undirected graph obtained from the mixed graph by forgetting the orientations of the directed edges.

A subset F of $E\cup A$ is called a *matching forest* if F does not contain any circuit (in the underlying undirected graph) and any vertex v is head of at most one edge in F. We call a vertex v a *root* of F if v is not the head of any edge in F. We denote the set of roots of F by $R(F)$.

It is convenient to consider the relations of matching forests with matchings in undirected graphs and branchings in directed graphs.

A *matching* in an undirected graph (V,E) is a set M of pairwise disjoint edges. So M is a matching if and only if M is a matching forest in the mixed graph $(V,E,\emptyset)$. In this case, the roots of M are the vertices not covered by M.

A *branching* in a directed graph (V,A) is a set B of edges being a forest in the underlying undirected graph, such that each vertex is head of at most one edge in B. So B is a branching if and only if B is a matching forest in the mixed graph $(V,\emptyset,A)$. An *arborescence* is a branching B with exactly one root. So for any branching B, each weak component of the directed graph (V,B) is an arborescence, containing precisely one root of B. (A *weak component* of a mixed graph is a component of the underlying undirected graph.)

In turn, we can characterize matching forests in terms of matchings and branchings: for any mixed graph (V,E,A), a subset F of $E\cup A$ is a matching forest if and only if $F\cap A$ is a branching in (V,A) and $F\cap E$ is a matching in (V,E) such that $F\cap E$ only covers roots of $F\cap A$.

It is useful to observe and remember the following formulas, for any matching forest F in a mixed graph (V,E,A), setting $M := F\cap E$ and

$B := F \cap A$:

(1) $$R(F) = R(M) \cap R(B) \quad \text{and} \quad V = R(M) \cup R(B).$$

In fact, for any matching M in (V,E) and any branching B in (V,A), the set $M \cup B$ is a matching forest if and only if $R(M) \cup R(B) = V$.

The *matching forest polytope* of a mixed graph (V,E,A) is the convex hull of the incidence vectors of the matching forests. The *incidence vector* of a subset F of $E \cup A$ is the vector χ^F in $\{0,1\}^{E \cup A}$ satisfying $\chi^F(e) = 1$ if and only if $e \in F$. So the matching forest polytope is a polytope in $\mathbb{R}^{E \cup A}$.

Giles showed that the matching forest polytope is determined by the following inequalities:

(2) (i) $x(e) \geq 0$ for each $e \in E \cup A$,
(ii) $x(\delta^{\text{head}}(v)) \leq 1$ for each $v \in V$,
(iii) $x(\gamma(\mathcal{L})) \leq \lfloor |\cup \mathcal{L}| - \frac{1}{2}|\mathcal{L}| \rfloor$ for each subpartition $\mathcal{L}$ with $|\mathcal{L}|$ odd.

Here we use the following notation and terminology. $\delta^{\text{head}}(v)$ denotes the set of edges with head v. A *subpartition* is a collection of pairwise disjoint nonempty subsets of V. As usual, $\cup\mathcal{L}$ denotes the he union of the sets in $\mathcal{L}$. For each subpartition $\mathcal{L}$ we set:

(3) $\gamma(\mathcal{L})$:= the set of undirected edges contained in $\cup\mathcal{L}$ and of directed edges contained in any set in $\mathcal{L}$.

Finally, $\lfloor x \rfloor$ denotes the lower integer part of a real number x.

The inequalities (i) and (ii) in (2) are trivially valid for the incidence vector of any matching forest F. To see that (iii) is valid, we can assume that $F \subseteq \gamma(\mathcal{L})$ and that $V = \cup\mathcal{L}$. Then $|R(F \cap A)| \geq |\mathcal{L}|$, since each set in $\mathcal{L}$ contains at least one root of $F \cap A$. Moreover, $|F \cap E| \leq \lfloor \frac{1}{2}|R(F \cap A)| \rfloor$, since $F \cap E$ is a matching on $R(F \cap A)$. As $|F \cap A| = |V| - |R(F \cap A)|$, this gives:

(4) $$\begin{aligned} |F| = |F \cap E| + |F \cap A| &\leq \left\lfloor \frac{1}{2}|R(F \cap A)| \right\rfloor + (|V| - |R(F \cap A)|) \\ &= \left\lfloor |V| - \frac{1}{2}|R(F \cap A)| \right\rfloor \leq \left\lfloor |\cup \mathcal{L}| - \frac{1}{2}|\mathcal{L}| \right\rfloor \end{aligned}$$

as required.

In this paper we show that system (2) is totally dual integral (a concept introduced by Edmonds and Giles [7]). It means that, for each 'weight' function $w : E \cup A \to \mathbb{Z}$, the linear program of maximizing $w^T x$ over (2) has integer optimum primal and dual solutions. It is equivalent to an "all-integer" min-max formula for the maximum weight of a matching forest (for integer weights).

Our proof implies Giles' characterization of the matching forest polytope, while we do not make use of the theorem of Edmonds and Giles that the existence of integer optimum dual solutions implies the existence of integer optimum primal solutions.

More precisely, our proof consists of showing that for any integer weight function w on $E \cup A$, there exists a matching forest F and an integer dual solution to the linear program of maximizing $w^T x$ over (2), of value equal to the weight of F. Since the incidence vector of F satisfies (2), standard polyhedral theory then gives that the vertices of the polytope P determined by (2) are exactly the incidence vectors of the matching forests (as any vertex of P is the unique optimum solution of maximizing $w^T x$ over (2), for some integer weight function w).

The total dual integrality of (2) has as special case a min-max formula for the maximum size of a matching forest in a mixed graph (V, E, A). However, a direct formula for this can be easily derived from the Tutte-Berge formula — see Section 4.

Our proof method in fact is a generalization of the proof in [14] of the result of Cunningham and Marsh [3] that the matching constraints are totally dual integral. The proof in [14] uses ideas of Lovász [12] on the characterization of the matching polytope.

2. An exchange property of matching forests

Bases of our proof are an exchange property for branchings, and one for matching forests. The latter generalizes a well-known and trivial exchange property for matchings in an undirected graph, based on considering the union of two matchings.

The proofs use the following lemma, that follows directly from a theorem of Edmonds [6] on edge-disjoint branchings. Here $d^{\text{in}}(K)$ denotes the number of edges entering K (a *strong component* of a directed graph is an inclusionwise maximal set of vertices inducing a strongly connected subgraph):

Lemma 1. *Let B_1 and B_2 be branchings partitioning the edge set of a directed graph $D = (V, A)$. Let R_1 and R_2 be sets with $R_1 \cup R_2 = R(B_1) \cup R(B_2)$ and $R_1 \cap R_2 = R(B_1) \cap R(B_2)$. Then A can be split into branchings B_1' and B_2' with $R(B_i') = R_i$ for $i = 1, 2$ if and only if each strong component K of D with $d^{\text{in}}(K) = 0$ intersects both R_1 and R_2.*

Proof. Necessity is easy, since the root set of any branching intersects any component K with $d^{\text{in}}(K) = 0$.

To see sufficiency, by the theorem of Edmonds [6], branchings B_1' and B_2' as required exist, if and only if for each nonempty subset U of V one has

$$d^{\text{in}}(U) \geq |\{i \in \{1,2\} | U \cap R_i = \emptyset\}|. \tag{5}$$

(Actually, Edmonds' theorem gives the existence of disjoint branchings B_1' and B_2' with $R(B_i') = R_i$ for $i = 1, 2$. The fact that $B_1' \cup B_2' = A$ follows from the fact that $|B_1'| + |B_2'| = |B_1| + |B_2|$, as $|R(B_1')| + |R(B_2')| = |R(B_1)| + |R(B_2)|$.)

Suppose that inequality (5) does not hold. Then the RHS is positive. If it is 2, then U is disjoint from both R_1 and R_2, and hence from both $R(B_1)$ and $R(B_2)$ (since $R_1 \cup R_2 = R(B_1) \cup R(B_2)$), implying that both B_1 and B_2 enter U, and so $d^{\text{in}}(U) \geq 2$.

So the RHS is 1, and hence the LHS is 0. We can assume that U is an inclusionwise minimal set with this property. It implies that U is a strong component of D. Then by the condition, U intersects both R_1 and R_2, contradicting the fact that the RHS in (5) is 1. ∎

First, this implies the following exchange property for branchings:

Theorem 1. *Let $D = (V, A)$ be a directed graph, and let B_1 and B_2 be branchings in D partitioning A. Let s be a root of B_2 but not of B_1, and let r be the root of the arborescence in B_1 containing s. Then A can be partitioned into branchings B_1' and B_2' with $R(B_1') = R(B_1) \cup \{s\}$ or $R(B_1') = (R(B_1) \setminus \{r\}) \cup \{s\}$.*

Proof. Let K be the strong component of D containing s. If no edge of D enters K, then $r \in K$ (as B_1 contains a directed path from r to s), and hence r is not a root of B_2 (as otherwise no arc enters r while K is strongly connected); define $R_1 := (R(B_1) \setminus \{r\}) \cup \{s\}$ and $R_2 := (R(B_2) \setminus \{s\}) \cup \{r\}$. Alternatively, if some edge of D enters K, define $R_1 := R(B_1) \cup \{s\}$ and $R_2 := R(B_2) \setminus \{s\}$. Then the Lemma implies that A can be split into branchings B_1' and B_2' with $R(B_i') = R_i$ for $i = 1, 2$. ∎

The lemma is also used to prove the following exchange property for matching forests (which can be generalized (with the methods below) so that it contains Theorem 1 — however, a most general statement seems not very attractive to formulate, and therefore we have restricted ourselves to stating what we need for our present purpose):

Theorem 2. *Let (V, E, A) be a mixed graph, and let F_1 and F_2 be matching forests partitioning $E \cup A$. Let $s \in R(F_2) \setminus R(F_1)$. Then there exist matching forests F_1' and F_2' partitioning $E \cup A$ such that $s \in R(F_1')$ and such that*

(6) *either* (i) $|F_1'| < |F_1|$,
or (ii) $|F_1'| = |F_1|$ *and* $|R(F_1')| > |R(F_1)|$,
or (iii) $|F_1'| = |F_1|$, $R(F_1') = (R(F_1) \setminus \{t\}) \cup \{s\}$ *for some* $t \in R(F_1)$, *and* $|R(F_1' \cap A) \cap K| = |R(F_1 \cap A) \cap K|$ *for each strong component* K *of the directed graph* (V, A).

Proof. Define $M_i := F_i \cap E$ and $B_i := F_i \cap A$ for $i = 1, 2$. Let $\mathcal{K}$ be the collection of those strong components K of the directed graph (V, A) such that no edge in A enters K. Note that each set in $\mathcal{K}$ intersects both $R(B_1)$ and $R(B_2)$, and that for any vertex $v \in R(B_1) \cap R(B_2)$ one has $\{v\} \in \mathcal{K}$.

So each $K \in \mathcal{K}$ with $|K| \geq 2$, intersects $R(B_1)$ and $R(B_2)$ in disjoint subsets. Hence we can choose for each such K

(7) a pair $e_K \subseteq K$ consisting of a vertex in $R(B_1) \setminus R(B_2)$ and a vertex in $R(B_2) \setminus R(B_1)$.

Let N be the set of pairs e_K for $K \in \mathcal{K}$ with $|K| \geq 2$. So N is a matching on V.

Then the undirected graph H on V with edge set

$$M_1 \cup M_2 \cup N \tag{8}$$

consists of a number of vertex-disjoint paths and circuits, since any vertex in $R(B_1) \setminus R(B_2)$ is not covered by M_2, and any vertex in $R(B_2) \setminus R(B_1)$ is not covered by M_1.

Moreover, s is the beginning vertex of a path component P of H (possibly only consisting of s). Indeed, s is not covered by M_2, as $s \in R(F_2) = R(M_2) \cap R(B_2)$. If s is covered by M_1, then $s \in R(B_1)$, and so $s \in R(B_1) \cap R(B_2)$, implying that s is not covered by N.

Let Y be the set of edges in $M_1 \cup M_2$ occurring in P, and set

(9) $M_1' := M_1 \triangle Y$ and $M_2' := M_2 \triangle Y$

(where $\triangle$ denotes symmetric difference). Since Y is the union of the edge set of some (path) components of the graph $(V, M_1 \cup M_2)$, we know that M_1' and M_2' are matchings again.

Then, obviously, $R(M_1')$ and $R(M_2')$ arise from $R(M_1)$ and $R(M_2)$ by exchanging these sets on VP (= the vertex set of P); that is:

$$\begin{aligned} R(M_1') &= (R(M_1) \setminus VP) \cup (R(M_2) \cap VP) \quad \text{and} \\ R(M_2') &= (R(M_2) \setminus VP) \cup (R(M_1) \cap VP). \end{aligned} \tag{10}$$

We show that a similar operation can be performed with respect to B_1 and B_2; that is, we show that there exist disjoint branchings B'_1 and B'_2 satisfying

(11)
$$R(B'_1) = (R(B_1) \setminus VP) \cup (R(B_2) \cap VP) \quad \text{and}$$
$$R(B'_2) = (R(B_2) \setminus VP) \cup (R(B_1) \cap VP).$$

By the Lemma, it suffices to show that each strong component K of (V, A) with $d^{\text{in}}(K) = 0$ intersects both sets in (11). If $|K| = 1$ then K is contained in both $R(B_1)$ and $R(B_2)$, and hence in both sets in (11). If $|K| \geq 2$, then e_K intersects both $R(B_1)$ and $R(B_2)$. Since e_K is either contained in VP or disjoint of VP, e_K intersects both sets in (11). Hence, as $e_K \subseteq K$, also K intersects both sets in (11). Therefore, branchings B'_1 and B'_2 satisfying (11) exist.

(10) and (11) imply:

(12) $\quad F'_1 := M'_1 \cup B'_1 \quad$ and $\quad F'_2 := M'_2 \cup B'_2 \quad$ are matching forests.

To see this, we must show that $R(M'_1) \cup R(B'_1) = V$ and $R(M'_2) \cup R(B'_2) = V$. Since $R(M_1) \cup R(B_1) = V$ and $R(M_2) \cup R(B_2) = V$, this follows directly from (10) and (11). This shows (12).

Since $R(F) = R(M) \cap R(B)$ for any matching forest F (with $M := F \cap E$ and $B := F \cap A$), (10) and (11) imply that also $R(F'_1)$ and $R(F'_2)$ arise from $R(F_1)$ and $R(F_2)$ by swapping on P; that is:

(13)
$$R(F'_1) = (R(F_1) \setminus VP) \cup (R(F_2) \cap VP) \quad \text{and}$$
$$R(F'_2) = (R(F_2) \setminus VP) \cup (R(F_1) \cap VP).$$

This implies:

(14)
$$s \in R(F'_1) \setminus R(F'_2),$$

since $s \in VP$ and $s \in R(F_2) \setminus R(F_1)$.

We study the effects of the exchanges (10) and (11), to show that one of the alternatives (6) holds. It is based on the following observations on the sizes of M'_1 and B'_1. Let t be the last vertex of P (possibly $t = s$).

Suppose that none of the alternatives (6) holds. If $s = t$, then s is not covered by M_1, and so $M'_1 = M_1$ and $R(B'_1) = R(B_1) \cup \{s\}$, implying $|F'_1| < |F_1|$, which is alternative (6)(i). So $s \neq t$.

By the exchanges we made, $|M_1| - |M'_1| = |M_1 \cap EP| - |M_2 \cap EP|$ (where $EP =$ the edge set of P), and $|R(F_1)| - |R(F'_1)| = |R(F_1) \cap VP| - |R(F_2) \cap VP|$. This gives, as alternative (6)(i) does not hold:

(15)
$$|M_1 \cap EP| - |M_2 \cap EP| + |R(F_1) \cap VP| - |R(F_2) \cap VP|$$
$$= |M_1| + |R(F_1)| - |M'_1| - |R(F'_1)| = |F'_1| - |F_1| \geq 0.$$

(The last equality follows from the fact that $|F'_i| = |V| - |M'_i| - |R(F'_i)|$ for $i=1,2$, since $|F'_i| + |M'_i|$ is the number of heads of edges in F'_i.)

We next note:

(16) any intermediate vertex v of P does not belong to $R(F_1) \cup R(F_2)$.

For suppose that $v \in R(F_1)$. Then (as v is an intermediate vertex of P) v is covered by M_2 and some $e_K \in N$. Hence $v \in R(B_2)$, and therefore $v \notin R(B_1)$ (by (7)), contradicting the fact that $v \in R(F_1)$. One similarly shows that $v \notin R(F_2)$, proving (16).

As $s \in R(F_2) \setminus R(F_1)$, (16) implies that

(17) $|R(F_1) \cap VP| \le |R(F_2) \cap VP|$, with equality only if $t \in R(F_1) \setminus R(F_2)$.

With (15) this gives that $|M_1 \cap EP| \ge |M_2 \cap EP|$.

Let k be the number of edges in $M_1 \cup M_2$ on P. Note that the edges in $M_1 \cup M_2$ occur along P alternatingly in M_1 and M_2, as any intermediate $e_K \in N$ on P connects an edge in M_1 and an edge in M_2 (as by (7), $e_K \in N$ consists of a vertex not in $R(B_2)$ and a vertex not in $R(B_1)$).

Suppose k is odd. Then $|M_1 \cap EP| = |M_2 \cap EP| + 1$. So the last edge in $M_1 \cup M_2$ along P (seen from s) belongs to M_1. Moreover, one has that $t \notin R(F_1)$. For if $t \in R(F_1)$, then t is not covered by M_1, and hence t belongs to some $e_K = \{v,t\} \in N$ with v covered by M_1. Hence $v \in R(B_1)$, and hence $t \notin R(B_1)$ (by (7)), contradicting the fact that $t \in R(F_1)$. So (17) implies that $|R(F_2) \cap VP| > |R(F_1) \cap VP|$. This implies with (15) that $|F'_1| = |F_1|$ (as $|M_1 \cap EP| = |M_2 \cap EP| + 1$), and with (13) that $|R(F'_1)| > |R(F_1)|$. So (6)(ii) holds, a contradiction.

So k is even, and hence $|M_1 \cap EP| = |M_2 \cap EP|$, which implies with (13), (15) and (17) that $|R(F_1)| = |R(F_2)|$ and $t \in R(F_1) \setminus R(F_2)$. Therefore, $R(F'_1) = (R(F_1) \setminus \{t\}) \cup \{s\}$.

Finally, $|R(B'_1) \cap K| = |R(B_1) \cap K|$ for each strong component K of D. This follows directly (with (11)) from the fact that for any $v \in K \cap VP$ one has either $K = \{v\}$ (if $|K| = 1$) or $v \in e_K$ (if $|K| \ge 2$). For suppose that $v \in VP$ is not incident with any $e_K \in N$. If v is an intermediate vertex of P, then v is covered by M_1 and M_2 and hence v belongs to $R(B_1)$ and $R(B_2)$. If $v = s$, then $v \in R(F_2)$ (so $v \in R(B_2)$) and v is covered by M_1, so $v \in R(B_1)$. If $v = t$, then $v \in R(F_1)$ (so $v \in R(B_1)$) and v is covered by M_2, so $v \in R(B_2)$. ∎

3. Total dual integrality of matching forest constraints

We show that system (2) is totally dual integral, which is equivalent to the following.

For any weight function $w:E\cup A\to\mathbb{Z}$, let ν_w denote the maximum weight of a matching forest. Call a matching forest F *w-maximal* if $w(F)=\nu_w$. Let Λ be the set of subpartitions $\mathcal{L}$ of V with $|\mathcal{L}|$ odd.

Then the total dual integrality of (2) is equivalent to: for any 'weight' function $w:E\cup A\to\mathbb{Z}$, there exist $y:V\to\mathbb{Z}_+$ and $z:\Lambda\to\mathbb{Z}_+$ satisfying

$$\sum_{v\in V} y(v)+\sum_{\mathcal{L}\in\Lambda} z(\mathcal{L})\lfloor|\cup\mathcal{L}|-\frac{1}{2}|\mathcal{L}|\rfloor\le\nu_w \tag{18}$$

and

$$\sum_{v\in V} y(v)\chi^{\delta^{\text{head}}(v)}+\sum_{\mathcal{L}\in\Lambda} z(\mathcal{L})\chi^{\gamma(\mathcal{L})}\ge w. \tag{19}$$

Theorem 3. *For any mixed graph (V,E,A), system (2) is totally dual integral.*

Proof. We must prove that for each mixed graph (V,E,A) and each function $w:E\cup A\to\mathbb{Z}$, there exist y,z satisfying (18) and (19).

In proving this, we can assume that w is nonnegative. For suppose that w has negative entries, and let w' be obtained from w by setting its negative entries to 0. As $\nu_{w'}=\nu_w$ and $w'\ge w$, any y, z satisfying (18) and (19) with respect to w', also satisfy (18) and (19) with respect to w.

Suppose the theorem is not true. Choose a counterexample (V,E,A) and $w:E\cup A\to\mathbb{Z}_+$ with $|V|+|E\cup A|+\sum_{e\in E\cup A} w(e)$ as small as possible.

Then the underlying undirected graph of (V,E,A) is connected, since otherwise one of the components will form a smaller counterexample. Moreover, $w(e)\ge 1$ for each edge e, since otherwise we can delete e to obtain a smaller counterexample.

Then:

(20) for each $v\in V$, there exists a w-maximal matching forest F with $v\in R(F)$.

For suppose that such a matching forest does not exist. For any edge e, let $w'(e):=w(e)-1$ if v is head of e and $w'(e):=w(e)$ otherwise. Then $\nu_{w'}=\nu_w-1$. By the minimality of w, there exist y,z satisfying (18) and (19) with respect to w'. Replacing $y(v)$ by $y(v)+1$ we obtain y,z satisfying (18) and (19) with respect to w, contradicting our assumption. This proves (20).

This implies:

(21) each weak component of the directed graph (V,A) is strongly connected.

To see this, it suffices to show that each directed edge $e=(u,v)$ is contained in some directed circuit. By (20) there exists a w-maximal matching forest

F with $v \in R(F)$. Then the weak component of F containing v is an arborescence rooted at v. As F has maximum-weight, $F \cup \{e\}$ is not a matching forest, and hence $F \cap A$ contains a directed $v-u$ path. This makes a directed circuit containing e, and proves (21).

Let $\mathcal{K}$ denote the collection of strong components of (V,A). Define $w'(e) := w(e) - 1$ for each edge e. The remainder of this proof consists of showing that $|\mathcal{K}|$ is odd (so $\mathcal{K} \in \Lambda$), and that

$$\nu_w \geq \nu_{w'} + \lfloor |V| - \frac{1}{2}|\mathcal{K}| \rfloor. \tag{22}$$

This is enough, since, by the minimality of w, there exist y, z satisfying (18) and (19) with respect to w'. Replacing $z(\mathcal{K})$ by $z(\mathcal{K}) + 1$ we obtain y, z satisfying (18) and (19) with respect to w (note that $\gamma(\mathcal{K}) = E \cup A$), contradicting our assumption.

To show (22), choose a w'-maximal matching forest F of maximum size $|F|$. Under this condition, choose F such that it maximizes $|R(F)|$.

We show that for each $s \in V$ the following holds, where r is the root of the arborescence in $F \cap A$ containing s:

(23) there exist a $t \in R(F)$ and a w'-maximal matching forest F' satisfying $|F'| = |F|$, $R(F') = (R(F) \setminus \{t\}) \cup \{s\}$, and $|R(F' \cap A) \cap K| = |R(F \cap A) \cap K|$ for each strong component K of (V, A); if $r \in R(F)$, then moreover $t = r$, and $R(F' \cap A) = (R(F \cap A) \setminus \{r\}) \cup \{s\}$.

Let $F_1 := F$, and let F_2 be a w-maximal forest with $s \in R(F_2)$ (which exists by (20)).

If $r \notin R(F)$, apply Theorem 2 to F_1 and F_2, in the mixed graph obtained from (V, E, A) by deleting any edge not in $F_1 \cup F_2$, and by replacing any edge in $F_1 \cap F_2$ by two parallel edges. It yields the matching forests F_1' and F_2'.

If $r \in R(F)$, apply Theorem 1 to $B_1 := F_1 \cap A$ and $B_2 := F_2 \cap A$, in the directed graph obtained from (V, A) by deleting any edge not in $B_1 \cup B_2$, and by replacing any edge in $B_1 \cap B_2$ by two parallel edges. It yields the branchings B_1' and B_2' in (V, A) satisfying $R(B_1') = R(B_1) \cup \{s\}$ or $R(B_1') = (R(B_1) \setminus \{r\}) \cup \{s\}$. This implies $R(B_2') = R(B_2) \setminus \{s\}$ or $R(B_2') = (R(B_2) \setminus \{s\}) \cup \{r\}$. Now define $F_i' := (F_i \cap E) \cup B_i'$ for $i = 1, 2$. Then the F_i' are matching forests, since $r \in R(F_1 \cap E)$ and $s \in R(F_2 \cap E)$.

Note that in both constructions, $|F_1'| \leq |F_1|$, and if $|F_1'| = |F_1|$ then $|R(F_1')| \geq |R(F_1)|$. Moreover,

$$\chi^{F_1'} + \chi^{F_2'} = \chi^{F_1} + \chi^{F_2}, \tag{24}$$

which implies that $w(F_1')+w(F_2')=w(F_1)+w(F_2)$. Hence

$$(25)\quad \begin{aligned} w'(F_1')+w(F_2') &= w(F_1')+w(F_2')-|F_1'| \geq w(F_1)+w(F_2)-|F_1| \\ &= w'(F_1)+w(F_2). \end{aligned}$$

Therefore, since F_1 is a w'-maximal matching forest and F_2 is a w-maximal matching forest, we have equality throughout in (25). So F_1' is w'-maximal and $|F_1'|=|F_1|$. Hence $|R(F_1')|\geq|R(F_1)|$. Then, by the maximality of $|R(F)|$, we know that $|R(F_1')|=|R(F_1)|$. Therefore, setting $F':=F_1'$, the conclusions in Theorems 2 and 1 give (23).

Note that (23) implies in particular that $R(F)\neq\emptyset$. Suppose $|R(F)|\geq 2$. Choose F under the additional condition that the minimum distance in (V,E,A) between distinct vertices $u,v\in R(F)$ is as small as possible. Here, the distance in (V,E,A) is the length of a shortest u–v path in the underlying undirected graph.

Necessarily, this distance is at least two, since otherwise we can extend F by an edge connecting u and v, thereby maintaining w'-maximality but increasing the size. This contradicts the maximality of $|F|$.

So we can choose an intermediate vertex s on a shortest $u-v$ path. Let F' be the matching forest described in (23), with $t\in R(F)$. By symmetry of u and v we can assume that $t\neq u$. Then $u,s\in R(F')$, contradicting the choice of F, as the distance of u and s is smaller than that of u and v.

This implies that $|R(F)|=1$. Let $R(F)=\{r\}$, and let K be the strong component of (V,A) containing r. We choose F under the additional constraint that $|R(F\cap A)\cap K|$ is as large as possible.

Suppose $|R(F\cap A)\cap K|\geq 2$. Choose F under the additional constraint that r has minimal distance in (V,A) from some root u of $F\cap A$ in $K\setminus\{r\}$. Here, the distance in (V,A) from u to r is the length of a shortest directed $u-r$ path.

Let T be the arborescence in $F\cap A$ containing r. Let s be the first vertex in T on a shortest directed $u-r$ path Q in (V,A). Necessarily $s\neq r$, since otherwise we can extend F by the last edge of Q, contradicting the maximality of $|F|$.

Let F' be the matching forest described in (23). Then $s\in R(F')$ and $R(F'\cap A)=(R(F\cap A)\setminus\{r\})\cup\{s\}$. Hence u remains a root of $F'\cap A$, while the distance in (V,A) from u to s is shorter than that from u to r. This contradicts our choice of F.

So $|R(F\cap A)\cap K|=1$. Suppose that there exists a component L of (V,A) with $|R(F\cap A)\cap L|\geq 2$. Choose s in L arbitrarily. Let F' be the matching forest described in (23). Then $s\in R(F')$ while $|R(F'\cap A)\cap L|\geq 2$, contradicting the choice of F.

So such a component L does not exist; that is, each $L \in \mathcal{K}$ contains exactly one root of $F \cap A$. So $|F \cap A| = |V| - |\mathcal{K}|$. Moreover, as $|R(F)| = 1$, $|\mathcal{K}|$ is odd and $|F \cap E| = \lfloor \frac{1}{2}|\mathcal{K}| \rfloor$. So $|F| = |F \cap A| + |F \cap E| = \lfloor |V| - \frac{1}{2}|\mathcal{K}| \rfloor$. Hence

(26) $$\nu_w \geq w(F) = w'(F) + |F| = \nu_{w'} + |F| = \nu_{w'} + \lfloor |V| - \frac{1}{2}|\mathcal{K}| \rfloor,$$

thus proving (22). ∎

We remark that the optimum dual solution y, z constructed in this proof has the following additional property: if $\mathcal{K}, \mathcal{L} \in \Lambda$ and $z(\mathcal{K}), z(\mathcal{L}) > 0$, then $\mathcal{K}$ and $\mathcal{L}$ are "laminar" in the following sense:

(27) $$\begin{aligned} &\forall K \in \mathcal{K} \exists L \in \mathcal{L} : K \subseteq L, \\ \text{or} \quad &\forall L \in \mathcal{L} \exists K \in \mathcal{K} : L \subseteq K, \\ \text{or} \quad &\forall K \in \mathcal{K} \forall L \in \mathcal{L} : K \cap L = \emptyset. \end{aligned}$$

4. The maximum size of a matching forest

The total dual integrality of (2) has as special case a min-max formula for the maximum size of a matching forest in a mixed graph (V, E, A). However, this min-max formula can be easily derived from the Tutte-Berge formula with the following direct formula.

Let $\mathcal{K}$ be the collection of strong components K of the directed graph (V, A) with the property that no directed edge enters K. Consider the undirected graph H with vertex set $\mathcal{K}$, where two distinct $K, L \in \mathcal{K}$ are adjacent if and only if there is an edge in E connecting K and L.

Then one has:

(28) The maximum size of a matching forest in (V, E, A) is equal to $\nu(H) + |V| - |\mathcal{K}|$,

where $\nu(H)$ denotes the maximum size of a matching in H.

To see (28), let M' be a maximum-size matching in H. Then M' yields a matching M in (V, E). Now there exists a branching B in (V, A) with the property that B has exactly $|\mathcal{K}|$ roots, such that each $K \in \mathcal{K}$ contains exactly one root and such that each vertex covered by M is a root of B. (To see that such a branching B exists, choose, for any $K \in \mathcal{K}$ not intersecting M, an arbitrary vertex in K. Let X be the set of chosen vertices together with the vertices covered by M. As X intersects each $K \in \mathcal{K}$, each vertex in V is reachable in (V, A) by a directed path from X. Hence there exists a branching B with root set X. This B has the required properties.)

Then $M \cup B$ is a matching forest, of size $\nu(H)+|V|-|\mathcal{K}|$ (as B has size $|V|-|\mathcal{K}|$).

To see that $M \cup B$ is a maximum-size matching forest, let F be any matching forest. Let $U := \cup\mathcal{K}$. Then F has at most $|V \setminus U|$ edges with at least one head in $V \setminus U$. Since no directed edge enters U, all other edges are contained in U. So it suffices to show that F has at most $\nu(H)+|U|-|\mathcal{K}|$ edges contained in U.

Let N be the set of (necessarily undirected) edges in F connecting two different components in $\mathcal{K}$. For each $K \in \mathcal{K}$ let α_K be the number of edges in N incident with K. Then

$$|N| - \sum_{K \in \mathcal{K}} \max\{0, \alpha_K - 1\} \leq \nu(H), \tag{29}$$

since by deleting, for each $K \in \mathcal{K}$, at most $\max\{0, \alpha_K - 1\}$ edges from N incident with K, we obtain a matching in the graph H defined above.

We have moreover that any $K \in \mathcal{K}$ contains at most $|K| - \max\{1, \alpha_K\}$ edges of F. With (29) this implies that the number of edges in F contained in U is at most

$$|N| + \sum_{K \in \mathcal{K}} (|K| - \max\{1, \alpha_K\}) \tag{30}$$

$$\leq \nu(H) + \sum_{K \in \mathcal{K}} (\max\{0, \alpha_K - 1\} + |K| - \max\{1, \alpha_K\}) = \nu(H) + |U| - |\mathcal{K}|,$$

as required.

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