Tensor subalgebras and First Fundamental Theorems in invariant theory

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Abstract. Let V be an n-dimensional complex inner product space and let $T := T(V)\otimes T(V^*)$ be the mixed tensor algebra over V. We characterize those subsets A of T for which there is a subgroup G of the unitary group $\mathcal{U}(n)$ such that $A = T^G$. They are precisely the nondegenerate contraction-closed graded $*$-subalgebras of T. While the proof makes use of the First Fundamental Theorem for $\mathrm{GL}(n,\mathbb{C})$ (in the sense of Weyl), the characterization has as direct consequences First Fundamental Theorems for several subgroups of $\mathrm{GL}(n,\mathbb{C})$. Moreover, a Galois correspondence between linear algebraic $*$-subgroups of $\mathrm{GL}(n,\mathbb{C})$ and nondegenerate contraction-closed graded $*$-subalgebras of T is derived. We also consider some combinatorial applications, viz. to self-dual codes and to combinatorial parameters.

1 Introduction

Let V be an n-dimensional complex inner product space, with inner product $\langle .,. \rangle$ and with dual space V^*. (The inner product is $\mathbb{C}$-linear in the first variable, and conjugate linear in the second variable.) Denote, as usual,

$$(1) \qquad T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k} \text{ and } T(V^*) := \bigoplus_{k=0}^{\infty} V^{*\otimes k},$$

where $V^{\otimes k}$ and $V^{*\otimes k}$ denote the tensor product of k copies of V and V^* respectively. Set

$$(2) \qquad T := T(V)\otimes T(V^*) \cong \bigoplus_{k,l=0}^{\infty} V^{\otimes k}\otimes V^{*\otimes l}.$$

This is the *mixed tensor algebra* over V (cf. [5]). (The multiplication is the usual tensor product of the rings $T(V)$ and $T(V^*)$, governed by the rule $(x\otimes y)\otimes(x'\otimes y') = (x\otimes x')\otimes(y\otimes y')$ for $x,x'\in T(V)$ and $y,y'\in T(V^*)$.)

Fixing an orthonormal basis $e_1,\ldots,e_n$ of V, we can identify V with $\mathbb{C}^n$ (with the inner product $\langle a,b\rangle = \overline{b}^{\mathsf{T}} a$). For any $U \in \mathrm{GL}(n,\mathbb{C})$, let $z \mapsto z^U$ be

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the linear right action of U on T, which is the unique algebra endomorphism on T satisfying $x^U = U^{-1}x$ and $y^U(x) = y(Ux)$ for $x \in V$ and $y \in V^*$. For any $G \subseteq \mathrm{GL}(n,\mathbb{C})$ and $A \subseteq T$, denote

(3) $$A^G := \{z \in A \mid z^U = z \text{ for all } U \in G\} \text{ and } G^A := \{U \in G \mid z^U = z \text{ for all } z \in A\}.$$

In this paper we characterize those subsets A of T for which there exists a subgroup G of the unitary group $\mathcal{U}(n)$ such that $A = T^G$. They turn out to be precisely the graded $*$-subalgebras of T that are nondegenerate and contraction-closed (for definitions, see Section 2). Our proof is based on the Stone-Weierstrass theorem, the First Fundamental Theorem (in the sense of Weyl [12]) for $\mathrm{GL}(n,\mathbb{C})$, and the existence of a Haar measure on $\mathcal{U}(n)$.

As consequences we derive the First Fundamental Theorem for a number of subgroups of $\mathrm{GL}(n,\mathbb{C})$. Indeed, our theorem directly implies that if some subgroup G of $\mathrm{GL}(n,\mathbb{C})$ satisfies $G = G^S$ for some subset S of T with $S = S^*$, then T^G is equal to the smallest nondegenerate contraction-closed graded subalgebra of T containing S. This often directly yields a spanning set for $(V^{\otimes k} \otimes V^{*\otimes l})^G$ for all k, l. That is, it implies a First Fundamental Theorem for G (the tensor version, which is equivalent to the polynomial version — cf. Goodman and Wallach [4] Section 4.2.3). We describe this in Section 5.

A subgroup G of $\mathrm{GL}(n,\mathbb{C})$ is called a *$*$-subgroup* if $G = G^* := \{U^* \mid U \in G\}$ (where U^* is the conjugate transpose of U). The following characterization is well-known: A linear algebraic subgroup $G \subseteq \mathrm{GL}(n,\mathbb{C})$ is a $*$-subgroup if and only if G is reductive and $G \cap \mathcal{U}(n)$ is a maximal compact and hence Zariski-dense subgroup. In Section 4 we show that if G is any $*$-subgroup of $\mathrm{GL}(n,\mathbb{C})$ and we set $A := T^G$, then $\mathrm{GL}(n,\mathbb{C})^A$ is equal to the smallest linear algebraic subgroup of $\mathrm{GL}(n,\mathbb{C})$ containing G. Together with the characterization above, this implies a Galois correspondence between linear algebraic $*$-subgroups of $\mathrm{GL}(n,\mathbb{C})$ and nondegenerate contraction-closed graded $*$-subalgebras of T.

In Sections 6 and 7 we give combinatorial applications of our theorem, viz. to self-dual codes and to combinatorial parameters. For the sake of exposition, we restrict ourselves to describing the most elementary of these applications. The application to combinatorial parameters in fact was our main motivation to prove Theorem 1.

2 Preliminaries

For any $A \subseteq T$ and $k, l \geq 0$, denote

$$(4) \qquad A_l^k := A \cap (V^{\otimes k} \otimes V^{*\otimes l}).$$

A subalgebra A of T is called *graded* if $A = \bigoplus_{k,l=0}^{\infty} A_l^k$.

For any $x \in V$, let $x^* \in V^*$ be defined by $x^*(z) = \langle z, x\rangle$ for all $z \in V$. This extends to a unique function $x \mapsto x^*$ on T satisfying $(x^*)^* = x$, $(\lambda x)^* = \overline{\lambda} x^*$, $(x+y)^* = x^* + y^*$, and $(x \otimes y)^* = y^* \otimes x^*$ for all $x, y \in T$ and $\lambda \in \mathbb{C}$. A subalgebra A of T is called a $*$*-subalgebra* if $A^* = A$.

$e_1^*, \ldots, e_n^*$ is equal to the usual dual basis of $e_1, \ldots, e_n$. The inner product $\langle .,. \rangle$ on V extends uniquely to an inner product on T for which all products

$$(5) \qquad e_{i_1} \otimes \cdots e_{i_k} \otimes e_{j_1}^* \otimes \cdots e_{j_l}^*$$

form an orthonormal basis, where k, l range over all nonnegative integers and where $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$ range over $1, \ldots, n$. (The inner product is independent of the choice of $e_1, \ldots, e_n$.) For all $x, y \in T$,

$$(6) \qquad \langle x^*, y^* \rangle = \langle y, x \rangle = \overline{\langle x, y \rangle}.$$

Moreover, $(z^U)^* = (z^*)^{U^{*-1}}$ for $z \in T$ and $U \in \mathrm{GL}(n, \mathbb{C})$. If we identify $V \otimes V^*$ with $\mathrm{End}(V)$ and with the $n \times n$ matrices, then $U^* = \overline{U}^{\mathsf{T}}$. For $U \in \mathcal{U}(n)$ we have $U^{*-1} = U$, hence

$$(7) \qquad (z^*)^U = (z^U)^* \text{ for all } U \in \mathcal{U}(n) \text{ and } z \in T.$$

Also,

$$(8) \qquad \langle x, y^U \rangle = \langle x^{U^*}, y \rangle \text{ for all } U \in \mathrm{GL}(n, \mathbb{C}) \text{ and } x, y \in T.$$

The identity matrix I in $V \otimes V^*$ is equal to

$$(9) \qquad I := \sum_{i=1}^{n} e_i \otimes e_i^*.$$

For $k,l \in \mathbb{N}$ and $1 \le i \le k$ and $1 \le j \le l$, the *contraction* $C^{k,l}_{i,j}$ is the unique linear transformation $V^{\otimes k} \otimes V^{*\otimes l} \to V^{\otimes k-1} \otimes V^{*\otimes l-1}$ satisfying

$$
\begin{aligned}
(10)\qquad & C^{k,l}_{i,j}(x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_l) = \\
& y_j(x_i)(x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_{j-1} \otimes y_{j+1} \otimes \cdots \otimes y_l)
\end{aligned}
$$

for all $x_1, \ldots, x_k \in V$ and $y_1, \ldots, y_l \in V^*$. It is useful to observe that, for any $k,l \in \mathbb{N}$, the function $(x,y) \mapsto \langle x,y \rangle$ on $T^k_l \times T^k_l$ is equal to a series of $k+l$ contractions applied to the tensor $x \otimes y^*$ (which belongs to T^{k+l}_{k+l}).

A graded subalgebra A of T is called *contraction-closed* if $C^{k,l}_{i,j}(A^k_l) \subseteq A^{k-1}_{l-1}$ for all $k,l \in \mathbb{N}$ and $1 \le i \le k$ and $1 \le j \le l$. The following is basic, and follows from the fact that $y^U(x^U) = y(x)$ for $x \in V$, $y \in V^*$:

(11) if $z \in T$ and $U \in \mathrm{GL}(n,\mathbb{C})$ satisfy $z^U = z$, then $w^U = w$ for any contraction w of z.

So T^G is contraction-closed for any $G \subseteq \mathrm{GL}(n,\mathbb{C})$.

We call $A \subseteq T$ *nondegenerate* if there is no proper subspace W of V such that $A \subseteq T(W) \otimes T(W^*)$. (Here $T(W^*)$ is taken as subspace of $T(V^*)$ with respect to the chosen inner product, which gives an orthogonal complement to W: it yields a natural isomorphism between W^* and $\{w^* \mid w \in W\}$, hence a natural embedding $W^* \hookrightarrow V^*$.) It follows from the proof of (13) below that a contraction-closed graded $*$-subalgebra A of T is nondegenerate if and only if $I \in A$.

We call a tensor $z \in T^k_l$ a *mutation* of a tensor $y \in T^k_l$ if z arises from y by permuting contravariant factors and permuting covariant factors. A useful observation is:

(12) Any contraction-closed graded $*$-subalgebra of T containing I is closed under taking mutations.

Indeed, any mutation of a tensor $z \in T^k_l$ can be obtained by applying a series of m contractions to $z \otimes I^{\otimes m}$ (for some m).

Most background on tensors, invariant theory, and linear algebraic groups can be found in the books of Goodman and Wallach [4] and Kraft [6] and in the survey article of Springer [10].

3 The characterization

Theorem 1. *Let $n \geq 1$ and $A \subseteq T$. Then there is a subgroup G of $\mathcal{U}(n)$ with $A = T^G$ if and only if A is a nondegenerate contraction-closed graded $*$-subalgebra of T.*

Proof. Necessity being direct, we show sufficiency. Let A be a nondegenerate contraction-closed graded $*$-subalgebra of T.

Consider again the elements of $V \otimes V^*$ as elements of $\mathrm{End}(V)$, or as the corresponding $n \times n$ matrices. Then A_1^1 is a subalgebra of $\mathrm{End}(V)$, since if $y, z \in A_1^1$ then the matrix product yz belongs to A_1^1, as it is a contraction of $y \otimes z$. We first show:

(13) $\qquad I \in A.$

As A_1^1 is a finite-dimensional C$*$-algebra, it contains an identity element e. In order to prove that $e = I$, it suffices to show that A_1^1 is nondegenerate as an operator algebra, that is, that $A_1^1 V = V$, since then $ev = v$ for each $v \in V$, as $v = av = eav = ev$ for some $a \in A_1^1$.

Define $W := A_1^1 V$. Then for (13) it suffices to show:

(14) $\qquad A \subseteq T(W) \otimes T(W^*),$

since it implies $W = V$ as A is nondegenerate.

To prove (14), we can assume that $W \cap \{e_1, \ldots, e_n\}$ is a basis of W, say it is $\{e_1, \ldots, e_m\}$. Express any $x \in A_l^k$ in the basis (5). If $x \notin W^{\otimes k} \otimes W^{*\otimes l}$, then we may assume (by the fact that $A = A^*$) that x uses a basis element (5) with $i_t > m$ for some $t \in \{1, \ldots, k\}$; say $i_t = m + 1$. Then there is a contraction of $x \otimes x^*$ to an element y of A_1^1 which uses $e_{m+1} \otimes e_{m+1}^*$. Hence $A_1^1 e_{m+1}$ uses e_{m+1}, contradicting the fact that $A_1^1 e_{m+1} \in W$.

This proves (14), and hence (13). It implies:

(15) $\qquad T^{\mathcal{U}(n)} \subseteq A.$

Indeed, the First Fundamental Theorem for $\mathcal{U}(n)$ (cf. [4]) states that, for each $k, l \in \mathbb{N}$, if $k \neq l$ then $(T_l^k)^{\mathcal{U}(n)}$ is equal to $\{0\}$, and if $k = l$ then it is spanned by all mutations of $I^{\otimes k}$. By (12), A contains all mutations of $I^{\otimes k}$, hence $(T_l^k)^{\mathcal{U}(n)} \subseteq A$, and we have (15).

Define $G := \mathcal{U}(n)^A$. To prove the theorem, it suffices to show $A = T^G$, where $A \subseteq T^G$ is direct.

Let $X := \mathcal{U}(n)/G$ be the set of right cosets of G, with the quotient topology. As $\mathcal{U}(n)$ is compact, X is compact. For $a \in A$ and $b \in T$, define a continuous function $\phi_{a,b} : X \to \mathbb{C}$ by

$$\phi_{a,b}(GU) := \langle a^U, b\rangle \tag{16}$$

for $U \in \mathcal{U}(n)$. This is well-defined, since if $GU' = GU$, then $U'U^{-1} \in G$, hence $a^{U'U^{-1}} = a$, and therefore $a^{U'} = a^U$.

Let $\mathcal{F}$ be the linear space spanned by the functions $\phi_{a,b}$ with $a \in A$ and $b \in T$. So $\mathcal{F}$ is in fact spanned by those $\phi_{a,b}$ with $a \in A_l^k$ and $b \in T_l^k$ for some k, l. We show

$$\overline{\mathcal{F}} = \mathcal{C}(X) \tag{17}$$

(with respect to the sup-norm topology on $\mathcal{C}(X)$), using the Stone-Weierstrass theorem (cf. for instance [1] Corollary 18.10). To this end, we check the conditions of the Stone-Weierstrass theorem.

First, $\mathcal{F}$ is a subalgebra of $\mathcal{C}(X)$ (with respect to pointwise multiplication). For let $a, b \in T_l^k$ and $a', b' \in T_{l'}^{k'}$ with $a, a' \in A$. Then for each $U \in \mathcal{U}(n)$:

$$\phi_{a,b}(GU)\phi_{a',b'}(GU) = \phi_{a\otimes a', b\otimes b'}(GU). \tag{18}$$

So $\phi_{a,b}\phi_{a',b'} = \phi_{a\otimes a', b\otimes b'}$. Moreover, $\mathcal{F}$ is self-conjugate: if $\phi \in \mathcal{F}$, also $\overline{\phi} \in \mathcal{F}$ (as $\overline{\phi_{a,b}} = \phi_{a^*,b^*}$, by (6) and (7)).

Finally, $\mathcal{F}$ is strongly separating. Indeed, for $U, U' \in \mathcal{U}(n)$ with $GU \neq GU'$ there exists $a \in A$ with $a^{U'U^{-1}} \neq a$ (as $U'U^{-1} \notin G$). So $a^{U'} \neq a^U$, and therefore $\langle a^{U'}, b\rangle \neq \langle a^U, b\rangle$ for some $b \in T$. Hence $\phi_{a,b}(GU') \neq \phi_{a,b}(GU)$. If $U \in \mathcal{U}(n)$, let a be a nonzero element in A (for instance, $a = I$ — here we use $n \geq 1$). Then $\langle a^U, b\rangle \neq 0$ for some $b \in T$, hence $\phi_{a,b}(GU) \neq 0$. This proves (17).

Now suppose that $T^G \not\subseteq A$. So there exist k, l such that $(T_l^k)^G \not\subseteq A_l^k$. Hence there exists a nonzero $z \in (T_l^k)^G$ orthogonal to A_l^k. As z is nonzero and $\langle z^U, z\rangle = \overline{\langle z, z^U\rangle}$,

$$\int_{\mathcal{U}(n)} \langle z^U, z\rangle\langle z, z^U\rangle d\mu(U) > 0, \tag{19}$$

where μ is a $\mathcal{U}(n)$-invariant Haar measure on $\mathcal{U}(n)$.

The function $\psi : X \to \mathbb{C}$ defined by $\psi(GU) := \langle z^U, z\rangle$ for $U \in \mathcal{U}(n)$ is continuous (and well-defined, as if $GU' = GU$, then $U'U^{-1} \in G$, hence $z^{U'U^{-1}} = z$ (as $z \in T^G$), therefore $z^{U'} = z^U$). So by (17), $\mathcal{F}$ contains functions arbitrarily close to ψ (in the sup-norm topology). With (19) this implies that there exist k', l' and $a \in A_{l'}^{k'}$ and $b \in T_{l'}^{k'}$ such that

$$\int_{\mathcal{U}(n)} \phi_{a,b}(GU)\langle z, z^U\rangle d\mu(U) > 0. \tag{20}$$

Hence, by definition of $\phi_{a,b}$, and using (8),

$$\begin{aligned} 0 \neq \int_{\mathcal{U}(n)} \langle a^U, b\rangle\langle z, z^U\rangle d\mu(U) = \int_{\mathcal{U}(n)} \langle a, b^{U^*}\rangle\langle z^{U^*}, z\rangle d\mu(U) = \\ \int_{\mathcal{U}(n)} \langle a, b^U\rangle\langle z^U, z\rangle d\mu(U) = \left\langle \int_{\mathcal{U}(n)} \langle a, b^U\rangle z^U d\mu(U), z \right\rangle. \end{aligned} \tag{21}$$

We will show that however

$$\int_{\mathcal{U}(n)} \langle a, b^U\rangle z^U d\mu(U) \in A, \tag{22}$$

which implies that (21) gives a contradiction, as z is orthogonal to A.

To show (22), note that (as observed above) $\langle a, b^U\rangle$ can be obtained by an appropriate series of $k' + l'$ contractions from $a \otimes (b^U)^*$ (this last tensor belongs to $T_{k'+l'}^{k'+l'}$). Hence

$$\langle a, b^U\rangle z^U = C(a \otimes (b^U)^* \otimes z^U), \tag{23}$$

where $C : T_{k'+l'+l}^{k'+l'+k} \to T_l^k$ consists of a series of $k' + l'$ contractions. Define

$$w := \int_{\mathcal{U}(n)} ((b^U)^* \otimes z^U) d\mu(U). \tag{24}$$

Then w belongs to $T^{\mathcal{U}(n)}$ (as $(b^U)^* = (b^*)^U$ by (7)), and hence, by (15), to A. Therefore,

$$\int_{\mathcal{U}(n)} \langle a, b^U\rangle z^U d\mu(U) = \int_{\mathcal{U}(n)} C(a \otimes (b^U)^* \otimes z^U) d\mu(U) = C(a \otimes w) \tag{25}$$

belongs to A, as $a, w \in A$ and as A is contraction-closed. This proves (22), and hence the theorem. ∎

4 A Galois correspondence and other corollaries

We formulate a few consequences of Theorem 1. The first consequence is implicit in the proof of Theorem 1, but it is convenient to state it explicitly.

Corollary 1a. *Let A be a nondegenerate contraction-closed graded $*$-subalgebra of T. Then*

$$(26) \qquad T^{\mathcal{U}(n)^A} = A.$$

Proof. Here $\supseteq$ is direct, while $\subseteq$ follows from the fact that if $A = T^G$ for some $*$-subgroup G, then $G \subseteq \mathcal{U}(n)^A$, hence $T^{\mathcal{U}(n)^A} \subseteq T^G \subseteq A$. ∎

For any $G \subseteq \mathrm{GL}(n, \mathbb{C})$, let $\overline{G}$ be the Zariski closure of G.

Corollary 1b. *For any $*$-subgroup G of $\mathrm{GL}(n, \mathbb{C})$:*

$$(27) \qquad \mathrm{GL}(n, \mathbb{C})^{T^G} = \overline{G}.$$

Proof. Set $A := T^G$. Then A is a nondegenerate contraction-closed graded $*$-subalgebra of T. So by (26), $T^{\mathcal{U}(n)^A} = A = T^G$.

Now, for any two groups $G, H \subseteq \mathrm{GL}(n, \mathbb{C})$, $T^G = T^H$ implies $\overline{G} = \overline{H}$ (cf. [6] or [10]). Hence

$$(28) \qquad \overline{G} = \overline{\mathcal{U}(n)^A} = \overline{\mathrm{GL}(n, \mathbb{C})^A \cap \mathcal{U}(n)} = \mathrm{GL}(n, \mathbb{C})^A.$$

The latter equality follows from the fact that for any Zariski-closed $*$-subgroup H of $\mathrm{GL}(n, \mathbb{C})$ one has $\overline{H \cap \mathcal{U}(n)} = H$. Now (28) gives (27). ∎

Theorem 1 and Corollary 1b imply that the relation $G \leftrightarrow T^G$ gives a one-to-one correspondence between the lattice of linear algebraic $*$-subgroups G of $\mathrm{GL}(n, \mathbb{C})$ and the lattice of nondegenerate contraction-closed graded $*$-subalgebras of T. It is a *Galois correspondence*: it reverses inclusion.

The following corollary is useful in deriving First Fundamental Theorems (as we do in Section 5).

Corollary 1c. *Let $S \subseteq T$ and let G be a $*$-subgroup of* $\mathrm{GL}(n,\mathbb{C})$ *with $\mathcal{U}(n)^S \subseteq G \subseteq \mathrm{GL}(n,\mathbb{C})^S$. Then T^G is equal to the smallest contraction-closed graded $*$-subalgebra of T containing $S \cup \{I\}$.*

Proof. Let A be the smallest contraction-closed graded $*$-subalgebra of T containing $S \cup \{I\}$. So A consists of those elements of T obtainable from $S \cup S^* \cup \{I\}$ by a series of linear combinations, tensor products, and contractions. Hence $\mathcal{U}(n)^S = \mathcal{U}(n)^A =: H$. Now T^G is a contraction-closed graded $*$-subalgebra containing $S \cup \{I\}$ (T^G is a $*$-subalgebra as G is a $*$-subgroup). So $A \subseteq T^G$. As $G \supseteq H$ this implies

$$(29) \qquad A \subseteq T^G \subseteq T^H = A,$$

by Corollary 1a. (A is nondegenerate as $I \in A$.) Therefore, we have equality throughout in (29), which proves the corollary. ∎

Incidentally, this corollary implies that each contraction-closed graded $*$-subalgebra A of T is finitely generated *as a contraction-closed algebra.* That is, there is a finite subset S of A such that each element of A can be obtained from S by a series of linear combinations, tensor products, and contractions:

Corollary 1d. *Each contraction-closed graded $*$-subalgebra A of T is finitely generated as a contraction-closed algebra.*

Proof. We may assume that A is nondegenerate. Let $G := \mathcal{U}(n)^A$, and for each $z \in A$, let $G_z := \mathcal{U}(n)^{\{z\}}$. So $G = \bigcap_{z \in A} G_z$. As each G_z is determined by polynomial equations, by Hilbert's finite basis theorem we know that $G = \bigcap_{z \in S} G_z$ for some finite subset S of A. So $G = \mathcal{U}(n)^S$. Hence

$$(30) \qquad A = T^G = T^{\mathcal{U}(n)^S}.$$

We can assume that $S^* = S$ (otherwise add S^* to S). Therefore, by Corollary 1c, A is the smallest contraction-closed graded subalgebra of T containing $S \cup \{I\}$. ∎

5 Applications to FFT's

We now apply Theorem 1 (more precisely, Corollary 1c) to derive a First Fundamental Theorem (FFT) in the sense of Weyl [12] for a number of subgroups of $\mathrm{GL}(n,\mathbb{C})$. The following lemma, which is straightforward to prove, will turn out to be useful. (An element z of T is *homogeneous* if $z \in T_l^k$ for some $k,l \in \mathbb{N}$.)

Lemma 1. *Let $S \subseteq T$ be a set of homogeneous elements and let A be the linear space spanned by all mutations of tensor products of elements of $S \cup \{I\}$. Then A is a graded subalgebra of T, and A is contraction-closed if each contraction of any element of S and of the tensor product of any two elements of S belongs to A.*

Proof. Easy. Note that any contraction of $z \otimes I$ is equal to $z' \otimes I$ for some contraction z' of z, or is $n \cdot z$, or is a mutation of z. Similarly for $I \otimes z$. ∎

FFT for $\mathrm{SL}(n,\mathbb{C}) = \{U \in \mathrm{GL}(n,\mathbb{C}) \mid \det U = 1\}$ (the special linear group). Define $\det \in V^{*\otimes n}$ by

$$\det := \sum_{\pi \in \mathcal{S}_n} \mathrm{sgn}(\pi) e^*_{\pi(1)} \otimes \cdots \otimes e^*_{\pi(n)}. \tag{31}$$

(We can consider det as element of $(V^{\otimes n})^*$, and then $\det(x_1 \otimes \cdots \otimes x_n)$ is equal to the usual determinant of the matrix with columns $x_1, \ldots, x_n$.)

One straighforwardly checks that $\det^U = \det(U) \cdot \det$ for any $U \in \mathrm{GL}(n,\mathbb{C})$. So $\mathrm{GL}(n,\mathbb{C})^{\{\det\}} = \mathrm{SL}(n,\mathbb{C})$. Hence by Corollary 1c, $T^{\mathrm{SL}(n,\mathbb{C})}$ is equal to the smallest contraction-closed subalgebra of T containing det, $\det^*$, and I. Lemma 1 then implies that $T^{\mathrm{SL}(n,\mathbb{C})}$ is equal to the linear space A spanned by all mutations of tensor products of det, $\det^*$, and I.

Indeed, set $S := \{\det, \det^*\}$. As det and $\det^*$ have only covariant or only contravariant factors, they cannot be contracted. Moreover, $\det \otimes \det^*$ is a linear combination of mutations of $I^{\otimes n}$, as it belongs to $T^{\mathrm{GL}(n,\mathbb{C})}$ (since $\det^U = \det(U) \cdot \det$ and $(\det^*)^U = \det(U)^{-1} \cdot \det^*$). So any contraction of $\det \otimes \det^*$ belongs to A.

FFT for $\mathrm{SL}_k(n,\mathbb{C}) = \{U \in \mathrm{GL}(n,\mathbb{C}) \mid \det U^k = 1\}$**.** The proof scheme is the same as for the FFT for $\mathrm{SL}(n,\mathbb{C})$ above. Since $(\det^{\otimes k})^U = (\det U)^k \cdot \det^{\otimes k}$, we know $\mathrm{GL}(n,\mathbb{C})^{\{\det^{\otimes k}\}} = \mathrm{SL}_k(n,\mathbb{C})$. So by Corollary 1c, $T^{\mathrm{SL}_k(n,\mathbb{C})}$ is equal to the smallest contraction-closed subalgebra of T containing $\det^{\otimes k}$,

$\det^{*\otimes k}$, and I. With Lemma 1 applied to $S := \{\det^{\otimes k}, \det^{*\otimes k}\}$, this again gives that $T^{\mathrm{SL}_k(n,\mathbb{C})}$ is spanned by mutations of tensor products of $\det^{\otimes k}$, $\det^{*\otimes k}$, and I.

FFT for $\mathcal{S}_n(\mathbb{C})$ = set of matrices in $\mathrm{GL}(n,\mathbb{C})$ with precisely one nonzero in each column (hence also in each row). For each k, let

$$(32) \qquad j_k := \sum_{i=1}^{n} (e_i \otimes e_i^*)^{\otimes k}$$

and define

$$(33) \qquad J := \{j_k \mid k \geq 1\}.$$

Then $\mathrm{GL}(n,\mathbb{C})^{\{j_2\}} = \mathcal{S}_n(\mathbb{C})$. Indeed, let $U = (u_{i,j})$ satisfy $j_2^U = j_2$. Choose a column index t and row indices $k \neq l$. Then, as $\langle e_i \otimes e_i, e_k \otimes e_l\rangle = 0$ for each i, we have

$$(34) \qquad 0 = \sum_{i=1}^{n} \langle e_i \otimes e_i \otimes e_i^* \otimes e_i^*, e_k \otimes e_l \otimes e_t^{*U^{*-1}} \otimes e_t^{*U^{*-1}}\rangle = \sum_{i=1}^{n} \langle e_i^U \otimes e_i^U \otimes e_i^{*U} \otimes e_i^{*U}, e_k \otimes e_l \otimes e_t^{*U^{*-1}} \otimes e_t^{*U^{*-1}}\rangle = \sum_{i=1}^{n} u_{k,i} u_{l,i} \delta_{i,t} = u_{k,t} u_{l,t}.$$

So $U \in \mathcal{S}_n(\mathbb{C})$. The reverse implication follows more directly.

Consequently, $T^{\mathcal{S}_n(\mathbb{C})}$ is the smallest contraction-closed graded subalgebra of T containing j_2 and I $(= j_1)$. Now the contractions of tensor powers of j_2 are precisely the mutations of tensor products of elements of J. Hence $\mathrm{GL}(n,\mathbb{C})^J = \mathcal{S}_n(\mathbb{C})$, and by taking $S := J$ in Lemma 1 it follows that $T^{\mathcal{S}_n(\mathbb{C})}$ is spanned by mutations of tensor products of elements of J.

FFT for $\mathrm{Sp}(n,\mathbb{C})$ = set of matrices $U \in \mathrm{GL}(n,\mathbb{C})$ with $UPU^{\mathsf{T}} = P$, where

$$(35) \qquad P = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

for $m := \frac{1}{2}n$ (assuming n to be even) (the symplectic group). Here I_m

denotes the $m \times m$ identity matrix. Define

$$p := \sum_{i=1}^{m} (e_i \otimes e_{m+i} - e_{m+i} \otimes e_i). \tag{36}$$

Then $\mathrm{GL}(n,\mathbb{C})^{\{p\}} = \mathrm{Sp}(n,\mathbb{C})$ (by definition of $\mathrm{Sp}(n,\mathbb{C})$). Any contraction of $p \otimes p^*$ is equal to $\pm I$. Hence $T^{\mathrm{Sp}(n,\mathbb{C})}$ is spanned by mutations of tensor products of p, p^*, and I.

FFT for $O(n,\mathbb{C}) = \{U \in \mathrm{GL}(n,\mathbb{C}) \mid UU^{\mathsf{T}} = I\}$ (the orthogonal group). Define

$$f := \sum_{i=1}^{n} e_i \otimes e_i. \tag{37}$$

Then $\mathrm{GL}(n,\mathbb{C})^{\{f\}} = O(n,\mathbb{C})$. So $T^{O(n,\mathbb{C})}$ is equal to the smallest contraction-closed algebra containing f, f^*, and I. Taking $S := \{f, f^*\}$ in Lemma 1, and observing that any contraction of $f \otimes f^*$ is equal to I, we see that $T^{O(n,\mathbb{C})}$ is spanned by mutations of tensor products of f, f^*, and I.

Note that this implies that $T(V)^{O(n,\mathbb{C})}$ is spanned by mutations of tensor powers of f. Since $O(n,\mathbb{C}) \cap \mathcal{U}(n) = O(n,\mathbb{R})$ (the real orthogonal group), we have as in Corollary 1c $T^{O(n,\mathbb{R})} = T^{O(n,\mathbb{C})}$.

In describing the FFT for subgroups of $O(n,\mathbb{C})$, it is convenient to introduce the concept of a 'flip'. A *flip* of an element $z \in T_l^k$ is obtained by applying the $\mathbb{C}$-linear transformation $e_i \mapsto e_i^*$ ($i = 1, \ldots, n$) to some (or none) of the contravariant factors of z, and the reverse transformation to some (or none) of the covariant factors of z. (So z is also flip of itself.)

Then f and f^* are flips of I. Hence another way of stating the FFT for $O(n,\mathbb{C})$ is that $T^{O(n,\mathbb{C})}$ is spanned by mutations of tensor products of flips of I. Note that for any $G \subseteq \mathrm{GL}(n,\mathbb{C})$,

$$G \subseteq O(n,\mathbb{C}) \iff T^G \text{ is invariant under taking flips,} \tag{38}$$

since $G \subseteq O(n,\mathbb{C}) \iff f \in T^G$. We can also formulate a lemma analogous to Lemma 1:

Lemma 2. *Let $S \subseteq T$ be a set of homogeneous elements and let A be the linear space spanned by all mutations of tensor products of flips of elements* *of $S\cup\{I\}$. Then A is a graded $*$-subalgebra of T, and A is contraction-closed if each contraction of any flip of any element of S and of the tensor product of flips of any two elements of S belongs to A.*

Proof. Here note that any contraction of $z\otimes g$ where g is a flip of I, is equal to $z'\otimes g$ for some contraction z' of z, or is $n\cdot z$, or is a mutation of a flip of z. Similarly for $g\otimes z$. ∎

FFT for $\mathrm{SO}(n,\mathbb{C}) = O(n,\mathbb{C})\cap \mathrm{SL}(n,\mathbb{C})$ (the special orthogonal group). Now $\mathrm{GL}(n,\mathbb{C})^{\{f,\det\}} = \mathrm{SO}(n,\mathbb{C})$ (as it is the intersection of $O(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$). So by Corollary 1c, $T^{\mathrm{SO}(n,\mathbb{C})}$ is equal to the smallest contraction-closed $*$-algebra containing f, det, and I. Taking $S:=\{\det\}$ in Lemma 2, we see that $T^{\mathrm{SO}(n,\mathbb{C})}$ is spanned by mutations of tensor products of flips of det and I.

FFT for $\mathcal{S}_n$ = set of $n\times n$ permutation matrices (the symmetric group). For each k, let

$$h_k := \sum_{i=1}^{n} e_i^{\otimes k} \tag{39}$$

and define

$$H := \{h_k \mid k\geq 1\}. \tag{40}$$

Then $\mathrm{GL}(n,\mathbb{C})^H = \mathcal{S}_n$. Hence (again with Corollary 1c and Lemma 2, taking $S:=H$) $T^{\mathcal{S}_n}$ is equal to the linear space A spanned by mutations of tensor products of flips of elements of H. (Any contraction of a flip of h_k or of $h_k\otimes h_l$ belongs to A.)

A second (but now finite), and more familiar, set of spanning tensors can be derived from it. For each k, define

$$g_k := \sum_{i_1,\ldots,i_k} e_{i_1}\otimes\cdots\otimes e_{i_k}, \tag{41}$$

where the sum ranges over all distinct $i_1,\ldots,i_k\in\{1,\ldots,n\}$. (So $g_k=0$ if $k>n$.) Then

$$g_k = \sum_{f:\{1,\ldots,k\}\to\{1,\ldots,n\}} \sum_{\substack{\pi\in S_k\\ f\circ\pi=f}} \mathrm{sgn}(\pi) e_{f(1)}\otimes\cdots\otimes e_{f(k)} = \tag{42}$$

$$\sum_{\pi\in S_k}\mathrm{sgn}(\pi)\sum_{\substack{f:\{1,\ldots,k\}\to\{1,\ldots,n\}\\ f\circ\pi=f}} e_{f(1)}\otimes\cdots\otimes e_{f(k)}.$$

Now, in the last expression, for each fixed $\pi\in S_k$, the inner sum is a mutation of $h_{i_1}\otimes\cdots\otimes h_{i_t}$, where $i_1,\ldots,i_t$ are the orbit sizes of π. So g_k belongs to $T^{\mathcal{S}_n}$.

Moreover, h_k itself occurs when π has precisely one orbit. As this holds for each k, it follows inductively that each h_k is spanned by mutations of tensor products of $g_0,\ldots,g_k$. This gives

(43) $T^{\mathcal{S}_n}=$ [linear space spanned by mutations of tensor products of flips of $h_1,h_2,\ldots$] $\subseteq$ [linear space spanned by mutations of tensor products of flips of $g_0,\ldots,g_n$] $\subseteq$ $T^{\mathcal{S}_n}$.

Hence we have equality throughout.

FFT for $\mathcal{S}_n^{\pm}=O(n,\mathbb{C})\cap\mathcal{S}_n(\mathbb{C})$ (so each nonzero entry of any matrix in $\mathcal{S}_n^{\pm}$ is ±1). Let

(44) $H':=\{h_k\mid k\text{ even},\ k\geq2\}$.

Then $\mathrm{GL}(n,\mathbb{C})^{H'}=\mathcal{S}_n^{\pm}$. Hence (as in the previous example) $T^{\mathcal{S}_n^{\pm}}$ is spanned by mutations of tensor products of flips of elements of H'.

As above, one may show that equivalently $T^{\mathcal{S}_n^{\pm}}$ is spanned by mutations of tensor products of flips of

(45) $$\sum_{i_1,\ldots,i_k} e_{i_1}^{\otimes2}\otimes\cdots\otimes e_{i_k}^{\otimes2}$$

(for $k=1,\ldots,n$), where the sum ranges over all distinct $i_1,\ldots,i_k\in\{1,\ldots,n\}$.

FFT for $\mathcal{A}_n=\mathcal{S}_n\cap\mathrm{SO}(n,\mathbb{C})$ (the alternating group). Let H be as in (40). Then $\mathrm{GL}(n,\mathbb{C})^{H\cup\{\det\}}=\mathcal{A}_n$. Hence $T^{\mathcal{A}_n}$ is equal to the linear space spanned by mutations of tensor products of flips of elements of H and of elements

(46) $$\sum_{\pi\in\mathcal{S}_n}\mathrm{sgn}(\pi)e_{\pi(1)}^{\otimes k_1}\otimes\cdots\otimes e_{\pi(n)}^{\otimes k_n},$$

ranging over all $k_1, \ldots, k_n \geq 0$. (To apply Lemma 2, take S equal to H joined with all elements (46), and check that any contraction of any flip of element of S or product of two elements of S belongs to A.)

FFT for $\mathcal{A}_n^{\pm} = \mathcal{S}_n^{\pm} \cap \mathrm{SO}(n, \mathbb{C})$. Let H' be as in (44). Then $\mathrm{GL}(n, \mathbb{C})^{H' \cup \{\det\}} = \mathcal{A}_n^{\pm}$. As in the previous example, $T^{\mathcal{A}_n^{\pm}}$ is spanned by mutations of tensor products of flips of elements of H' and of elements (46), ranging over all odd $k_1, \ldots, k_n \geq 1$.

The examples of FFT's for subgroups of the orthogonal group can in fact also be derived from the following consequence of Theorem 1. Let V be an n-dimensional real inner product space. For $1 \leq i < j \leq k$, let $C_{i,j}^k : V^{\otimes k} \to V^{\otimes k-2}$ be the operator contracting the ith and jth factor in $V^{\otimes k}$. (So $C_{i,j}^k(a \otimes b \otimes c \otimes d \otimes e) = \langle b, d \rangle (a \otimes c \otimes e)$ for $a \in V^{\otimes i-1}$, $b, d \in V$, $c \in V^{\otimes j-i-1}$, $e \in V^{\otimes k-j}$, where $\langle ., . \rangle$ is the inner product.) Call $A \subseteq T(V)$ *contraction-closed* if $C_{i,j}^k(A \cap V^{\otimes k}) \subseteq A$ for all k and $1 \leq i < j \leq k$. Call A *nondegenerate* if A is not a subset of $T(W)$ for some proper subspace W of V.

Corollary 1e. *Let $n \geq 1$ and $A \subseteq T(V)$. Then there is a subgroup G of $O(n, \mathbb{R})$ with $A = T(V)^G$ if and only if A is a nondegenerate contraction-closed graded subalgebra of $T(V)$.*

Proof. This follows by applying Theorem 1 to the set of all flips of elements of $A + \mathrm{i}\mathrm{i}A$, seen as subset of $T(V + \mathrm{i}\mathrm{i}V) \otimes T((V + \mathrm{i}\mathrm{i}V)^*)$. Here we need that A is closed under mutations, which follows (cf. (12)) from the fact that the identity matrix I in $V^{\otimes 2}$ belongs to A. This can be derived similarly as (13) from the nondegeneracy of A and from the fact if $M \in A \cap V^{\otimes 2}$ then $M^{\mathsf{T}} M \in A \cap V^{\otimes 2}$, as $M^{\mathsf{T}} M = C_{1,3}^4(M \otimes M)$. ∎

6 Application to self-dual codes

The results above also apply to the study of weight enumerators of self-dual codes, as initiated by Gleason [3] (cf. MacWilliams and Sloane [7] Chapter 19 for background). To give the idea, we just describe the most elementary application.

Let $\mathbb{F}$ be a finite field, with q elements. For $k, l \geq 0$ and $C \subseteq \mathbb{F}^k \times \mathbb{F}^l$, define $C^{\perp}$ by

$$C^{\perp} := \{(z, w) \in \mathbb{F}^k \otimes \mathbb{F}^l \mid z^T x = w^T y \text{ for each } (x, y) \in C\}. \tag{47}$$

Here $z^T x := \sum_{i=1}^k z_i x_i$, taking addition and multiplication in the field $\mathbb{F}$; $w^T y$ is defined similarly. Call C *self-dual* if $C^\perp = C$. In that case, C is a linear subspace of $\mathbb{F}^k \times \mathbb{F}^l$.

Let $V := \mathbb{C}^q$, and encode the coordinates of $\mathbb{C}^q$ by the elements of $\mathbb{F}$. For $k, l \geq 0$ and $C \subseteq \mathbb{F}^k \times \mathbb{F}^l$, define the following tensor τ_C in $V^{\otimes k} \otimes V^{*\otimes l}$:

$$\tau_C := \sum_{(x,y)\in C} \bigotimes_{i=1}^{k} e_{x_i} \otimes \bigotimes_{j=1}^{l} e^*_{y_j}. \tag{48}$$

Let A be the linear space spanned by all τ_C, taken over all k, l and all self-dual codes $C \subseteq \mathbb{F}^k \times \mathbb{F}^l$. Then one easily checks that A is a nondegenerate contraction-closed graded $*$-subalgebra of $T = T(V) \otimes T(V^*)$. Hence, by Theorem 1, $A = T^G$ for some subgroup G of $\mathcal{U}(q)$.

This applies to self-dual codes, as follows. Let $\xi : T(V) \to \mathbb{R}[x_i \mid i \in \mathbb{F}]$ be the symmetrization operator (bringing e_i to x_i). Set $w_C := \xi(\tau_C)$, the *weight enumerator* of C. So $\xi(A \cap T(V))$ is spanned by the weight enumerators of all self-dual codes over $\mathbb{F}$. Moreover, $\xi(A \cap T(V)) = \mathbb{R}[x_i \mid i \in \mathbb{F}]^G$. Hence, the weight enumerators of the self-dual codes over $\mathbb{F}$ span an invariant subring of $\mathbb{R}[x_i \mid i \in \mathbb{F}]$.

We illustrate the use of this with the very simple case $\mathbb{F} = \{0, 1\}$. Let J be the trivial code $\{(0,0),(1,1)\}$ and let H be the $[8,4,4]$ Hamming code. Then $w_J = x_0^2 + x_1^2$ and $w_H = x_0^8 + 14x_0^4x_1^4 + x_1^8$. One may check that the group G of unitary matrices U with $(w_J)^U = w_J$ and $(w_H)^U = w_H$ is generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $2^{-\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Moreover, for each self-dual code C we have $\tau_C^U = \tau_C$ for each $U \in G$. So A as defined above is equal to the smallest contraction-closed graded $*$-subalgebra of $T = T(V) \otimes T(V^*)$ that contains τ_J and τ_H. It implies that A is equal to the smallest graded subalgebra of T containing I, τ_H, and τ_H^*. Therefore, we obtain the result of Gleason [3] that the G-invariant subring of $\mathbb{R}[x_0, x_1]$ is spanned by the weight enumerators of self-dual codes, and is generated by w_J and w_H.

One may next apply Molien's theorem to derive that w_H and w_J are algebraically independent. Conversely, the algebraic independence of w_H and w_J gives Molien's theorem for this invariant ring.

A generalization is obtained as follows. Consider again a finite field $\mathbb{F}$ and moreover some $m \in \mathbb{N}$. Now let A be the linear space spanned by all τ_C, taken over all k, l and those self-dual codes $C \subseteq \mathbb{F}^k \times \mathbb{F}^l$ for which $\text{weight}(x) - \text{weight}(y)$ is a multiple of m for each $(x, y) \in C$. Here the weight $\text{weight}(x)$ of x is the number of nonzero components of x. Then A is a nondegenerate contraction-closed graded $*$-subalgebra of T, hence $A = T^G$ for some subgroup G of $\mathcal{U}(q)$ (for $q := |\mathbb{F}|$). Similarly to above, it yields for instance the characterization of Gleason [3] of the weight enumerators of even self-dual binary codes. Here *even* means that the weight of each word is a multiple of 4.

7 Application to combinatorial parameters

We finally describe an application of Theorem 1 to combinatorial parameters. This application was in fact our main motivation to prove Theorem 1. We sketch a simple special case of this application, which case is a variant of a theorem of Freedman, Lovász, and Schrijver [2]. The method is inspired by Szegedy [11]. Full proofs and more general applications are given in [9].

Let $\mathcal{G}$ be the collection of (undirected) graphs (loops and multiple edges allowed). A (real-valued) *graph parameter* is a function $f : \mathcal{G} \to \mathbb{R}$ such that if G and H are isomorphic graphs then $f(G) = f(H)$.

For any $n \in \mathbb{N}$ and any symmetric matrix $M \in \mathbb{R}^{n\times n}$, define a graph parameter $f_M : \mathcal{G} \to \mathbb{R}$ by

$$f_M(G) := \sum_{\phi:VG\to[n]} \prod_{uv\in EG} M_{\phi(u),\phi(v)} \tag{49}$$

for $G \in \mathcal{G}$. Here VG and EG denote the vertex and edge set of G, respectively, and $[n] := \{1, \dots, n\}$. By uv we denote an edge connecting u and v.

We characterize for which graph parameters $f : \mathcal{G} \to \mathbb{R}$ there is an $n \in \mathbb{N}$ and a symmetric matrix $M \in \mathbb{R}^{n\times n}$ with $f = f_M$. To this end, define a *k-labeled graph* to be a pair (G, λ) of a graph G and a function $\lambda : [k] \to VG$ (not necessarily injective). Let $\mathcal{G}_k$ be the collection of k-labeled graphs.

For two k-labeled graphs (G, λ) and (G', λ'), let $(G, \lambda) \cdot (G', \lambda')$ be the graph obtained by making the disjoint union of G and G' and identifying $\lambda(i)$ and $\lambda'(i)$ for $i = 1, \dots, k$. (Since λ and λ' need not be injective, this might mean repeated identification.)

For any graph parameter $f : \mathcal{G} \to \mathbb{R}$ and any $k \in \mathbb{N}$, define a function $N_{f,k} : \mathcal{G}_k \times \mathcal{G}_k \to \mathbb{R}$ by

$$N_{f,k}((G, \lambda), (G', \lambda')) := f((G, \lambda) \cdot (G', \lambda')). \tag{50}$$

We can consider $N_{f,k}$ as a matrix. Call f *reflection positive* if $N_{f,k}$ is positive semidefinite. Call f *multiplicative* if $f(\emptyset) = 1$ and $f(G \cup G') = f(G)f(G')$ for disjoint graphs G and G'. (Here $\emptyset$ is the graph with no vertices.)

Theorem 2. *Let $f : \mathcal{G} \to \mathbb{R}$ be a graph parameter. Then $f = f_M$ for some $n \in \mathbb{N}$ and some symmetric matrix $M \in \mathbb{R}^{n\times n}$ if and only if f is multiplicative and reflection positive.*

The full proof of Theorem 2 is too long to give here (see [9]), but we will give the point where Theorem 1 is used.

Let $n \in \mathbb{N}$, and introduce variables x_{ij} for $1 \le i \le j \le n$. For any graph G, define the polynomial p_G in $\mathbb{R}[x_{11}, x_{12}, \ldots, x_{nn}]$ by

$$p_G(x_{11}, x_{12}, \ldots, x_{nn}) := \sum_{\phi: VG \to [n]} \prod_{uv \in EG} x_{\phi(u)\phi(v)}. \tag{51}$$

(Here $x_{ij} = x_{ji}$ if $i > j$.) So $f_M(G) = p_G(M)$ for any symmetric matrix $M \in \mathbb{R}^{n\times n}$.

Consider the subalgebra R of $\mathbb{R}[x_{11}, x_{12}, \ldots, x_{nn}]$ spanned by the polynomials p_G. Set $V := \mathbb{C}^n$. An easy construction shows that R is the set of real symmetric tensors in $A \cap T(V)$ for some nondegenerate contraction-closed graded $*$-subalgebra A of $T(V) \otimes T(V^*)$. Corollary 1e then implies that $R = \mathbb{R}[x_{11}, x_{12}, \ldots, x_{nn}]^H$ for some subgroup H of $O(n, \mathbb{R})$.

To obtain a real matrix M with $p_G(M) = f(G)$ for each graph G, the proof uses the Positivstellensatz. The existence of the group H enables to project the polynomials that arise in the Positivstellensatz, onto R, by which the sufficiency in Theorem 2 follows. (A sharpening of the theorem can be obtained by using the theorem of Procesi and Schwarz [8].)

While in the graph case the group H above can in fact be described quite directly, similar theorems for more general combinatorial structures can be derived where the corresponding group is not explicitly known — see [9].

Note. Harm Derksen generalized Theorem 1 by giving a correspondence between reductive subgroups of $\mathrm{GL}(V)$ and nondegenerate contraction-closed graded subalgebras of T containing the identity matrix I. In this case, 'nondegenerate' means that the natural $\mathbb{C}$-bilinear form on T is nondegenerate on the subalgebra.

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