Strongly regular graphs

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Preface

The present volume is a monograph on the topic of Strongly Regular Graphs. So far, no book-length treatment of this subject area has been available.

The topic of strongly regular graphs is an area where statistics, Euclidean geometry, group theory, finite geometry, and extremal combinatorics meet. The subject concerns beautifully regular structures, studied mostly using spectral methods, group theory, geometry and sometimes lattice theory.

Roughly around 1970–1980, Algebraic Combinatorics came up as a separate branch in mathematics. It turned out that the same structures were studied in statistics (for the design of experiments), in Euclidean geometry (e.g. in the construction of systems of equiangular lines), in group theory (where several sporadic groups arise as automorphism groups of a strongly regular graph), in coding theory (where association schemes provide a tool for obtaining bounds on the size of codes, and beautiful structures give rise to good and easy-to-decode codes), in the theory of special functions (where the spectral data of association schemes give rise to series of orthogonal polynomials), in finite geometry (where collinearity graphs of polar spaces are strongly regular), in extremal combinatorics, in cryptography, and elsewhere. More recently such very regular structures find some application in the theory of quantum computation (e.g. for mutually unbiased bases (MUBs) and symmetric, informationally complete, positive operator-valued measures (SICPOVMs)).

Axiomatizing the combinatorial information in the action of a finite permutation group G on a set X yields a hierarchy of combinatorial structures. A general group gives the structure of coherent configuration. For a transitive group one finds an association scheme. If the representation is multiplicity-free, the pair (G, K), where K is the point stabilizer in G, is called a Gelfand pair. The corresponding combinatorial object is a commutative association scheme. If G is generously transitive, one finds a symmetric association scheme. The simplest nontrivial case is that of a strongly regular graph, the combinatorial analog of a rank 3 group, where K has three orbits on $X \times X$.

Delsarte's 1973 thesis[1] defined the concept of (commutative) association scheme and showed the use of the linear programming bound. Bannai & Ito[2] introduced the term 'algebraic combinatorics', described as 'character-theoretical study of combinatorial objects', or 'group theory without groups'. Brouwer, Cohen & Neumaier[3] published a monograph on distance-regular graphs (that is, P- and Q-polynomial association schemes) of diameter at least 3 (where the

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[1] Ph. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Res. Rep. Suppl. **10** (1973).

[2] E. Bannai & T. Ito, *Algebraic Combinatorics I*, Benjamin, 1984.

[3] A. E. Brouwer, A. M. Cohen & A. Neumaier, *Distance-Regular Graphs*, Springer, 1989.

strongly regular graphs are precisely the distance-regular graphs of diameter 2). They wrote 'Another book would be required to cover the present knowledge about strongly regular graphs (no such book is available at present)'. The present monograph fills this gap.

Various teams of authors, starting around 1980 with Van Lint and the present first author, contemplated writing such a book, but for various reasons such a project was never completed. Many years later J. I. Hall, at a 2011 meeting in Oisterwijk, again commented on the lack of a good source of information about strongly regular graphs more recent than Hubaut's 1975 survey,[4] and the project was rekindled.

This book was started with the aim to give the classification of rank 3 graphs and to describe these graphs, possibly as members of larger families, and give information such as parameters, group, cliques, cocliques, local structure, and characterization. Later, the project was widened to include the theory of general strongly regular graphs.

The bulk of the material is more or less well known. Many details are new. In particular, we give information about regular subsets that is often new. Our approach to the (affine) half spin graphs of rank 5 hyperbolic polar spaces is original and based on the idea of 'thickening' the Clebsch graph. We felt free to omit proofs that are rather technical, or that do not fit naturally into the line of development of the book.

Chapter 1 contains the fundaments. Chapters 2 and 3 find the finite polar geometries in a uniform way and describe the related graphs and substructures. Chapter 4 is a brief introduction to buildings,[5] and provides an explicit and elementary construction of the finite buildings of types E_6 and G_2. Chapter 5 is a very short introduction to the geometry related to the Fischer groups.[6] For later use, lax embeddings of symplectic copolar spaces are studied. Chapter 6 gives the main facts on the Golay codes and Witt designs, and contains a very short introduction to the Leech lattice.[7] Chapter 7 is about cyclotomy and difference sets, and the relation to two-weight codes. Chapter 8 contains combinatorial material that is partly new, with, for example, discussions of orthogonal arrays, quasi-symmetric designs, partial geometries, regular two-graphs, spherical designs, randomness properties and much more. Chapter 9 discusses the p-rank of the adjacency matrix, in some cases a useful invariant that may distinguish graphs with the same parameters. The long Chapter 10 consists of a hundred sections discussing (more than) a hundred individual graphs in some more detail. In Chapter 11 we give the classification of rank 3 groups, and identify in each case the corresponding strongly regular graph. Everywhere there are extensive tables. Chapter 12 is just a table, listing all feasible parameter sets of strongly regular graphs with at most 512 vertices together with some information about existence and other details, with references to other parts of the book.

We would like to especially thank Jon Hall, Ferdinand Ihringer, Alexander Gavrilyuk, Dima Pasechnik, and the anonymous referees for detailed comments

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[4] X. L. Hubaut, *Strongly regular graphs*, Discr. Math. **13** (1975) 357–381.

[5] For a monograph, see P. Abramenko & K. S. Brown, *Buildings, Theory and Applications*, Springer, 2008.

[6] For a monograph on the group theoretical side, see M. Aschbacher, *3-Transposition Groups*, Cambridge University Press, 1997.

[7] For a monograph, see J. H. Conway & N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer, 1988.

on earlier drafts. We are also grateful to Maarten De Boeck, Ludmila Tsiovkina, Paulien Jansen, Jeroen Schillewaert, Anneleen De Schepper, Sam Mattheus, and Jan De Beule for reading, and commenting on, draft versions of specific chapters. Finally, we acknowledge with thanks the help of the librarian, Samuel Perez, for locating several difficult-to-find papers.

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Contents

Chapter 1

Graphs

This chapter collects some basic material on strongly regular graphs and gives some information about more general objects (distance-regular graphs and association schemes) that will be needed later.

1.1 Strongly regular graphs

A *graph* is a set X of *vertices* provided with a symmetric relation $\sim$ on X called *adjacency*, such that no $x \in X$ is adjacent to itself. If the graph is denoted Γ, then its vertex set X is also denoted by $\mathsf{V}\Gamma$. A pair of adjacent vertices is called an *edge*. If xy is an edge, then y is called a *neighbor* of x.

Let Γ be a finite graph. The *adjacency matrix* A of Γ is the square matrix indexed by the vertices of Γ such that $A_{xy} = 1$ when $x \sim y$, and $A_{xy} = 0$ otherwise. The *spectrum* of Γ is by definition the spectrum (eigenvalues and multiplicities) of A, considered as a real matrix. A nonzero (column) vector u, indexed by $\mathsf{V}\Gamma$, is an eigenvector of A with eigenvalue θ when $Au = \theta u$, i.e., when $\sum_{y \sim x} u_y = \theta u_x$ for all x.

A graph Γ is *regular* of *degree* (or *valency*) k, for some integer k, when every vertex has precisely k neighbors.

Let Γ be finite with adjacency matrix A. The all-1 vector $\mathbf{1}$ (of appropriate length) is an eigenvector (with eigenvalue k) if and only if Γ is regular (of valency k). If Γ is regular of valency k, then the multiplicity of the eigenvalue k is the number of connected components of Γ. An eigenvalue θ of a regular graph is called *restricted* if it has an eigenvector orthogonal to $\mathbf{1}$.

A finite regular graph without restricted eigenvalues has at most one vertex. A finite regular graph with only one restricted eigenvalue is complete or edgeless. A *strongly regular graph* is a finite regular graph with precisely two restricted eigenvalues.

History

The term 'strongly regular graph' was first used by Bose [92]. An equivalent concept was studied by Bose & Shimamoto [97].

1.1.1 Parameters

Let Γ be a strongly regular graph, regular of valency k, with adjacency matrix A and restricted eigenvalues r, s, where $r > s$. Let J be the all-1 matrix of suitable size, so that $AJ = JA = kJ$. We have $(A - rI)(A - sI) = \mu J$ for some constant μ, so that $A^2 = \kappa I + \lambda A + \mu(J - I - A)$ for certain constants κ, λ, μ. Apparently $\kappa = k$ and $\lambda = \mu + r + s$ and $k - \mu = -rs$.

This can be stated in a combinatorial way: For $x, y \in \mathsf{V}\Gamma$, the number of common neighbors of x, y is k when $x = y$, and λ when $x \sim y$, and μ when $x \not\sim y$. One says that the strongly regular graph Γ has *parameters* (v, k, λ, μ), where $v = |\mathsf{V}\Gamma|$ is the number of vertices. Conversely, if in a finite graph Γ, not complete and not edgeless, the number of common neighbors of two vertices is k, λ, μ depending on whether they are equal, adjacent or nonadjacent, then Γ is strongly regular, and the restricted eigenvalues r, s are found as the roots of $x^2 + (\mu - \lambda)x + (\mu - k) = 0$.

The combinatorial definition of k, λ, μ shows that these are nonnegative integers, and $0 \le \lambda \le k - 1$ and $0 \le \mu \le k$. By Perron-Frobenius' theorem, $k \ge r$. Since $\operatorname{tr} A = 0$ it follows that $s < 0$ and $r \ge 0$.

If $\mu \ne 0$, then the parameters are related by $v = 1 + k + k(k - 1 - \lambda)/\mu$.

From $(A - rI)(A - sI) = \mu J$ one gets the identity $(k - r)(k - s) = \mu v$.

History

The parameters $n, k, l, \lambda, \mu, r, s, f, g$ (with $n = v$ and $l = v - k - 1$) were perhaps first used in [419]. Earlier, Bose [92] used $v, n_1, n_2, p^1_{11}, p^2_{11}$.

1.1.2 Complement

If Γ is a strongly regular graph with parameters (v, k, λ, μ) and restricted eigenvalues r, s, then the complementary graph $\overline{\Gamma}$ (with the same vertex set as Γ, and where distinct vertices are adjacent if and only if they are nonadjacent in Γ) is also strongly regular, with parameters $(v, \bar{k}, \bar{\lambda}, \bar{\mu})$ and restricted eigenvalues $\bar{r}, \bar{s}$, where $\bar{k} = v - k - 1$, $\bar{\lambda} = v - 2k + \mu - 2$, $\bar{\mu} = v - 2k + \lambda$, $\bar{r} = -1 - s$, $\bar{s} = -1 - r$, as is immediately clear from the definitions and the fact that $\overline{\Gamma}$ has adjacency matrix $\overline{A} = J - I - A$.

1.1.3 Imprimitivity

A strongly regular graph Γ is called *imprimitive* when Γ or $\overline{\Gamma}$ is a nontrivial equivalence relation, equivalently, when $\lambda = k - 1$ or $\mu = k$, equivalently, when $\mu = 0$ or $v = 2k - \lambda$, equivalently, when $s = -1$ or $r = 0$.

In the former case Γ is a disjoint union aK_m of a complete graphs of size m (and $v = am$, $k = m - 1$, $\lambda = m - 2$, $\mu = 0$, $r = m - 1$, $s = -1$), where $a > 1$.

In the latter case Γ is a complete multipartite graph $K_{a\times m}$ (and $v = am$, $k = (a - 1)m$, $\lambda = (a - 2)m$, $\mu = (a - 1)m$, $r = 0$, $s = -m$), again with $a > 1$.

(The graphs K_m and $K_{1\times m} = \overline{K_m}$ have only one restricted eigenvalue, namely -1 and 0 respectively, and hence are not strongly regular.)

For a primitive strongly regular graph it follows that $0 \le \lambda < k - 1$ and $0 < \mu < k$ and $r > 0$ and $s < -1$. A primitive strongly regular graph is connected, and hence $k > r$.

The graph nK_2 is sometimes called a *ladder graph*. Its complement $\overline{nK_2} = K_{n\times 2}$ a *cocktail party graph*.

1.1.4 Spectrum

Let Γ be strongly regular, with spectrum k, r (with multiplicity f) and s (with multiplicity g). Then f, g can be solved from $1 + f + g = v$ and $k + fr + gs = \operatorname{tr} A = 0$. The fact that f, g must be integers is a strong restriction on possible parameter sets.

If $f \neq g$, then one can also solve r, s from $r + s = \lambda - \mu$ and $fr + gs = -k$, and it follows that r, s are rational. Since they are also algebraic integers, they are integral in this case. On the other hand, if $f = g$, then $f = g = (v-1)/2$. Now $k = (\mu - \lambda)f = (\mu - \lambda)(v-1)/2$, and since $0 < k < v - 1$ it follows that $k = (v-1)/2$ and $\mu = \lambda + 1$. Now $v = 1 + k + k(k - 1 - \lambda)/\mu$ yields $\mu = k - 1 - \lambda = k/2$, so that $(v, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)$ for a suitable integer t, and $r, s = (-1 \pm \sqrt{v})/2$. This is known as the 'half case'. It occurs e.g. for the Paley graphs (see §1.1.9). For further details, see §8.2.

Summary: if we are not in the half case, then the spectrum is integral.

Explicit expressions for f, g are $f = \frac{(s+1)k(k-s)}{\mu(s-r)}$ and $g = \frac{(r+1)k(k-r)}{\mu(r-s)}$.

The identity $\frac{vk(v-1-k)}{fg} = (r-s)^2$ (known as the *Frame quotient*, cf. [123] §2.2A, 2.7A) follows.

In particular, $v = (r-s)^2$ if and only if $\{f, g\} = \{k, v - k - 1\}$.

1.1.5 Rank 3 permutation groups

A *permutation group* is a group G together with an action of G on some set X, that is, together with a map $G \times X \to X$ written $(g, x) \mapsto gx$, such that $1x = x$ and $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$, where 1 is the identity element of G.

An *orbit* of G on X is a set of the form Gx for some $x \in X$. The G-orbits form a partition of X. The action (or the group) is called *transitive* when this partition has a single element only, that is, when $Gx = X$ for all $x \in X$. A set A is *preserved* by G when $gA = A$ for all $g \in G$.

The action of G on X induces an action of G on $X \times X$ via $g(x, y) = (gx, gy)$. If G is transitive, then it is said to be *of (permutation) rank* r when it has precisely r orbits on $X \times X$.

The action (or the group) is called *primitive* when there is no nontrivial equivalence relation $R \subseteq X \times X$ that is preserved by G. The trivial equivalence relations are the full set $X \times X$ and the diagonal $D = \{(x, x) \mid x \in X\}$.

Suppose G is a rank 3 permutation group on the set X. Then G has three orbits D, E, F on $X \times X$, where D is the diagonal. Now either E and F are inverse relations: $F = \{(y, x) \mid (x, y) \in E\}$, or E and F are symmetric. In the former case (X, E) is a complete directed graph, a tournament (and (X, F) is the opposite tournament). In the latter case (X, E) and (X, F) are a complementary pair of graphs. When X is finite, they are a complementary pair of strongly regular graphs: the group G acts as a group of automorphisms on the graphs (X, E) and (X, F), and since E and F are single orbits, G is transitive on ordered pairs of adjacent (nonadjacent) vertices, and the number of common neighbors of two vertices does not depend on the vertices chosen, but only on whether they are equal, adjacent or nonadjacent.

History

The study of rank 3 permutation groups was initiated by Higman [420].

1.1.6 Local graphs

If Γ is a graph, and x a vertex of Γ, then the *local graph* of Γ at x is the graph induced by Γ on the set of neighbors of x in Γ.

A graph Γ is called *locally* Δ (or *locally* X) where Δ is a graph and X a graph property, when all local graphs are isomorphic to Δ (or have property X).

For example, the icosahedron is the unique connected locally pentagon graph. Hall [391] determined all locally Δ graphs on at most 11 vertices, for all possible Δ, and determined for each graph Δ on at most 6 vertices whether there exists a locally Δ graph.

If Γ is a connected graph, and x a vertex of Γ, then the i-th *subconstituent* of Γ (at x) is the graph induced on the set of vertices at (graph) distance i from x. If Γ is a strongly regular graph, and x a vertex of Γ, then the second subconstituent of Γ (at x) is the graph induced on the set of vertices other than x and nonadjacent to x.

1.1.7 Johnson graphs

Let Ω be a set, and $d \geq 0$ an integer. The *Johnson graph* $J(\Omega, d)$ is the graph that has as vertex set the set $\binom{\Omega}{d}$ of d-subsets of Ω, where two d-sets D, E are adjacent when $|D \cap E| = d - 1$. Suppose $|\Omega| \geq 2d$. Then $J(\Omega, d)$ has diameter d, and the symmetric group $\mathsf{Sym}(\Omega)$ acts as a group of automorphisms that is transitive of rank $d+1$. If $|\Omega| = m$ one writes $J(m, d)$ instead of $J(\Omega, d)$.

The full group of automorphisms of $J(\Omega, d)$ is $\mathsf{Sym}(\Omega)$ when $|\Omega| > 2d > 0$, but $\mathsf{Sym}(\Omega) \times 2$ when $|\Omega| = 2d > 0$, and 1 when $d = 0$.

In particular, the graph $J(m, 2)$ (also called the *triangular graph* $T(m)$), where $m \geq 4$, is strongly regular. It has parameters $v = m(m-1)/2$, $k = 2(m-2)$, $\lambda = m-2$, $\mu = 4$ and eigenvalues k, $r = m-4$, $s = -2$ with multiplicities 1, $f = m-1$, $g = m(m-3)/2$. The graph $T(m)$ is the line graph of the complete graph K_m on m vertices. The complement $\overline{T(5)}$ of $T(5)$ is the *Petersen graph* (§10.3).

These graphs are characterized by their parameters, except when $m = 8$. There are four graphs with the parameters $(v, k, \lambda, \mu) = (28, 12, 6, 4)$ of $T(8)$, namely $T(8)$ itself and three graphs known as the *Chang graphs* ([191, 192]), cf. §10.11.

1.1.8 Hamming graphs

Let Ω be a set, and $d \geq 0$ an integer. The *Hamming graph* $H(d, \Omega)$ is the graph that has as vertex set the set Ω^d of d-tuples of elements of Ω, where two d-tuples $(a_1, \ldots, a_d)$, $(b_1, \ldots, b_d)$ are adjacent when they have *Hamming distance* 1, i.e., when $a_i \neq b_i$ for a unique i. Suppose $|\Omega| \geq 2$. Then $H(d, \Omega)$ has diameter d, and its full group of automorphisms is the wreath product $\mathsf{Sym}(\Omega) \,\mathrm{wr}\, \mathsf{Sym}(d)$. This group is transitive of rank $d+1$. If $|\Omega| = q$ one writes $H(d, q)$ instead of $H(d, \Omega)$.

In particular, the graph $H(2,q)$ (also called the *lattice graph* $L_2(q)$ or the $q \times q$ *grid*), where $q \geq 2$, is strongly regular. It has parameters $v = q^2$, $k = 2(q-1)$, $\lambda = q-2$, $\mu = 2$ and eigenvalues k, $r = q-2$, $s = -2$ with multiplicities 1, $f = 2(q-1)$, $g = (q-1)^2$. The graph $H(2,q)$ is the line graph of the complete bipartite graph $K_{q,q}$. The graph $L_2(3)$ is isomorphic to its complement. It is the Paley graph (see §1.1.9) of order 9.

These graphs are characterized by their parameters, except when $q = 4$. There are two graphs with the parameters $(v,k,\lambda,\mu) = (16,6,2,2)$, namely $L_2(4)$ and the *Shrikhande graph* ([649]), cf. §10.6.

The graph $H(d,q)$ is locally dK_{q-1}, the disjoint union of d complete graphs of size $q-1$. The Shrikhande graph is locally a hexagon.

1.1.9 Paley graphs

Let $q = 4t+1$ be a prime power. The *Paley graph* Paley(q) is the graph with the finite field $\mathbb{F}_q$ as vertex set, where two vertices are adjacent when they differ by a nonzero square. It is strongly regular with parameters $(4t+1, 2t, t-1, t)$. (The restriction $q \equiv 1 \pmod 4$ is to ensure that -1 is a square, so that the resulting graphs are undirected.)

Let $q = p^e$, where p is prime. The full group of automorphisms consists of the maps $x \mapsto ax^\sigma + b$ where $a, b \in \mathbb{F}_q$, a a nonzero square, and $\sigma = p^i$ with $0 \leq i < e$ ([186]). It has order $eq(q-1)/2$.

Paley(5) is the pentagon. Paley(9) is the 3×3 grid. Paley(13) is a graph that is locally a hexagon. For a more detailed discussion, see §7.4.4.

1.1.10 Strongly regular graphs with smallest eigenvalue -2

A disjoint union of cliques has smallest eigenvalue $s = -1$. The pentagon has smallest eigenvalue $(-1-\sqrt{5})/2$. All other strongly regular graphs satisfy $s \leq -2$. SEIDEL [642] determined the strongly regular graphs with smallest eigenvalue $s = -2$. There are three infinite families and seven more graphs:

(i) the complete n-partite graph $K_{n\times 2}$, with parameters $(v,k,\lambda,\mu) = (2n, 2n-2, 2n-4, 2n-2)$, $n \geq 2$,

(ii) the lattice graph $L_2(n)$, that is, the Hamming graph $H(2,n)$, that is, the $n \times n$ grid, with parameters $(v,k,\lambda,\mu) = (n^2, 2(n-1), n-2, 2)$, $n \geq 3$,

(iii) the triangular graph $T(n)$ with parameters $(v,k,\lambda,\mu) = (\binom{n}{2}, 2(n-2), n-2, 4)$, $n \geq 5$,

(iv) the Shrikhande graph (cf. §10.6), with parameters $(v,k,\lambda,\mu) = (16,6,2,2)$,

(v) the three Chang graphs (cf. §10.11), with parameters $(v,k,\lambda,\mu) = (28,12, 6,4)$,

(vi) the Petersen graph (cf. §10.3), with parameters $(v,k,\lambda,\mu) = (10,3,0,1)$,

(vii) the Clebsch graph (cf. §10.7), with parameters $(v,k,\lambda,\mu) = (16,10,6,6)$,

(viii) the Schläfli graph (cf. §10.10), with parameters $(v,k,\lambda,\mu) = (27,16,10,8)$.

More generally, the strongly regular graphs with fixed smallest eigenvalue are (i) complete multipartite graphs, (ii) Latin square graphs, (iii) block graphs of Steiner systems, (iv) finitely many further graphs, see Theorem 8.6.4.

We include a proof of Seidel's classification. (For different proofs, see [419] and [123], Theorem 3.12.4. See also below.)

Theorem 1.1.1 *A strongly regular graph with smallest eigenvalue* -2 *is one of the examples in (i)–(viii) above.*

Proof. We shall assume the classification of the graphs with the parameters of the examples. The proof here derives the possible parameters.

Let Γ be a strongly regular graph with parameters v, k, λ, μ and spectrum $k^1\ r^f\ s^g$, where $s = -2$. Then $\lambda = \mu + r - 2$ and $k = \mu + 2r$ (by §1.1.1), so that $k = 2\lambda - \mu + 4$.

If $\mu = 2$, then Γ has the parameters of $L_2(n)$ (for $n = r + 2$), and hence is $L_2(n)$, or (if $n = 4$) the Shrikhande graph (cases (ii) and (iv)). If $\mu = 4$, then Γ has the parameters of $T(n)$ (for $n = r + 4$), and hence is $T(n)$, or (if $n = 8$) a Chang graph (cases (iii) and (v)). Assume $\mu \neq 2, 4$.

From $1 + f + g = v$ and $k + fr - 2g = 0$ and $\mu v = (k - r)(k + 2)$, we find $f = \frac{2v-k-2}{r+2} = \frac{(\mu+2r)(\mu+2r+2)}{\mu(r+2)}$.

Let an *m-claw* be an induced $K_{1,m}$ subgraph. Let a *quadrangle* be an induced C_4 subgraph. Let $x \sim a, b$ with $a \not\sim b$. If $\{x, a, b\}$ is contained in c 3-claws and in q quadrangles, then $k = 2 + 2\lambda - (\mu - 1 - q) + c$ so that $c + q = 1$.

First consider the case where the graph contains a 3-claw. Let $x \sim a, b, c$ with mutually nonadjacent a, b, c. We shall show that $v = 2k + 4$ and Γ is one of the examples (iv)–(vi).

For a list of vertices Z, let $N(Z)$ ('near') be the set of vertices adjacent to each z in Z, and $F(Z)$ ('far') the set of vertices not in Z and nonadjacent to each z in Z. Since the $k - \lambda - 1 = r + 1$ vertices in $N(x) \cap F(a)$ are in $\{b, c\} \cup N(b, c) \setminus \{x\}$, we have $r \leq \mu$. Since the $k - \lambda$ vertices in $(N(a) \cap F(x)) \cup \{a\}$ are among the $\bar{\lambda} = v - 2k + \mu - 2$ vertices of $F(b, c)$, we have $v \geq 5r + \mu + 4$. Since $\mu v = (k - r)(k + 2)$ we have $v = 3r + \mu + 2 + \frac{2r(r+1)}{\mu}$ so that $\mu \leq r$. It follows that $\mu = r$, $\lambda = 2r - 2$, $k = 3r$, $v = 6r + 4 = 2k + 4$, $f = 9 - \frac{12}{r+2}$ so that $r \in \{1, 2, 4, 10\}$. For $r = 1, 2, 4$ we are in case (vi), (iv), (v), respectively. The case $(v, k, \lambda, \mu) = (64, 30, 18, 10)$ has $f = 8$, which violates the absolute bound $v \leq \frac{1}{2}f(f + 3)$ (Proposition 1.3.14 below).

Now assume that Γ does not contain 3-claws. Since $c + q = 1$, each 2-claw is in a unique quadrangle. It follows that μ is even, say $\mu = 2m$, and if $a \not\sim b$, then $N(a, b)$ induces a $K_{m\times 2}$. If moreover $d \sim a$, $d \not\sim b$, then d is adjacent to precisely m vertices of $N(a, b)$. (If $x, y \in N(a, b)$ with $x \not\sim y$, then d cannot be nonadjacent to both x and y, since $(a; x, y, d)$ would be a 3-claw, and d cannot be adjacent to both x and y, since we already see the μ common neighbors of x and y in $N(a, b) \cup \{a, b\}$.)

Let b be a vertex, and consider the graph induced on $F(b)$. It is strongly regular or complete or edgeless with parameters $(v_0, k_0, \lambda_0, \mu_0) = (v - k - 1, k - \mu, \lambda - m, \mu)$. If it is edgeless, then $k = \mu$, so that Γ is imprimitive, and we are in case (i). If it is complete, then $v - k - 1 = k - \mu + 1$ so that $(\mu + 2r)(r + 1) = \mu(2r + 1)$, hence $\mu = 2(r + 1)$ and $f = 8 - \frac{12}{r+2}$, so that

$r \in \{1, 2, 4, 10\}$. For $r = 1$ we have $T(5)$ (in case (iii)), for $r = 2$ the Clebsch graph (case (vii)), and $r = 4$ ($v = 28$, $f = 6$) and $r = 10$ ($v = 64$, $f = 7$) both violate the absolute bound.

So we may assume that $F(b)$ induces a strongly regular graph Δ. Since $k_0 = 2\lambda_0 - \mu_0 + 4$, also Δ has smallest eigenvalue -2, and the other restricted eigenvalue is $r_0 = r - m$ with multiplicity $f_0 = \frac{2r(r+1)}{m(r-m+2)}$. By induction we already know Δ (and it does not contain 3-claws) so either $\mu \in \{6, 8\}$, or Δ is $K_{n\times 2}$. For $\mu = 6$ there are no feasible parameters. For $\mu = 8$ we find the Schläfli graph (case (viii)). If Δ is $K_{n\times 2}$, then $(v, k, \lambda, \mu) = (6n-3, 4n-4, 3n-5, 2n-2)$, $r = n - 1$, $f = 8 - \frac{12}{r+2}$, so that $r \in \{1, 2, 4, 10\}$. For $r = 1$ we have $L_2(3)$ (in case (ii)), for $r = 2$ we have $T(6)$ (in case (iii)), for $r = 4$ the Schläfli graph (case (viii)), and $r = 10$ ($v = 63$, $f = 7$) violates the absolute bound. □

Root systems

In fact it is possible to find all graphs with smallest eigenvalue ≥ -2. By the beautiful theorem of CAMERON, GOETHALS, SEIDEL & SHULT [179] (see also [123], §3.12 and [132], §8.4) such a graph is either a generalized line graph or is one in a finite (but large) collection.

(Sketch of the proof: Consider $A + 2I$. It is positive semidefinite, so one can write $A + 2I = M^\top M$. Now the columns of M are vectors of squared length 2 with integral inner products, and this set of vectors can be completed to a root system. By the classification of root systems one gets one of A_n, D_n, E_6, E_7 or E_8. In the first two cases the graph was a generalized line graph. In the latter three cases the graph is finite: at most 36 vertices, each vertex of degree at most 28. If the graph was regular, it has at most 28 vertices, and each vertex has degree at most 16. For details, see [123], Theorem 3.12.2 or [132], Chapter 8.)

There is a lot of literature describing manageable parts of this large collection, and related problems. A book-length treatment is CVETKOVIĆ et al. [249].

1.1.11 Seidel switching

Instead of the ordinary adjacency matrix A, Seidel considered the *Seidel matrix* S of a graph, with zero diagonal, where $S_{xy} = -1$ if $x \sim y$, and $S_{xy} = 1$ otherwise. These matrices are related by $S = J - I - 2A$.

Let Γ be a graph with vertex set X. Let $Y \subseteq X$. The graph Γ' obtained by *switching* Γ with respect to Y is the graph with vertex set X, where two vertices that are both inside or both outside Y are adjacent in Γ' when they are adjacent in Γ, while a vertex inside Y is adjacent in Γ' to a vertex outside Y when they are not adjacent in Γ. If Γ has Seidel matrix S, then Γ' has Seidel matrix S' where S' is obtained from S by multiplying each row and each column with index in Y by -1. It follows that S and S' have the same spectrum.

If Γ' is obtained from Γ by switching w.r.t. Y, and Γ'' is obtained from Γ' by switching w.r.t. Z, then Γ'' is obtained from Γ by switching w.r.t. $Y \triangle Z$. It follows that graphs related by switching fall into equivalence classes (called *switching classes*). Two graphs in the same switching class are called *switching equivalent*.

If two regular graphs of the same valency are switching equivalent, then they have the same ordinary spectrum. This happens precisely when each vertex inside (outside) the switching set is adjacent to half of the vertices outside (resp. inside) the switching set. For example, the Shrikhande graph is obtained from the 4×4 grid by switching w.r.t. a diagonal.

It may happen that two strongly regular graphs of different valencies are switching equivalent. If that happens, then they are related to regular 2-graphs (see §1.1.12).

Proposition 1.1.2 *Let* Γ *be a strongly regular graph with parameters* (v, k, λ, μ) *and spectrum* $k^1\ r^f\ s^g$. *Let* Δ *be a strongly regular graph of valency* $\ell > k$ *switching equivalent to* Γ. *Then (i)* Δ *has spectrum* $\ell^1\ r^{f-1}\ s^{g+1}$, *(ii)* $\frac{1}{2}v = k - s = \ell - r$, *(iii)* $k - r = 2\mu$, *(iv)* $\frac{1}{2}v = 2k - \lambda - \mu$, *(v) any switching set from* Γ *to* Δ *has size* $\frac{1}{2}v$ *and is regular of degree* $k - \mu$.

Proof. (i)–(iv) The Seidel matrices $S = J - I - 2A$ of Γ and Δ have the same spectrum $(v-1-2k)^1\ (-1-2r)^f\ (-1-2s)^g$, and if $k < l$ it follows that $v - 1 - 2k = -1 - 2s$ and $v - 1 - 2\ell = -1 - 2r$. Since $(k-r)(k-s) = \mu v$ for all strongly regular graphs, it follows from $k - s = \frac{1}{2}v$ that $k - r = 2\mu$. Since $r + s = \lambda - \mu$ for all strongly regular graphs, we find $\frac{1}{2}v = k - s = k - r - s + r = k + \mu - \lambda + k - 2\mu = 2k - \lambda - \mu$. (v) Suppose Δ is obtained from Γ by switching w.r.t. a set U of size u. Let $x \in U$ have k_1 neighbors in U and k_2 outside. Then $k = k_1 + k_2$ and $\ell = k_1 + v - u - k_2$, so that k_1 and k_2 can be expressed in terms of k, ℓ, u, v and are independent of x. Similarly, if $y \notin U$ has k_3 neighbors in U and k_4 outside, then $k = k_3 + k_4$ and $\ell = u - k_3 + k_4$, so that k_3 and k_4 are independent of y. Counting the number of edges with one end in U in two ways, we find $k_2 u = k_3(v - u)$, and since $k_2 = \frac{1}{2}(k - \ell - u + v)$ and $k_3 = \frac{1}{2}(k - \ell + u)$ this simplifies to $(k - \ell)u = (k - \ell)(v - u)$, so that $u = \frac{1}{2}v$, $k_2 = k_3$, $k_1 = k_4$. □

The Seidel matrix plays a role in the description of regular two-graphs and of sets of equiangular lines, cf. [132], Chapter 10. The condition $\frac{1}{2}v = 2k - \lambda - \mu$ is necessary and sufficient for a strongly regular graph to be associated to a regular two-graph, cf. [132], 10.3.2(i), and see below.

History

The Seidel matrix was introduced in Seidel [641].

1.1.12 Regular two-graphs

A *two-graph* $\Omega = (V, \Delta)$ is a finite set V provided with a collection Δ of unordered triples from V, such that every 4-subset of V contains an even number of triples from Δ. The triples from Δ are called *coherent*.

From a graph $\Gamma = (V, E)$, one can construct a two-graph $\Omega = (V, \Delta)$ by calling a triple from V coherent if the three vertices induce a subgraph in Γ with an odd number of edges. One checks that Ω is a two-graph. It is called the two-graph associated to Γ. Switching equivalent graphs have the same associated two-graph.

Conversely, from any two-graph $\Omega = (V, \Delta)$, and any fixed $w \in V$, we can construct a graph $\Gamma = \Omega_w$ with vertex set V as follows: let w be an isolated vertex in Γ, and let any two other vertices x, y be adjacent in Γ if $\{w, x, y\} \in \Delta$. Then Ω is the two-graph associated to Γ.

Thus we have established a one-to-one correspondence between two-graphs and switching classes of graphs.

Let $\Omega = (V, \Delta)$ be a two-graph, and $w \in V$. The *descendant* of Ω at w is the graph Ω_w^*, obtained from Ω_w by deleting the isolated vertex w.

A two-graph (V, Δ) is called *regular* (of degree a) if every unordered pair from V is contained in exactly a triples from Δ. The two-graph $\Omega = (V, \Delta)$ with $v = |V|$ vertices and $0 < |\Delta| < \binom{v}{3}$ is regular if and only if any descendant is strongly regular with parameters $(v-1, k, \lambda, \mu)$ where $\mu = k/2$ (and then this holds for all descendants). If this is the case, then $a = k$ and $v = 3k - 2\lambda$.

See also §8.10 and [132], §10.3.

History

Regular two-graphs were introduced by G. Higman. See also Taylor [677].

1.1.13 Regular partitions and regular sets

Let Γ be a finite graph with vertex set X. A partition $\{X_1, \ldots, X_m\}$ of X is called *regular* or *equitable* when there are numbers e_{ij}, $1 \le i, j \le m$, such that each vertex of X_i is adjacent to precisely e_{ij} vertices in X_j. In this situation the matrix $E = (e_{ij})$ of order m is called the *quotient matrix* of the partition.

If θ is an eigenvalue of E, say $Eu = \theta u$, then θ is also an eigenvalue of Γ, for the eigenvector that is constant u_i on X_i. And conversely, the eigenvalues of Γ that belong to eigenvectors constant on each X_i are eigenvalues of E.

Let Γ be finite and regular of valency k. A subset Y of the vertex set X is called *regular* (of *degree* d and *nexus* e) when the partition $\{Y, X \setminus Y\}$ is regular (and $e_{11} = d$, $e_{21} = e$ where $X_1 = Y$). Now the quotient matrix $E = \left(\begin{smallmatrix} d & k-d \\ e & k-e \end{smallmatrix}\right)$ has eigenvalues k and $d - e$, so that $d - e$ is an (integral) eigenvalue of Γ.

A regular set is also called an *intriguing set* ([263]).

Proposition 1.1.3 *Let Γ be strongly regular with parameters (v, k, λ, μ). If Y and Y' are regular sets of degrees d, d' and nexus e, e' belonging to different eigenvalues $d - e$ and $d' - e'$ other than k, then $|Y \cap Y'| = ee'/\mu$.*

Proof. The vector u that is 1 on Y and $a := \frac{-e}{k-d}$ outside Y is an eigenvector of the adjacency matrix A of Γ with eigenvalue $\theta := d - e$. Here $a \ne 1$ since $\theta \ne k$. The characteristic vector of Y is $\chi_Y = \frac{1}{1-a}u - \frac{a}{1-a}\mathbf{1}$, where $\frac{a}{1-a} = \frac{-e}{k-\theta}$. Similarly for Y'. Since $u, u', \mathbf{1}$ are mutually orthogonal, $(\mathbf{1}, \mathbf{1}) = v$, and $\mu v = (k-\theta)(k-\theta')$, we have $|Y \cap Y'| = (\chi_Y, \chi_{Y'}) = \frac{ee'}{(k-\theta)(k-\theta')}v = ee'/\mu$. □

We also see that $|Y| = (\chi_Y, \mathbf{1}) = \frac{ev}{k-\theta}$ with $\theta = d - e$.

The collection of regular sets belonging to the same eigenvalue $\theta = d - e$ (together with $\emptyset$ and X) is closed under taking complements, under taking disjoint unions, and under removal of one set from one containing it.

In descendants of regular two-graphs, switching sets are regular sets.

Proposition 1.1.4 *Let Γ be strongly regular with parameters (v, k, λ, μ) and restricted eigenvalues r, s, where $k = 2\mu$. Let Y be a regular set in Γ of degree d and nexus e. If $|Y| = k - c$, where $\{c, d - e\} = \{r, s\}$, then adding an isolated vertex and switching w.r.t. Y yields a strongly regular graph with parameters $(v + 1, k - c, \lambda - c, \mu - c)$.* □

1.1.14 Inequalities for subgraphs

We give inequalities that must hold for a graph Γ to have certain induced subgraphs. Additional regularity holds when there are such subgraphs and the inequality holds with equality.

Interlacing

Let Γ be a finite graph with adjacency matrix A, and let $\Pi = \{X_1, \ldots, X_m\}$ be a partition of a subset of $\mathsf{V}\Gamma$. The *quotient matrix* of A w.r.t. Π is the matrix B of order m where B_{ij} is the average row sum of the submatrix $A(i,j)$ of A that has rows indexed by X_i and columns indexed by X_j. If each $A(i,j)$ has constant row sums, and Π partitions $\mathsf{V}\Gamma$, then Π is an equitable partition of Γ, and B is a quotient matrix in the sense of §1.1.13 (hence the present definition generalizes the previous one).

Theorem 1.1.5 *Let Γ be a graph with adjacency matrix A and v vertices. Let $\Pi = \{X_1, \ldots, X_m\}$ be a partition of a subset of $\mathsf{V}\Gamma$ with quotient matrix B. Then the eigenvalues of B interlace those of A. That is, if A has eigenvalues $\theta_1 \geq \cdots \geq \theta_v$ and B has eigenvalues $\eta_1 \geq \cdots \geq \eta_m$, then $\theta_i \geq \eta_i$ $(1 \leq i \leq m)$ and $\eta_{m-i} \geq \theta_{v-i}$ $(0 \leq i \leq m-1)$.*

If the interlacing is tight, that is, if there is an h such that $\eta_i = \theta_i$ for $1 \leq i \leq h$ and $\eta_i = \theta_{v-m+i}$ for $h+1 \leq i \leq m$, then the partition is equitable.

For a proof, and related results, see [132], §2.5.

Note that this theorem applies to an (induced) subgraph Δ of Γ with adjacency matrix B. Indeed, one can take for Π the partition of $\mathsf{V}\Delta$ into singletons.

Bounds on the size of regular subgraphs

As an application of interlacing, we find bounds on the size of regular subgraphs of a graph.

Proposition 1.1.6 *Let Γ be a regular graph with v vertices, valency k, second largest eigenvalue r and smallest eigenvalue s. Let Y be a nonempty proper subset of $X := \mathsf{V}\Gamma$ inducing a subgraph that is regular of degree d. Then*

(i) $|Y| \leq \frac{v(d-s)}{k-s}$, and

(ii) $|Y| \geq \frac{v(d-r)}{k-r}$ if $r < k$.

(iii) If equality holds in either (i) or (ii), then each vertex in $X \setminus Y$ has the same number $e = d - \theta$ of neighbors in Y, where $\theta = s$ in case (i), and $\theta = r$ in case (ii).

Proof. Apply Theorem 1.1.5 with $\Pi = \{Y, X \setminus Y\}$. Put $u = |Y|$. The quotient matrix is

$$B = \begin{pmatrix} d & k-d \\ \frac{u(k-d)}{v-u} & k - \frac{u(k-d)}{v-u} \end{pmatrix}$$

with eigenvalues k and $d - \frac{u(k-d)}{v-u}$. By interlacing we have $s \leq d - \frac{u(k-d)}{v-u} \leq r$, which gives (i) and (ii). If equality holds on either side, then the partition is equitable, and $e = \frac{u(k-d)}{v-u}$. $\square$

One sees that in case of equality the vector $\chi_Y - \frac{u}{v}\mathbf{1}$ is an eigenvector of A with eigenvalue $\theta = d - e$. If θ has small multiplicity this allows a computer search for all such subgraphs Y.

Hoffman bound

In particular, we have the so-called Hoffman bounds (due to Delsarte for strongly regular graphs, generalized by Hoffman to arbitrary regular graphs, then further by Haemers to arbitrary graphs) on the sizes of cliques and cocliques.

Proposition 1.1.7 *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue s. Then*

(i) If C is a coclique in Γ, then $|C| \le v/(1 + \frac{k}{-s})$. If equality holds, then each vertex outside C has the same number $-s$ of neighbors inside.

(ii) If D is a clique in Γ, then $|D| \le 1 + \frac{k}{-s}$. If equality holds, then each vertex outside D has the same number $\mu/(-s)$ of neighbors inside.

(iii) If a coclique C and a clique D both meet the bounds of (i) and (ii), then $|C \cap D| = 1$.

Proof. Part (i) is the special case $d = 0$ of Proposition 1.1.6. Part (ii) is part (i) applied to $\overline{\Gamma}$. For part (iii), clearly C and D cannot have more than one point in common. If they are disjoint, then the number of edges joining C and D is both $k - s$ and $\mu v/(k - s) = k - r$, a contradiction. □

If a regular set in a strongly regular graph is a coclique or a clique, then it has equality in (i) or (ii), respectively.

The bound on cocliques for Γ equals the bound on cliques for the complementary graph $\overline{\Gamma}$, i.e., $v/(1 + \frac{k}{-s}) = 1 + \frac{v-k-1}{r+1}$.

Quadratic counting

Similar results are obtained by combinatorial methods. Consider a strongly regular graph with parameters (v, k, λ, μ) and an induced subgraph with u vertices, e edges, and degree sequence $d_1, \ldots, d_u$. Let there be x_i vertices outside the subgraph that are adjacent to precisely i vertices inside. Then

$$\sum_i x_i = A = v - u,$$
$$\sum_i i x_i = B = ku - 2e,$$
$$\sum_i \binom{i}{2} x_i = C = \lambda e + \mu(\binom{u}{2} - e) - \sum_{i=1}^{u} \binom{d_i}{2}.$$

Let $\gamma = B/A$ and put $\underline{i} = \lfloor \gamma \rfloor$ and $\bar{i} = \lceil \gamma \rceil$. Then

$$(B + 2C) - (\underline{i} + \bar{i})B + \underline{i}\bar{i}A = \sum_i (i - \underline{i})(i - \bar{i})x_i \ge 0. \qquad (*)$$

Equality holds if and only if every vertex outside the subgraph is adjacent to either $\underline{i}$ or $\bar{i}$ vertices inside.

If the subgraph is a clique or a coclique, the inequality $\sum_i (i - \gamma)^2 x_i \ge 0$ is equivalent to the Hoffman bound. When γ is nonintegral, inequality (*) is slightly stronger.

This inequality is folklore. For the case of cliques an equivalent inequality was rediscovered in [364]. Sometimes combinatorial bounds are stronger than the Hoffman bound. For example, for the parameter set $(v,k,\lambda,\mu) = (400,21,2,1)$ with $s=-4$, the Hoffman bound for the size of cliques is 6.25, but the obvious upper bound $\lambda+2$ is 4.

For the case of cliques of size u, the above counts become $\sum x_i = v-u$, $\sum ix_i = u(k-u+1)$, $\sum \binom{i}{2}x_i = \binom{u}{2}(\lambda-u+2)$. For example, for $(v,k,\lambda,\mu) = (235,42,9,7)$ with $s=-5$, the Hoffman bound is 9.4, but the above counting also rules out size 9. And for example for $(v,k,\lambda,\mu) = (11124,882,45,72)$, with $s=-45$, the Hoffman bound is 20.6, but the above counting rules out size 18 so that the upper bound for clique sizes becomes 17.

Cvetković bound

Let Γ be a graph on v vertices, and let A be a matrix indexed by $V\Gamma$ such that $A_{xy}=0$ when $x \not\sim y$. Let $n^+(A)$ (resp. $n^-(A)$) be the number of positive (resp. negative) eigenvalues of A. For the independence number $\alpha(\Gamma)$ of Γ we have the bound (known as *Cvetković bound* or *inertia bound*)

$$\alpha(\Gamma) \le \min(v-n^+(A), v-n^-(A)).$$

One has additional regularity in case both the Hoffman and the Cvetković bound are tight.

Proposition 1.1.8 (Haemers [376], Theorem 2.1.7) *Let Γ be a strongly regular graph with point set X, and C a coclique in Γ with $|C| = 1+\frac{v-k-1}{r+1} = g$. Then the graph induced on $X\setminus C$ is strongly regular.*

This happens for example for a 21-coclique in a graph with parameters $(v,k,\lambda,\mu) = (77,16,0,4)$. See also §8.5.8.

Greaves-Koolen-Park bound

Greaves, Koolen & Park [363] derived a bound on the size of maximal cliques that rules out an interval of values. In some cases that interval extends past the Hoffman upper bound, so that the upper bound is greatly strengthened. If in addition one can show that cliques must exist with a size past the start of the interval, then the corresponding parameter set is ruled out.

Denote by $H(a,t)$ the graph on $a+t+1$ vertices consisting of a clique K_{a+t} together with a vertex that is adjacent to precisely a vertices of the clique. The graph $H(a,t)$ has an obvious equitable partition with quotient matrix

$$Q = \begin{bmatrix} 0 & a & 0 \\ 1 & a-1 & t \\ 0 & a & t-1 \end{bmatrix}.$$

Lemma 1.1.9 *Let Γ be a graph having smallest eigenvalue $-m$ that contains $H(a,t)$ as an induced subgraph. Then*

$$(a-m(m-1))(t-(m-1)^2) \le (m(m-1))^2.$$

Proof. This inequality expresses $\det(Q+mI) \ge 0$. □

If a strongly regular graph Γ has a maximal clique C of size c, and a vertex outside adjacent to a vertices of the clique, then $a \le \mu$. The above lemma (with $t = c-a$) gives a quadratic inequality on a, and if the quadratic has two roots r_1, r_2, then $r_1 < a < r_2$ is excluded. If $r_1 \le \mu < r_2$, it follows that

$a \le r_1$. On the other hand, there are certainly vertices outside C that have at least $\alpha = 1 + \frac{(c-1)(\lambda-c+2)}{k-c+1}$ neighbors in C. The inequality $\alpha \le a \le r_1$ gives a condition on c.

Lemma 1.1.10 *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue $-m$, where $\mu > m(m-1)$. If Γ has a maximal clique C of order $c > \max\{(m-1)(4m-1), \frac{\mu^2}{\mu-m(m-1)} - m + 1\}$ and $D = (c+m-1)(c-(m-1)(4m-1))$ then $(2(c-1)(\lambda-c+2)-(c+m-3)(k-c+1))^2-(k-c+1)^2D \ge 0$.*

This lemma gives a cubic condition on c.

For example, consider the case $(v, k, \lambda, \mu) = (1344, 221, 88, 26)$ where $m = 3$. The Hoffman bound is $c \le 74$. Lemma 1.1.10 says that $32 \le c \le 80$ is impossible for maximal cliques. So a maximal clique has size at most 31.

By the usual claw-and-clique method (cf. §8.6.5) one finds a lower bound for the size of maximum cliques.

Lemma 1.1.11 *Let Γ be a strongly regular graph with parameters (v, k, λ, μ). If e is a nonnegative integer such that $(\mu - 1)\binom{e}{2} < e(\lambda + 1) - k$, then Γ has a clique of size at least $\lambda + 2 - (e - 2)(\mu - 1)$.*

Together with the above, this sometimes suffices to rule out a parameter set.

For example, consider the case $(v, k, \lambda, \mu) = (23276, 1330, 372, 58)$ with $m = 4$. The Hoffman bound says that cliques have sizes $c \le 333$. By Lemma 1.1.10, for maximal cliques $71 \le c \le 340$ is impossible. By Lemma 1.1.11 with $e = 6$, there is a clique of size $c \ge 146$. It follows that no such graph exists.

Various refinements are possible.

1.1.15 Connectivity

For a graph Γ, let $\Gamma_i(x)$ denote the set of vertices at distance i from x in Γ. Instead of $\Gamma_1(x)$ we write $\Gamma(x)$. Using interlacing, we see that the 2nd subconstituent of a primitive strongly regular graph is connected.

Proposition 1.1.12 *If Γ is a primitive strongly regular graph, then the subgraph $\Gamma_2(x)$ is connected for each vertex x.*

Proof. Note that $\Gamma_2(x)$ is regular of valency $k-\mu$. If it is not connected, then its eigenvalue $k-\mu$ would have multiplicity at least two, and hence would be not larger than the second largest eigenvalue r of Γ. Then $x^2 + (\mu - \lambda)x + \mu - k \le 0$ for $x = k - \mu$, i.e., $(k - \mu)(k - \lambda - 1) \le 0$, a contradiction. □

The *vertex connectivity* $\kappa(\Gamma)$ of a connected non-complete graph Γ is the smallest integer m such that Γ can be disconnected by removing m vertices.

Theorem 1.1.13 (Brouwer & Mesner [138]) *Let Γ be a connected strongly regular graph of valency k. Then $\kappa(\Gamma) = k$, and the only disconnecting sets of size k are the sets of all neighbors of some vertex x.* □

One might guess that the cheapest way to disconnect a strongly regular graph such that all components have at least two vertices would be by removing the $2k-\lambda-2$ neighbors of an edge. Cioabă, Kim & Koolen [198] observed that this is false (the simplest counterexample is probably $T(6)$, where edges have 10 neighbors and certain triangles only 9), but proved it for several infinite classes of strongly regular graphs and conjectured that any counterexample must have $\lambda \ge k/2$. See also [199].

1.1.16 Graphs induced on complementary subsets of the vertex set of a graph

For a real symmetric matrix with two distinct eigenvalues, and with a symmetric 2×2 partition of rows and columns, the spectrum of the upper left-hand corner determines the spectrum of the lower right-hand corner (cf. [132], Lemma 2.11.1).

For strongly regular graphs with adjacency matrix A this applies to $A - aJ$ for suitable a, so that the spectrum of a regular induced subgraph determines the spectrum of the subgraph induced on the complementary set of vertices, when that is also regular. More generally, one has

Proposition 1.1.14 (DE CAEN [264]) *Let Γ be strongly regular on v vertices, with spectrum $k^1\ r^f\ s^g$, and suppose that $V\Gamma$ has a partition $\{C, D\}$ such that the graph $\Gamma[C]$ induced by Γ on C is regular, with valency k_C. Let $c = |C|$. If $\Gamma[C]$ has eigenvalues $k_C, \lambda_1, \ldots, \lambda_{c-1}$, then the graph $\Gamma[D]$ has eigenvalues r (with multiplicity $f - c$), s (with multiplicity $g - c$), $r + s - \lambda_j$ $(1 \le j \le c - 1)$, together with the two roots of $(X - k)(X - r - s + k_C) + \mu c = 0$.*

In the case of a regular partition, these two roots can be given explicitly:

Proposition 1.1.15 *If also the graph $\Gamma[D]$ is regular, with valency k_D, then $k_C + k_D - k \in \{r, s\}$, and $(X - k)(X - r - s + k_C) + \mu c = \frac{(X-k_D)(X-r)(X-s)}{X-k_C-k_D+k}$.*

For example, if Γ is a strongly regular graph with parameters $(v, k, \lambda, \mu) = (28, 9, 0, 4)$ and C is a point-neighborhood (a 9-coclique), then D has spectrum $1^{12}\ (-5)^{-3}\ (-4)^8\ 5^1\ 0^1$, a contradiction. So no such graph exists.

For example, if Γ is the unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (77, 16, 0, 4)$ and C is a 21-coclique, then D induces a Gewirtz subgraph (with parameters $(v, k, \lambda, \mu) = (56, 10, 0, 2)$ and spectrum $10^1\ 2^{35}(-4)^{20}$, see §10.20).

For example, if Γ is the $\mathsf{O}_6^-(3)$ graph on 112 vertices (with parameters $(v, k, \lambda, \mu) = (112, 30, 2, 10)$ and spectrum $30^1\ 2^{90}\ (-10)^{21}$, see §10.34), and C induces a Gewirtz subgraph, then the subgraph induced on the remaining 56 vertices has the same spectrum, and hence is also a Gewirtz subgraph.

See also [178], [381].

1.1.17 Enumeration

For some smallish parameter sets, a complete enumeration of all strongly regular graphs has been made. We list only one graph from a complementary pair. Triangular graphs and $n \times n$ grids on more than 50 vertices are not listed.

count	v	k	λ	μ	ref
1	5	2	0	1	pentagon
1	9	4	1	2	3×3 grid
1	10	3	0	1	Petersen graph, $\overline{T(5)}$
1	13	6	2	3	Paley
1	15	6	1	3	GQ(2,2), $\overline{T(6)}$
1	16	5	0	2	folded 5-cube, complement of the Clebsch graph
2	16	6	2	2	4×4 grid, Shrikhande graph

continued...

count	v	k	λ	μ	ref
1	17	8	3	4	Paley
1	21	10	3	6	$\overline{T(7)}$
1	25	8	3	2	5×5 grid
15	25	12	5	6	Paulus [606]; enumerated by Rozenfel'd [632]
10	26	10	3	4	Paulus [606]; enumerated by Rozenfel'd [632]
1	27	10	1	5	GQ(2,4), complement of the Schläfli graph
4	28	12	6	4	$T(8)$, 3 Chang graphs
41	29	14	6	7	enumerated by Bussemaker and by Spence
3854	35	16	6	8	enumerated by McKay & Spence [556]
1	36	10	4	2	6×6 grid
180	36	14	4	6	enumerated by McKay & Spence [556]
1	36	14	7	4	$T(9)$
32548	36	15	6	6	enumerated by McKay & Spence [556]
28	40	12	2	4	enumerated by Spence [670]
78	45	12	3	3	enumerated by Coolsaet, Degraer & Spence [223]
1	45	16	8	4	$T(10)$
1	49	12	5	2	7×7 grid
1	50	7	0	1	Hoffman & Singleton [436]
1	56	10	0	2	Gewirtz [342]
167	64	18	2	6	enumerated by Haemers & Spence [384]
1	77	16	0	4	Brouwer [111]
1	81	20	1	6	Brouwer & Haemers [130]
1	100	22	0	6	Gewirtz [341]
1	105	32	4	12	Coolsaet [221]
1	112	30	2	10	Cameron, Goethals & Seidel [178]
1	120	42	8	18	Degraer & Coolsaet [274]
1	126	50	13	24	Coolsaet & Degraer [222]
1	162	56	10	24	Cameron, Goethals & Seidel [178]
1	176	70	18	34	Degraer & Coolsaet [274]
1	275	112	30	56	Goethals & Seidel [356]

Table 1.1: Number of nonisomorphic strongly regular graphs

Let us call a parameter set (v, k, λ, μ) *feasible* when it and its complement satisfy the conditions of §1.1.1 and §1.1.4. There are further general conditions on strongly regular graphs, such as the absolute bound (§1.3.7), the Krein conditions (§1.3.4), the claw bound (§8.6.4), and the condition on conference graphs (§8.2), and on graphs with $\mu = 1$ or $\mu = 2$ (§8.18). For a few sets of parameters there is an ad hoc proof that no such graph exists. Below the current list of such cases.

v	k	λ	μ	ref
49	16	3	6	Bussemaker et al. [162]
57	14	1	4	Wilbrink & Brouwer [732]
75	32	10	16	Azarija & Marc [20]
76	21	2	7	Haemers [378]; see also [8]
76	30	8	14	Bondarenko et al. [89]
95	40	12	20	Azarija & Marc [21]
96	38	10	18	Degraer [273]
289	54	1	12	Bondarenko & Radchenko [90]
324	57	0	12	Gavrilyuk & Makhnev [336],

continued...

v	k	λ	μ	ref
				Kaski & Östergård [483]
460	153	32	60	Bondarenko et al. [88]
486	165	36	66	Makhnev [534]
1127	486	165	243	Makhnev [534]
1911	270	105	27	Koolen & Gebremichel[1]
3159	1408	532	704	Bannai et al. [49], [646]

Table 1.2: Sporadic parameter sets for which no srg exists

Makhnev [535] purports to show the nonexistence of graphs with parameters $(v, k, \lambda, \mu) = (784, 116, 0, 20)$, but the proof is wrong. Also the proof in Makhnev [536] of the nonexistence of graphs with parameters $(3250, 57, 0, 1)$ is wrong.

Money

J. H. Conway [214] offered \$1000 for the construction or nonexistence proof of a strongly regular graph with parameters $(v, k, \lambda, \mu) = (99, 14, 1, 2)$.

Wilbrink [731] showed that such a graph cannot have an automorphism of order 11, and hence cannot have a transitive group. Behbahani & Lam [55] show that any automorphism of prime order must have order 2 or 3. Crnković & Maksimović [240] rule out groups of order six or nine.

History

Uniqueness of the triangular graphs $T(n)$, given the parameters, was shown for $n \geq 9$ by Connor [211], for $n \leq 6$ by Shrikhande [648], and for $n \neq 8$ by Hoffman [432]. The latter also found a counterexample for $n = 8$. Independently, Chang [191, 192] settled all cases and found the three counterexamples for $n = 8$.

Uniqueness of the lattice graph $L_2(n)$, $n \neq 4$ was shown by Mesner [559]. Shrikhande [649] gave a shorter proof and also found the single exception.

1.1.18 Prolific constructions

Strongly regular graphs live on the boundary between the crystalline world and the random world. For some parameters there is no graph, or a unique graph. For other parameters the number of examples is exponentially large. Constructions that produce hyperexponentially many strongly regular graphs for certain parameters have been given by Wallis [718] and Fon-Der-Flaass [328]. See also [184], [176], [580].

1.2 Distance-regular graphs

A finite connected graph Γ of diameter d is called *distance-regular* with *parameters* a_i, b_i, c_i $(0 \leq i \leq d)$ if for any two vertices x, y with mutual distance $d(x, y) = i$ the number of vertices z adjacent to y and at distance $i - 1$ or i or $i + 1$ from x equals c_i or a_i or b_i, respectively.

A distance-regular graph is regular with valency $k = b_0$, and $a_i + b_i + c_i = k$ for all i. Obviously $c_0 = a_0 = b_d = 0$ and $c_1 = 1$. The *intersection array* is the symbol $\{b_0, b_1, \ldots, b_{d-1};\ c_1, c_2, \ldots, c_d\}$ that suffices to determine all parameters.

[1] Pers. comm., Aug. 2021.

The distance-regular graphs of diameter 2 are precisely the connected strongly regular graphs. A connected strongly regular graph with parameters (v, k, λ, μ) is distance-regular with intersection array $\{k, k-\lambda-1;\ 1, \mu\}$.

Let Γ be distance-regular, with vertex x. The number k_i of vertices at distance i from x is found by $k_0 = 1$ and $k_{i+1} = k_i b_i / c_{i+1}$ for $0 \le i \le d-1$, and is independent of the choice of x. The total number of vertices is $v = k_0 + \cdots + k_d$.

Let $v = |\mathsf{V}\Gamma|$. Let A_i be the matrix of order v indexed by $\mathsf{V}\Gamma$ with $(A_i)_{xy} = 1$ when $d(x, y) = i$ and $(A_i)_{xy} = 0$ otherwise. Clearly $A_0 = I$. Let $A = A_1$ be the adjacency matrix of Γ. Then $AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}$ for $0 \le i \le d$, if we agree that $b_{-1}A_{-1} = c_{d+1}A_{d+1} = 0$. We find an expression for A_i of degree i in A, and then an equation of degree $d+1$ for A, so that A has precisely $d+1$ distinct eigenvalues (since the matrices A_i are linearly independent).

Biggs' multiplicity formula

The previous paragraph implies (for a precise argument see also below) that the $d+1$ eigenvalues of A are the $d+1$ eigenvalues of the matrix L, where

$$L = \begin{pmatrix} 0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & b_2 & \\ & & \cdots & \cdots & \cdots \\ 0 & & & c_d & a_d \end{pmatrix}.$$

Theorem 1.2.1 (Biggs [67], Theorem 21.4) *If $Lu = \theta u$ and $u_0 = 1$, then the multiplicity of θ as eigenvalue of Γ equals*

$$m(\theta) = v / (\textstyle\sum k_i u_i^2).$$

Proof. We have $A_i = p_i(A)$ where the polynomials p_i are defined by $p_0(x) = 1$, $p_1(x) = x$, $xp_i(x) = b_{i-1}p_{i-1}(x) + a_ip_i(x) + c_{i+1}p_{i+1}(x)$. If η is an eigenvalue of A, then $p(\eta) = (p_0(\eta), \ldots, p_d(\eta))$ is a left eigenvector of L and $p(\eta)L = \eta p(\eta)$. The u_i satisfy $c_iu_{i-1} + a_iu_i + b_iu_{i+1} = \theta u_i$, so that $p_i(\theta) = k_iu_i$. Now $v = \mathrm{tr} \sum_i u_iA_i = \sum_{i,\eta} u_i m(\eta) p_i(\eta) = m(\theta) \sum_i k_iu_i^2$, where the sum is over the eigenvalues η of A, and the last equality holds because left and right eigenvectors for different eigenvalues are orthogonal. □

Thus, the parameters of a distance-regular graph determine eigenvalues and multiplicities. The fact that the multiplicities must be integers is a strong restriction on candidate parameter sets.

A comprehensive monograph on the topic of distance-regular graphs, complete up to 1989, is Brouwer, Cohen & Neumaier [123]. An update to the state of affairs in 2016 is Van Dam, Koolen & Tanaka [252].

1.2.1 Distance-transitive graphs

A connected graph Γ is called *distance-transitive* if for any vertices x, y, z, w with $d(x, y) = d(z, w)$ there is an automorphism g of Γ such that $g(x) = z$ and $g(y) = w$. If Γ is distance-transitive of diameter d, then its group of automorphisms is transitive, of rank $d+1$.

Every finite distance-transitive graph is distance-regular.

The classification of distance-transitive graphs of diameter $d > 2$ is unfinished. For a survey of the status in 2007, see Van Bon [86].

1.2.2 Johnson graphs

The Johnson graph $J(m, d)$, where $m \geq 2d$, is distance-transitive of diameter d. It has parameters $b_i = (d-i)(m-d-i)$, $c_i = i^2$ and eigenvalues $b_i - i$ with multiplicity $\binom{m}{i} - \binom{m}{i-1}$ $(0 \leq i \leq d)$ and $v = \binom{m}{d}$ vertices.

1.2.3 Hamming graphs

The Hamming graph $H(d, q)$, where $q > 1$, is distance-transitive of diameter d. It has parameters $b_i = (q-1)(d-i)$, $c_i = i$ and eigenvalues $b_i - i$ with multiplicity $\binom{d}{i}(q-1)^i$ $(0 \leq i \leq d)$ and $v = q^d$ vertices.

1.2.4 Grassmann graphs

Let V be a vector space of dimension n over the field $\mathbb{F}_q$. The *Grassmann graph* $J_q(n, m)$ is the graph with vertex set $\begin{bmatrix} V \\ m \end{bmatrix}$, the set of all m-subspaces of V, where two m-subspaces are adjacent when they intersect in an $(m-1)$-space. This graph is distance-transitive, with parameters $b_i = q^{2i+1}\begin{bmatrix} m-i \\ 1 \end{bmatrix}\begin{bmatrix} n-m-i \\ 1 \end{bmatrix}$, $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}^2$, diameter $d = \min(m, n-m)$, and eigenvalues $q^{i+1}\begin{bmatrix} m-i \\ 1 \end{bmatrix}\begin{bmatrix} n-m-i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}$ with multiplicity $\begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix}$. (Here $\begin{bmatrix} n \\ i \end{bmatrix} = (q^n - 1) \cdots (q^{n-i+1} - 1)/(q^i - 1) \cdots (q-1)$ is the q-binomial coefficient, the number of i-subspaces of an n-space.)

In particular, for $m = 2$, $n \geq 4$, we find the graph $J_q(n, 2)$ of lines in a projective space, adjacent when they meet. This graph is strongly regular, with parameters $v = \begin{bmatrix} n \\ 2 \end{bmatrix}$, $k = (q+1)(\begin{bmatrix} n-1 \\ 1 \end{bmatrix} - 1)$, $\lambda = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} + q^2 - 2$, $\mu = (q+1)^2$, and eigenvalues k, $r = q^2\begin{bmatrix} n-3 \\ 1 \end{bmatrix} - 1$, $s = -q-1$ with multiplicities, 1, $f = \begin{bmatrix} n \\ 1 \end{bmatrix} - 1$, $g = \begin{bmatrix} n \\ 2 \end{bmatrix} - \begin{bmatrix} n \\ 1 \end{bmatrix}$.

1.2.5 Van Dam-Koolen graphs

Van Dam & Koolen [251] construct distance-regular graphs vDK(q, m) with the same parameters as $J_q(2m+1, m)$. (They call them the *twisted Grassmann graphs*.) The group of automorphisms of these graphs is not transitive.

The construction is as follows. Let V be a vector space of dimension $2m+1$ over $\mathbb{F}_q$, and let H be a hyperplane of V. Take as vertices the $(m+1)$-subspaces of V not contained in H, and the $(m-1)$-subspaces contained in H, where two subspaces of the same dimension are adjacent when their intersection has codimension 1 in both, and two subspaces of different dimension are adjacent when one contains the other.

It follows that Grassmann graphs need not be determined by their parameters. Also, that the combinatorial definition of distance-regular graphs does not directly imply the existence of a nice group of automorphisms, not even when the diameter is large.

For $m = 2$, these graphs are strongly regular.

1.2.6 Imprimitive distance-regular graphs

Let Γ be a distance-regular graph of diameter d, and let Γ_i be the graph with the same vertex set, where two vertices are adjacent when they have distance i in Γ, so that A_i is the adjacency matrix of Γ_i $(0 \le i \le d)$. The graph Γ is called *imprimitive* if Γ_i is disconnected for some i, $2 \le i \le d$.

If Γ is a *polygon* (i.e., if it has valency 2) then Γ_i is disconnected for each i with $i \,|\, v$, $1 < i < v$. The only imprimitive distance-regular graphs of valency $k > 2$ are the bipartite and the antipodal graphs.

A graph is called *bipartite* if it does not contain an odd cycle. A *halved graph* of a connected bipartite graph Γ is the graph with as vertex set one of the two bipartite classes, where two vertices are adjacent when they have distance 2 in Γ.

A distance-regular graph of diameter d is called *antipodal* when having distance 0 or d is an equivalence relation on its vertex set. The *folded graph* of an antipodal distance-regular graph is the graph with as vertices the equivalence classes of Γ_d, where two equivalence classes are adjacent when they contain adjacent vertices.

Theorem 1.2.2 *An imprimitive distance-regular graph of valency k, $k > 2$, is bipartite or antipodal (or both). Let Γ be distance-regular of diameter d with intersection array $\{b_0, \ldots, b_{d-1};\ c_1, \ldots, c_d\}$, and put $\mu = c_2$ and $m = \lfloor d/2 \rfloor$.*

(i) Γ is bipartite if and only if $a_i = 0$ (i.e., $b_i + c_i = k$) for all i. If Γ is bipartite, then its halved graphs are distance-regular of diameter m with intersection array

$$\{\frac{b_0 b_1}{\mu}, \frac{b_2 b_3}{\mu}, \ldots, \frac{b_{2m-2} b_{2m-1}}{\mu};\ \frac{c_1 c_2}{\mu}, \frac{c_3 c_4}{\mu}, \ldots, \frac{c_{2m-1} c_{2m}}{\mu}\}.$$

(ii) Γ is antipodal if and only if $b_i = c_{d-i}$ for all $i \neq m$. If Γ is antipodal, then its antipodal classes have size $r = 1 + b_m / c_{d-m}$, and the folded graph is distance-regular of diameter m with intersection array

$$\{b_0, \ldots, b_{m-1};\ c_1, \ldots, c_{m-1}, \gamma c_m\}$$

where $\gamma = r$ if $d = 2m$, and $\gamma = 1$ if $d = 2m + 1$. □

For example, the Johnson graph $J(2d, d)$ is antipodal. The *folded Johnson graph* $\overline{J}(2d, d)$ (of which the vertices are the partitions of a $2d$-set into two d-sets) is distance-regular of diameter $\lfloor d/2 \rfloor$, and in particular is strongly regular for $d = 4, 5$.

1.2.7 Taylor graphs

A distance-regular graph Γ with intersection array $\{k, \mu, 1;\ 1, \mu, k\}$ is called a *Taylor graph.* Such a graph is an antipodal double cover of the complete graph K_{k+1}. The local graphs $\Delta = \Gamma(x)$ are strongly regular, and satisfy $v_\Delta = k$, $k_\Delta = \lambda_\Gamma = k - \mu - 1 = 2\mu_\Delta$, $\lambda_\Delta = \frac{1}{2}(3k_\Delta - k - 1)$. See §8.10.4.

Given a graph Σ with vertex set X, its *Taylor double* is the graph with vertex set $\{x^\varepsilon \mid x \in X, \varepsilon = \pm 1\}$ and edges $x^\delta y^\varepsilon$ (for $x \neq y$) with $\delta\varepsilon = 1$ when $x \sim y$ and $\delta\varepsilon = -1$ otherwise.

Given a strongly regular graph Δ with $k_\Delta = 2\mu_\Delta$, its *Taylor extension* $\mathsf{T}\Delta$ is the Taylor double of $\{\infty\} + \Delta$. It is a Taylor graph.

1.3 Association schemes and coherent configurations

We briefly state the main facts for symmetric association schemes. For more details, see [123], Chapter 2, and [132], Chapter 11. Results proved there are given here without proof.

1.3.1 Association schemes

A (symmetric) *association scheme with d classes* is a finite set X together with $d+1$ relations R_i on X such that

(i) $\{R_0, R_1, \ldots, R_d\}$ is a partition of $X \times X$;

(ii) $R_0 = \{(x,x) \mid x \in X\}$;

(iii) if $(x,y) \in R_i$, then also $(y,x) \in R_i$, for all $x, y \in X$ and $i \in \{0, \ldots, d\}$;

(iv) for any $(x,y) \in R_k$ the number p^k_{ij} of $z \in X$ with $(x,z) \in R_i$ and $(z,y) \in R_j$ depends only on i, j and k.

The numbers p^k_{ij} are called the *intersection numbers* of the association scheme.

Define $n = |X|$, and $n_i = p^0_{ii}$. Clearly, for each $i \in \{1, \ldots, d\}$, (X, R_i) is a simple graph which is regular of degree n_i.

Proposition 1.3.1 *The intersection numbers of an association scheme satisfy*

(i) $p^k_{0j} = \delta_{jk},\ p^0_{ij} = \delta_{ij} n_j,\ p^k_{ij} = p^k_{ji}$,

(ii) $\sum_i p^k_{ij} = n_j,\ \sum_j n_j = n$,

(iii) $p^k_{ij} n_k = p^j_{ik} n_j$,

(iv) $\sum_l p^l_{ij} p^m_{kl} = \sum_l p^l_{kj} p^m_{il}$. □

It is convenient to write the intersection numbers as entries of the so-called *intersection matrices* $L_0, \ldots, L_d$ defined by

$$(L_i)_{kj} = p^k_{ij}.$$

Note that $L_0 = I$ and $L_i L_j = \sum p^k_{ij} L_k$.

From the definition it is clear that an association scheme with two classes is the same as a pair of complementary strongly regular graphs. If (X, R_1) is strongly regular with parameters (v, k, λ, μ), then the intersection matrices of the scheme are

$$L_1 = \begin{bmatrix} 0 & k & 0 \\ 1 & \lambda & k-\lambda-1 \\ 0 & \mu & k-\mu \end{bmatrix},\quad L_2 = \begin{bmatrix} 0 & 0 & v-k-1 \\ 0 & k-\lambda-1 & v-2k+\lambda \\ 1 & k-\mu & v-2k+\mu-2 \end{bmatrix}.$$

History

Association schemes as defined above (also known as 'symmetric association schemes') were introduced in Bose & Shimamoto [97] as one of the ingredients for a PBIBD (partially balanced incomplete block design). Almost the same definition of PBIBD occurs already in Bose & Nair [96].

1.3.2 The Bose-Mesner algebra

The relations R_i of an association scheme are described by their adjacency matrices A_i of order n defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{whenever } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, A_i is the adjacency matrix of the graph (X, R_i). In terms of the adjacency matrices, the axioms (i)–(iv) become

(i) $\sum_{i=0}^{d} A_i = J$,

(ii) $A_0 = I$,

(iii) $A_i = A_i^\top$, for all $i \in \{0, \dots, d\}$,

(iv) $A_i A_j = \sum_k p_{ij}^k A_k$, for all $i, j \in \{0, \dots, d\}$.

From (i) we see that the $(0,1)$ matrices A_i are linearly independent, and by use of (ii)–(iv) we see that they generate a commutative $(d+1)$-dimensional algebra $\mathscr{A}$ of symmetric matrices with constant diagonal. This algebra was first studied by Bose & Mesner [95] and is called the *Bose-Mesner algebra* of the association scheme.

Since the matrices A_i commute, they can be diagonalized simultaneously. It follows that the algebra $\mathscr{A}$ is semisimple and has a unique basis of minimal idempotents $E_0, \dots, E_d$, where $E_i E_j = \delta_{ij} E_i$ and $\sum_{i=0}^{d} E_i = I$.

The matrix $\frac{1}{n} J$ is a minimal idempotent. We shall fix the numbering so that $E_0 = \frac{1}{n} J$. Let P and $\frac{1}{n} Q$ be the matrices relating our two bases for $\mathscr{A}$:

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \ \ E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

Then clearly

$$PQ = QP = nI.$$

It also follows that

$$A_j E_i = P_{ij} E_i,$$

which shows that the P_{ij} are the eigenvalues of A_j and that the columns of E_i are the corresponding eigenvectors. Thus $m_i = \operatorname{rk} E_i$ is the multiplicity of the eigenvalue P_{ij} of A_j (provided that $P_{ij} \neq P_{kj}$ for $k \neq i$). We see that $m_0 = 1$, $\sum_i m_i = n$, and $m_i = \operatorname{trace} E_i = n \cdot (E_i)_{jj}$ (indeed, E_i has only eigenvalues 0 and 1, so rk E_i equals the sum of the eigenvalues).

Proposition 1.3.2 *The numbers P_{ij} and Q_{ij} satisfy*

(i) $P_{i0} = Q_{i0} = 1$, $P_{0i} = n_i$, $Q_{0i} = m_i$,

(ii) $P_{ij} P_{ik} = \sum_{l=0}^{d} p_{jk}^l P_{il}$,

(iii) $m_i P_{ij} = n_j Q_{ji}$, $\sum_i m_i P_{ij} P_{ik} = n n_j \delta_{jk}$, $\sum_i n_i Q_{ij} Q_{ik} = n m_j \delta_{jk}$,

(iv) $|P_{ij}| \le n_j$, $|Q_{ij}| \le m_j$. □

An association scheme is called *primitive* if no union of the relations is a nontrivial equivalence relation. Equivalently, if no graph (X, R_i) with $i \neq 0$ is disconnected. For a primitive association scheme, (iv) above can be sharpened to $|P_{ij}| < n_j$ and $|Q_{ij}| < m_j$ for $j \neq 0$.

If $d = 2$, and (X, R_1) is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ s^g$, the matrices P and Q are

$$P = \begin{bmatrix} 1 & k & v-k-1 \\ 1 & r & -r-1 \\ 1 & s & -s-1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & f & g \\ 1 & fr/k & gs/k \\ 1 & -f\frac{r+1}{v-k-1} & -g\frac{s+1}{v-k-1} \end{bmatrix}.$$

The matrices P and Q can be computed from the intersection numbers of the scheme.

Proposition 1.3.3 *For $j = 0, \ldots, d$, the intersection matrix L_j has eigenvalues P_{ij} $(0 \leq i \leq d)$.* □

The fact that the multiplicities $m_i = Q_{0i}$ must be nonnegative integers is a powerful restriction on the parameters of an association scheme.

1.3.3 Linear programming bound and code-clique theorem

Being symmetric and idempotent, the matrices E_j are positive semidefinite. (Indeed, for any vector $x \in \mathbb{R}^n$ and any E with $E^\top = E = E^2$ one has $x^\top Ex = x^\top E^2 x = x^\top E^\top Ex = \|Ex\|^2 \geq 0$.) This leads to inequalities.

First of all, we have the linear programming bound for subsets of an association scheme. Consider a nonempty subset C of X. Its *inner distribution* a is the row vector with entries $a_i = \frac{1}{|C|}\chi^\top A_i \chi$, where $\chi = \chi_C$ is the characteristic vector of C. The value a_i is the average number of points of C in relation R_i to a given point of C. Note that $a_0 = 1$ and $|C| = \sum_i a_i$.

Theorem 1.3.4 *The inner distribution a of a nonempty subset C of X satisfies $aQ \geq 0$. Moreover, $(aQ)_j = 0$ if and only if $E_j \chi_C = 0$.*

Proof. Let $\chi = \chi_C$. Then

$$|C|(aQ)_j = \chi^\top \sum_i Q_{ij} A_i \chi = n\chi^\top E_j \chi = n\|E_j \chi\|^2 \geq 0. \qquad \square$$

This theorem gives inequalities on subsets when information on their inner distribution is given. For example, let Γ be the graph (X, R_j) defined by relation R_j in an association scheme. Let it have valency k (namely, n_j) and smallest eigenvalue s (namely, some P_{ij}) with $k > 0$. A clique C of size c in Γ has inner distribution a with $a_0 = 1$, $a_j = c - 1$ and $a_h = 0$ for $h \neq 0, j$. The inequality $(aQ)_i \geq 0$ yields $1 + \frac{s}{k}(c-1) \geq 0$, that is, $c \leq 1 + \frac{k}{-s}$. For strongly regular graphs this is the Hoffman bound on cliques.

One can also give results for pairs of subsets. First a lemma.

Lemma 1.3.5 (Roos [630]) *For any vectors $x, y \in \mathbb{R}^v$, we have*

$$\sum_{i=0}^{d} \frac{x^\top A_i y}{n n_i} A_i = \sum_{j=0}^{d} \frac{x^\top E_j y}{m_j} E_j.$$

Proof.

$$\sum_i \frac{x^\top A_i y}{n n_i} A_i = \sum_{i,j} \frac{x^\top P_{ji} E_j y}{n n_i} A_i = \sum_{i,j} \frac{x^\top Q_{ij} E_j y}{n m_j} A_i = \sum_j \frac{x^\top E_j y}{m_j} E_j. \quad \square$$

Let $T \subseteq \{1, \ldots, d\}$. A nonempty subset C of X with characteristic vector χ and inner distribution a is called a *T-code* when $a_i = 0$ for all $i \in T$. It is called a *T-anticode* when $a_i = 0$ for all $i \in \{1, \ldots, d\} \setminus T$. It is called a *$T$-design* when $E_j \chi = 0$ for all $j \in T$. It is called a *T-antidesign* when $E_j \chi = 0$ for all $j \in \{1, \ldots, d\} \setminus T$.

Theorem 1.3.6 *Let C be a T-design and D a T-antidesign in X, where $T \subseteq \{1, \ldots, d\}$. Then $|C \cap D| = |C| \cdot |D|/n$.*

Proof. Let C and D have characteristic vectors χ and η. Then $n\chi^\top A_i \eta = n \sum_j P_{ji} \chi^\top E_j \eta = n_i |C| \cdot |D|$. The theorem is the special case $i = 0$. $\square$

Theorem 1.3.7 *Let C be a T-code and D a T-anticode in X, where $T \subseteq \{1, \ldots, d\}$. Then $|C| \cdot |D| \leq n$. When equality holds, $|C \cap D| = 1$.*

Proof. Let χ and η be the characteristic vectors of C and D, respectively. Apply Roos' lemma with $x = y = \chi$, and pre- and post-multiply by $\eta^\top$ and η to find $\sum_i \frac{1}{n_i} (\chi^\top A_i \chi)(\eta^\top A_i \eta) = n \sum_j \frac{1}{m_j} (\chi^\top E_j \chi)(\eta^\top E_j \eta)$. The only nonzero term on the left-hand side is that for $i = 0$, which is $|C| \cdot |D|$. The right hand side is bounded below by the term for $j = 0$, which is $\frac{1}{n} |C|^2 |D|^2$. When equality holds, C and D are an S-design and S-antidesign for some $S \subseteq \{1, \ldots, d\}$, and the previous theorem yields the desired conclusion. $\square$

For example, if in a strongly regular graph a clique and a coclique both meet the Hoffman bound, then they meet in a single point (see also Proposition 1.1.7(iii)).

History

The linear programming bound is due to Delsarte [276].

1.3.4 Krein parameters

The Bose-Mesner algebra $\mathscr{A}$ is not only closed under ordinary matrix multiplication, but also under componentwise (Hadamard, Schur) multiplication (denoted $\circ$). Clearly $\{A_0, \ldots, A_d\}$ is the basis of minimal idempotents with respect to this multiplication. Write

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^{d} q_{ij}^k E_k.$$

The numbers q_{ij}^k thus defined are called the *Krein parameters*.

Proposition 1.3.8 *The Krein parameters of an association scheme satisfy*

(i) $q_{0j}^k = \delta_{jk},\ q_{ij}^0 = \delta_{ij} m_j,\ q_{ij}^k = q_{ji}^k,$

(ii) $\sum_i q_{ij}^k = m_j,\ \ \sum_j m_j = n,$

(iii) $q_{ij}^k m_k = q_{ik}^j m_j,$

(iv) $\sum_l q_{ij}^l q_{kl}^m = \sum_l q_{kj}^l q_{il}^m,$

(v) $Q_{ij} Q_{ik} = \sum_{l=0}^d q_{jk}^l Q_{il},$

(vi) $n m_k q_{ij}^k = \sum_l n_l Q_{li} Q_{lj} Q_{lk}.$ □

The main use of the Krein parameters is the fact that they are nonnegative, and that the scheme satisfies additional regularity properties when some Krein parameter is zero.

Theorem 1.3.9 (Scott [638, 639]) *The Krein parameters of an association scheme satisfy* $q_{ij}^k \geq 0$ *for all* $i, j, k \in \{0, \ldots, d\}$. □

Theorem 1.3.10 ([123], Theorem 2.3.2) *For given* $i, j, k \in \{0, \ldots, d\}$ *one has* $q_{ij}^k = 0$ *if and only if*

$$\sum_{x \in X} E_i(u,x) E_j(v,x) E_k(w,x) = 0$$

for all $u, v, w \in X$. □

The Krein parameters can be computed by use of equation (vi) above. In the case of a strongly regular graph we obtain

$$q_{11}^1 = \frac{f^2}{v}\left(1 + \frac{r^3}{k^2} - \frac{(r+1)^3}{(v-k-1)^2}\right) \geq 0,$$

$$q_{22}^2 = \frac{g^2}{v}\left(1 + \frac{s^3}{k^2} - \frac{(s+1)^3}{(v-k-1)^2}\right) \geq 0$$

or, equivalently (assuming $r \neq k$ and $s \neq -1$),

$$(r+1)(k+r+2rs) \leq (k+r)(s+1)^2,$$

$$(s+1)(k+s+2rs) \leq (k+s)(r+1)^2$$

(the other Krein conditions are trivially satisfied in this case).

History

L. L. Scott, jr. gave the Krein conditions in the case of finite groups with abelian centralizer algebra, and credited C. Dunkl, who in turn quoted Kreĭn [502], p. 139. See also [422].

Smith graphs and graphs with strongly regular subconstituents

A strongly regular graph is called a *Smith graph* when $q_{22}^2 = 0$, or, equivalently, when $k = \frac{s^2(2r+1)-r^2s}{(r+1)^2-s-1}$.

Theorem 1.3.11 (Cameron, Goethals & Seidel [178]) *Let Γ be a strongly regular graph with $q_{11}^1 = 0$ or $q_{22}^2 = 0$. Then for each vertex x both subconstituents of x are themselves strongly regular or complete or edgeless.*

Proof. Given three vertices u, v, w, let $p_{ijk}(u, v, w)$ be the number of vertices x at distances i, j, k from u, v, w, respectively. When one of i, j, k is 0, the numbers $p_{ijk}(u, v, w)$ do not depend on u, v, w but only on their mutual distances. E.g., $p_{ij0}(u, v, w) = 1$ if $d(u, w) = i$ and $d(v, w) = j$, and $p_{ij0}(u, v, w) = 0$ otherwise. We also have identities like $\sum_k p_{ijk}(u, v, w) = p_{ij}^h$ when $d(u, v) = h$. It follows that all $p_{ijk}(u, v, w)$ can be expressed in the single value $p_{111}(u, v, w)$ (given the mutual distances of u, v, w). Since $E_j = \frac{1}{n} \sum Q_{hj} A_h$, we have $E_j(u, x) = \frac{Q_{hj}}{n}$ if $d(u, x) = h$. Now by Theorem 1.3.10, if $q_{11}^1 = 0$, then $\sum_{i,j,k} p_{ijk}(u, v, w) Q_{i1} Q_{j1} Q_{k1} = 0$. One checks that this equation is independent of the previous identities for the $p_{ijk}(u, v, w)$‡, so that all values $p_{ijk}(u, v, w)$ are determined. □

For example, there is no graph with parameters $(2950, 891, 204, 297)$ since it would have $q_{22}^2 = 0$ but there is no feasible parameter set on 891 vertices with valency 204 ([88]). There are various generalizations of this theorem to distance-regular graphs of small diameter. See, e.g., [224].

Conversely, the authors of [178] investigated in what cases both subconstituents of a strongly regular graph are themselves strongly regular or complete or edgeless. The primitive strongly regular graphs in question are the Smith graphs and their complements, and possibly graphs with Latin square or negative Latin square parameters (cf. §8.4.2).

Examples of Smith graphs are the pentagon, mK_2, $K_{m,m}$, the complement of the Clebsch graph (§10.7), the complement of the Schläfli graph (§10.10), the Higman-Sims graph (§10.31), the McLaughlin graph (§10.61), and both of its subconstituents (§§10.34, 10.48). An infinite family of examples is that of the strongly regular graphs with the parameters of the collinearity graph of a generalized quadrangle of order (q, q^2). It follows that these graphs are collinearity graphs of such generalized quadrangles ([178], Theorem 7.9).

Graphs with negative Latin square parameters $\mathrm{NL}_r(r^2 + 3r)$ are Smith graphs (with $\lambda = 0$, cf. p. 203). All further known strongly regular graphs with parameters $\mathrm{LS}_m(n)$ or $\mathrm{NL}_m(n)$ and strongly regular subconstituents are the grids $m \times m$ or have parameters $\mathrm{LS}_t(2t)$ or $\mathrm{NL}_t(2t)$ (that is, $(v, k, \lambda) = (4t^2, t(2t \pm 1), t(t \pm 1))$. The authors of [178] conjecture that there are no further example parameters. Examples are the graphs $\overline{VO_{2m}^{\varepsilon}(2)}$.

1.3.5 Euclidean representation

Let $(X, \mathscr{R})$ be an association scheme with d classes. Fix a primitive idempotent E of the scheme. Let $m := \operatorname{rk} E$. Now the map $x \mapsto \bar{x}$ that maps $x \in X$ to

‡After eliminating the p_{ijk} where some index is 0, the equations are of the form $\sum a_{ijk} p_{ijk} = 0$ with $a_{111} + a_{122} + a_{212} + a_{221} = a_{112} + a_{121} + a_{211} + a_{222}$. But the final equation has $a_{ijk} = Q_{i1} Q_{j1} Q_{k1}$ and is of this form only when $Q_{11} = Q_{21}$, impossible.

column x of E maps X into a system of vectors in $\mathbb{R}^m$ (namely, the column space of E) with the property that the inner product $\langle \bar{x}, \bar{y} \rangle$ only depends on the relation x, y are in, and not on the choice of x, y. Indeed, if $E = \sum c_i A_i$, and $(x, y) \in R_i$, then $\langle \bar{x}, \bar{y} \rangle = (E^\top E)_{xy} = E_{xy} = c_i$, since $E^\top = E$ and $E^2 = E$. This allows one to use Euclidean geometry to study $(X, \mathscr{R})$.

In particular we find, after scaling, that if θ is an eigenvalue $\neq k$ of a primitive strongly regular graph Γ of multiplicity m, then the vertices x of Γ have a representation in $\mathbb{R}^m$ by unit vectors $\bar{x}$ such that $\langle \bar{x}, \bar{y} \rangle = \frac{\theta}{k}$ if $x \sim y$ and $\langle \bar{x}, \bar{y} \rangle = \frac{-\theta-1}{v-k-1}$ if $x \not\sim y$.

There are many applications.

1.3.6 Subschemes

An association scheme $(X, \mathscr{S})$ is called a *subscheme* (or *fusion scheme*) of the association scheme $(X, \mathscr{R})$ (with $\mathscr{R} = \{R_0, \ldots, R_d\}$) when each $S \in \mathscr{S}$ is the union of a subset of $\mathscr{R}$, that is, when the partition $\mathscr{R}$ of $X \times X$ is a refinement of $\mathscr{S}$. Equivalently, when the Bose-Mesner algebra of $(X, \mathscr{S})$ is a subalgebra of the Bose-Mesner algebra of $(X, \mathscr{R})$.

Given the P matrix of $(X, \mathscr{R})$ one can find all subschemes with e classes by considering all possible partitions Π of $\{0, \ldots, d\}$ into $e+1$ parts, one of which is $\{0\}$. Let Z be the $(d+1) \times (e+1)$ $(0,1)$-matrix with columns indexed by Π with entries $Z_{i\pi} = 1$ if $i \in \pi$. (Then Z has row sums 1.) The partition Π defines a subscheme if and only if the $(d+1) \times (e+1)$ matrix PZ has precisely $e+1$ distinct rows.

For example, the P matrix of $J(13, 6)$ is

$$\begin{pmatrix} 1 & 42 & 315 & 700 & 525 & 126 & 7 \\ 1 & 29 & 120 & 50 & -125 & -69 & -6 \\ 1 & 18 & 21 & -60 & -15 & 30 & 5 \\ 1 & 9 & -15 & -15 & 30 & -6 & -4 \\ 1 & 2 & -15 & 20 & -5 & -6 & 3 \\ 1 & -3 & 0 & 10 & -15 & 9 & -2 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{pmatrix}$$

and with $\Pi = \{\{0\}, \{1, 2, 4\}, \{3, 5, 6\}\}$ one finds the P matrix of a subscheme by taking the rows of PZ, deleting duplicates:

$$\begin{pmatrix} 1 & 882 & 833 \\ 1 & 24 & -25 \\ 1 & -18 & 17 \end{pmatrix}.$$

With $\Pi = \{\{0\}, \{1, 6\}, \{2, 5\}, \{3, 4\}\}$ one finds

$$\begin{pmatrix} 1 & 49 & 441 & 1225 \\ 1 & 23 & 51 & -75 \\ 1 & 5 & -21 & 15 \\ 1 & -5 & 9 & -5 \end{pmatrix}.$$

Subschemes of the Johnson scheme

Trivially, $J(2m, m)$ has the subscheme with $\Pi = \{\{0\}, \{1, \ldots, m-1\}, \{m\}\}$, where R_m has valency 1. The scheme $J(2m+1, m)$ has the subscheme with

$\Pi = \{\{0\},\{1,m\},\{2,m-1\},\dots,\{\lfloor\frac{m+1}{2}\rfloor,\lceil\frac{m+1}{2}\rceil\}\}$ isomorphic to the folded scheme $\overline{J}(2m+2,m+1)$. Sporadic examples are due to Mathon and Klin: the distance 1-or-3 graphs of $J(10,3)$ and $J(12,4)$, the distance 1-or-4 graph of $J(11,4)$, and the distance 1-or-2-or-4 graph of $J(13,6)$ are strongly regular with parameters $(v,k,\lambda,\mu) = (120,56,28,24)$, $(495,256,136,128)$, $(330,63,24,9)$, and $(1716,882,456,450)$, respectively. For $m = 3,4$, the distance 1-or-m graph of $J(2m+1,m)$ is strongly regular with parameters $(v,k,\lambda,\mu) = (35,16,6,8)$ and $(126,25,8,4)$, respectively.

Muzychuk [579] and Uchida [707] showed that $J(n,m)$ does not have a nontrivial subscheme for $n \geq f(m)$, where $f(3) = 11$, $f(4) = 13$, $f(5) = 15$, $f(6) = 18$ and $f(m) = 3m-1$ for $m \geq 7$.

Subschemes of distance-regular graphs of diameter 3

Proposition 1.3.12 *Let Γ be a distance-regular graph of diameter 3. Then*

(i) the distance-2 graph Γ_2 of Γ is strongly regular if and only if $c_3(a_3+a_2-a_1) = b_1a_2$, and

(ii) the distance-3 graph Γ_3 of Γ is strongly regular if and only if Γ has eigenvalue -1, that is, if and only if $k = b_2+c_3-1$.

Proof. See [123], 4.2.17. □

For example, the distance-3 graph of the collinearity graph of a generalized hexagon of order s is strongly regular.

1.3.7 Absolute bound and μ-bound

The absolute bound

The absolute bound expresses the fact that in a Euclidean representation the dimension cannot be too small.

Proposition 1.3.13 *The multiplicities m_i $(0 \leq i \leq d)$ of a d-class association scheme satisfy*

$$\sum_{q_{ij}^k \neq 0} m_k \leq \begin{cases} m_i m_j & \text{if } i \neq j, \\ \frac{1}{2}m_i(m_i+1) & \text{if } i = j. \end{cases}$$

Proof. See [123], 2.3.3. □

In particular, one finds for strongly regular graphs:

Proposition 1.3.14 (Absolute bound) *The multiplicities f, g of a primitive strongly regular graph satisfy $v \leq \frac{1}{2}f(f+3)$ and $v \leq \frac{1}{2}g(g+3)$.*

Proof. See [132], 10.6.8. □

Proposition 1.3.13 implies 'if $q_{11}^1 \neq 0$ then $v \leq \frac{1}{2}f(f+1)$'. It follows that if $v = \frac{1}{2}f(f+3)$ then $q_{11}^1 = 0$ and if $v = \frac{1}{2}g(g+3)$ then $q_{22}^2 = 0$.

This rules out, e.g., $(v,k,\lambda,\mu) = (841,200,87,35)$ with $f = 40$ and $q_{11}^1 \neq 0$.

The μ-bound

For primitive strongly regular graphs with smallest eigenvalue $s = -m$, the value of μ is bounded as a function of m.

This was first shown by Hoffman who developed a structure theory for families of graphs with lower bounded smallest eigenvalue (cf. [433, 435]). An explicit (and sharp) bound was given by Neumaier.

Theorem 1.3.15 (Neumaier [587]) *Let Γ be a primitive strongly regular graph with integral smallest eigenvalue $s = -m$. Then $\mu \le m^3(2m-3)$. If equality holds, then $n = m(m-1)(2m-1)$, where $n = r - s$.* □

This bound is proved as a consequence of the Krein condition and the absolute bound. It does not yield new feasibility conditions. Equality holds for the Schläfli graph ($m = 2$) and the McLaughlin graph ($m = 3$).

Equality in Krein condition or absolute bound

With the notation from Theorem 1.3.15, the three independent parameters of a strongly regular graph can be taken to be m, n, μ.

Proposition 1.3.16 *For a primitive strongly regular graph:*
(i) We have $q^1_{11} = 0$ if and only if

$$\mu = \frac{(n+m^2-m)(n-m)(m-1)}{n-m^2+m}.$$

(ii) We have $v = \frac{1}{2}f(f+3)$ if and only if μ has the value given in (i), and $n = m(m-1)(2m-1)$. Now $\mu = m^3(2m-3)$.
(iii) We have $q^2_{22} = 0$ if and only if

$$\mu = \frac{(r+1)(r^2+s)s}{r^2+2r-s}.$$

(iv) We have $v = \frac{1}{2}g(g+3)$ if and only if μ has the value given in (iii), and $-s = r^2(2r+3)$. Now $\mu = r^3(2r+3)$. □

If the graph does not have integral eigenvalues, then these conditions hold if and only if the graph is the pentagon. The graphs from (iii) are the Smith graphs.

1.3.8 Coherent configurations

Above we gave the definition of a symmetric association scheme. It is the combinatorial analog of the permutation group-theoretical situation of a transitive permutation group with only self-paired orbits. More general schemes are the analogs of more general group actions.

Coherent configurations

Consider a permutation group G acting on a finite set X. For $g \in G$, let P_g be the permutation matrix indexed by X with entries $(P_g)_{xy} = 1$ if $x = gy$, so that $P_g P_h = P_{gh}$. The *centralizer algebra* $\mathscr{A}$ of G is the algebra consisting of the matrices M such that $P_g M = M P_g$ for all $g \in G$. A basis for $\mathscr{A}$ is the set of 0-1 matrices A_O, where O is a G-orbit on $X \times X$, and A_O is the 0-1 matrix with $(A_O)_{xy} = 1$ if $(x, y) \in O$.

The corresponding combinatorial analog is called *coherent configuration*, see Higman [423, 424]. Thus, a coherent configuration is described by a collection of 0-1 matrices $\{A_0, \ldots, A_d\}$ that is a basis for an algebra $\mathscr{A}$ such that (i) $\sum_i A_i = J$, (ii) $I \in \mathscr{A}$, (iii) for each i there is a j such that $A_i^\top = A_j$. As before, the A_i are viewed as adjacency matrices for relations. The fact that $\mathscr{A}$ is an algebra means that $A_i A_j = \sum p_{ij}^k A_k$ for certain constants p_{ij}^k.

A coherent configuration is called *Schurian* if it is derived from a permutation group as above.

Homogeneous coherent configurations

If G is transitive, then the diagonal of $X \times X$ is a single relation, so that we can take $A_0 = I$. Now $\dim \mathscr{A} = d + 1$ is the *rank* of the permutation action.

A coherent configuration with $A_0 = I$ is called *homogeneous*, or also a (general) association scheme, following Delsarte [276].

Commutative association schemes

For transitive G, the action of G on X is isomorphic to the action of G on G/K (by left multiplication), where $K = G_a$ is a point stabilizer. The algebra $\mathscr{A}$ is commutative precisely when this action is multiplicity-free, that is, when all irreducible constituents of the permutation character $\pi = (1_K)^G$ are distinct (see [729], §29). Now (G, K) is a *Gelfand pair*.

An association scheme is called *commutative* when $\mathscr{A}$ is commutative. Now $p_{ij}^k = p_{ji}^k$ for all i, j, k.

Symmetric association schemes

A permutation group G is called *generously transitive* if for arbitrary elements $x, y \in X$ there is a $g \in G$ with $gx = y$ and $gy = x$. This happens if and only if G is transitive and all its suborbits are self-paired (Neumann [591]).

An association scheme is called *symmetric* when $A_i^\top = A_i$ for all i, so that $M^\top = M$ for all $M \in \mathscr{A}$. This was the original definition of association scheme (Bose & Shimamoto [97]). A symmetric association scheme is commutative since $MN = (MN)^\top = N^\top M^\top = NM$.

Linear programming bound

Hobart [431] proved an analog of Delsarte's linear programming bound for general coherent configurations.

The Weisfeiler-Leman algorithm

The k-dimensional Weisfeiler-Leman algorithm is a procedure that given a graph Γ with vertex set X computes a canonical partition (coloring) Π of X^k.

Compute for $h \geq 0$ partitions Π_h of X^k by successive refinement. Start with Π_0, the partition according to the ordered isomorphism type of the k-tuples, so that u, v are in the same part if and only if the map $u_i \mapsto v_i$ $(1 \leq i \leq k)$ preserves identity, adjacency and nonadjacency.

For $u \in X^k$ and $x \in X$ and $1 \leq i \leq k$, let $f_i^x(u)$ be the k-tuple v with $v_i = x$ and $v_j = u_j$ for $j \neq i$. For $h \geq 0$ and $u \in X^k$, let $c_h(u)$ be the color of u after step h, that is, the part of Π_h containing u. Given Π_h, compute the refinement Π_{h+1} by splitting each part according to the value of the map that assigns to the k-tuple u the multiset $\{(c_h(f_1^x(u)), \ldots, c_h(f_k^x(u))) \mid x \in X\}$ of k-tuples of colors of neighboring k-tuples. Repeat this step until no further splitting occurs. Put $\Pi = \Pi_h$ when $\Pi_h = \Pi_{h+1}$.

This algorithm is efficient (takes time $O(v^{k+1} \log v)$), and any automorphism of Γ must preserve Π. The special case $k = 2$ of this algorithm computes the coarsest coherent configuration that is a refinement of Γ, in the sense that its algebra $\mathscr{A}$ contains the adjacency matrix A of Γ.

Chapter 2

Polar spaces

In this chapter we define and study (finite) polar spaces and their collinearity graphs. We first give a geometric description of polar spaces embedded in a finite-dimensional vector space, and classify them, next describe the same spaces in terms of bilinear, sesquilinear, and quadratic forms, and finally look in detail at the geometries and graphs of each of the three families, the symplectic, unitary, and orthogonal spaces. We give parameter information, and state what is currently known about substructures like caps, ovoids, spreads, and tight sets.

2.1 Polar spaces

Generalities

A *partial linear space* is a set of *points* together with a collection of subsets (called *lines*) such that two points are on at most one line and each line has at least two points.

The *collinearity graph* (or *point graph*) of a partial linear space is the graph with the points as vertices, where two (distinct) points are adjacent when they are collinear.

The *incidence graph* of a partial linear space is the bipartite graph with the points and lines as vertices, where a point is adjacent to a line when it is on the line. The geometry is *connected* when its incidence graph is connected.

A *flag* is an incident point-line pair. An *antiflag* is a nonincident point-line pair.

A *subspace* is a subset of the point set that contains each line that meets it in at least two points. A *singular subspace* is a subspace such that any two of its points are collinear. A *(geometric) hyperplane* is a proper subspace that meets each line.

Polar spaces

A *polar space* is a partial linear space $(X, \mathscr{L})$ such that for each line L and point $x \notin L$, the point x is collinear with either 1 or all points of L. This is known as the *Buekenhout-Shult axiom.* The polar spaces where the second alternative of the Buekenhout-Shult axiom does occur, every line contains at least 3 points, every nested family of singular subspaces is finite, and no point is collinear to all other points, were classified by Buekenhout & Shult [158]. They showed that such a polar space is equivalent to a polar space in the sense of Veldkamp [715] and Tits [694] and then one can use the classification in [694]. Below we study embedded polar spaces, i.e., polar spaces embedded in a finite-dimensional vector space. This covers all nondegenerate finite polar spaces containing proper projective planes. In the infinite case further examples arise.

Comments

The results of Veldkamp [715] also include a classification, but restricted to the case where all planes of the polar space are Desarguesian. This is enough for the finite case. In the general case, there is one class of polar space (of rank 3) which does not have Desarguesian planes; these polar spaces are usually referred to as *non-embeddable polar spaces*. They are related to octonion division rings and algebraic groups of type E_7. The corresponding planes still satisfy the so-called Moufang condition (every line is a translation line).

In the Desarguesian case, besides the line Grassmannian of any projective 3-space, the polar spaces (of rank at least 3) are classified by (nondegenerate) pseudo-quadratic forms. However, in the case the characteristic of the skew field underlying the projective planes is different from 2, the nondegenerate pseudo-quadratic forms are equivalent to ordinary nondegenerate reflexive sesquilinear forms (and these are equivalent to nondegenerate Hermitian forms and nondegenerate symmetric and alternating bilinear forms). Hence in this case the polar space is fully embedded in a projective space in such a way that its point set is the set of absolute points of a polarity of that projective space. In the characteristic 2 case this is not true and the situation is more complicated. Roughly, besides the polar spaces arising from nondegenerate reflexive forms (for a suitable definition of nondegeneracy), there are also polar spaces contained in such a polar space that cannot be described by reflexive forms, but only by pseudo-quadratic forms. As a result, all polar spaces in characteristic 2 can be fully embedded in a projective space, but in some cases only as a proper subset of the set of absolute points of a polarity.

In the finite case, the anomalies in characteristic 2 do not appear, and every Moufang plane is Desarguesian, even Pappian (corresponding to a field). Hence all finite polar spaces of rank at least 3 are embeddable in a finite projective space. Moreover, all examples of polar spaces of rank 2 that produce rank 3 graphs are also embeddable in a projective space. The embeddings also provide other rank 3 graphs by looking at points off the polar space, lines not belonging to polar space, etc. Reasons enough to introduce embedded polar spaces and explicitly classify the finite ones. We consider a slight variation of the definitions of Veldkamp and Tits to adapt them to the embeddable setting. Along the way we also show that the Buekenhout-Shult axiom is satisfied.

2.2 Embedded polar spaces

2.2.1 Projective spaces

Let V be a vector space. The *projective space* $\mathsf{P}V$ is the collection of subspaces of V. A *point* (*line*, *plane*) of $\mathsf{P}V$ is a 1-space (2-space, 3-space) in V. A *hyperplane* of $\mathsf{P}V$ is a hyperplane (subspace of codimension 1) of V.

If S is a subset of V, or of $\mathsf{P}V$, then $\langle S\rangle$ denotes the subspace of V spanned by S. We write $\langle v\rangle$ instead of $\langle\{v\}\rangle$. The empty set of vectors spans the 0-dimensional subspace 0.

Suppose V has finite dimension n over the finite field $\mathbb{F}_q$. Then V has q^n vectors, and $\mathsf{P}V$ has $(q^n-1)/(q-1)$ points. More generally, V has $\begin{bmatrix} n \\ m \end{bmatrix}_q$ subspaces of dimension m, where $\begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_{i=0}^{m-1} \frac{q^{n-i}-1}{q^{m-i}-1}$ is the Gaussian (or q-binomial) coefficient. The subscript q is usually omitted.

2.2.2 Definition of embedded polar spaces

Let V be a vector space and $\mathsf{P}V$ the corresponding projective space. A pair (X, Ω) is a *polar space (embedded) in* $\mathsf{P}V$ if X is a set of points spanning $\mathsf{P}V$ and Ω is a nonempty family of finite-dimensional subspaces of V satisfying conditions (EPS1) and (EPS2) below. We shall view the members of Ω as sets of projective points, and write $x \in \omega$ and $\omega \cap \omega' = \emptyset$ instead of $x \subseteq \omega$ and $\omega \cap \omega' = 0$ for $x \in X$ and $\omega, \omega' \in \Omega$.

Two points x, y of X are called *collinear (in* (X, Ω)*)*, notation $x \perp y$, if they are contained in a common member of Ω. Otherwise they are called *opposite.*

(EPS1) For every $\omega \in \Omega$, the set of points of ω is contained in X.

(EPS2) For every $x \in X$ and every $\omega \in \Omega$ with $x \notin \omega$, the set U of points of ω collinear with x is a codimension 1 subspace of ω and $\langle x, U \rangle \in \Omega$.

Let (X, Ω) be an embedded polar space. By (EPS1), $\bigcup \Omega \subseteq X$; by (EPS2), $X \subseteq \bigcup \Omega$, since $\Omega \neq \emptyset$. Hence X is the union of all elements of Ω.

The intersection $R := \bigcap \Omega$ of all members of Ω is called the *radical* of (X, Ω). The space (X, Ω) is called *nondegenerate* if

(EPS3) The intersection of all members of Ω is empty.

The *collinearity graph* $\Gamma(X, \Omega)$ of (X, Ω) is the graph with vertex set X with collinearity as adjacency. We shall show that the collinearity graph of a nondegenerate embedded polar space with finite set of points is strongly regular.

2.2.3 Rank and radical

Let (X, Ω) be an embedded polar space. Let $\Delta = \Delta(X, \Omega)$ be the graph with vertex set Ω, where $\omega \sim \omega'$ when $\omega \cap \omega'$ has codimension 1 in both ω and ω'. Vertices in the same connected component of Δ have the same dimension.

Lemma 2.2.1 *The graph Δ is connected. In particular, the dimensions of all members of Ω are the same.*

Proof. By (EPS1) and (EPS2), no member of Ω is strictly contained in another. (If $\omega \subset \omega'$ and $x \in \omega' \setminus \omega$ then $x \in X$ by (EPS1) and x is collinear with all of ω, contradicting (EPS2).)

Hence, if ω and ξ are distinct members of Ω, then we can find $x \in \xi \setminus \omega$ and by (EPS2) there is an $\omega' \in \Omega$ that is adjacent to ω and such that ω' contains $\langle x, \omega \cap \xi \rangle$. Since ξ is finite-dimensional, an induction on $\dim(\omega \cap \xi)$ implies that ξ and ω are in the same connected component of Δ. $\square$

The common (vector space) dimension n of all members of Ω is called the *(polar) rank* of (X, Ω).

We show some equivalent forms of axiom (EPS3).

Lemma 2.2.2 *Equivalent are*

(i) (EPS3),

(ii) *Every point of X is opposite some other point of X,*

(iii) *Every member of Ω is disjoint from some other member of Ω,*

(iv) *There exist two disjoint members of Ω.*

More generally, if R is the radical of (X, Ω), then
(a) Every point of $X \setminus R$ is opposite some other point of $X \setminus R$.
(b) Every member of Ω meets some member of Ω in precisely R.

Proof. (a). Let $x \in X \setminus R$. Then by definition of R there is an $\omega \in \Omega$ with $x \notin \omega$. By (EPS2) the point x is not collinear to all points of ω.

(a)⇒(b). Let $\xi \in \Omega$ be given, and $\omega \in \Omega$ be arbitrary. If $x, x' \in X$, where $x \in \omega \cap \xi$ and x' is opposite x (by (a), there is such a pair when $\omega \cap \xi \neq R$), then by (EPS2) there is an ω' containing x' adjacent to ω in Δ. Now $\omega' \cap \xi$ is strictly contained in $\omega \cap \xi$ since $x \notin \omega'$ and x is already collinear with the points of the hyperplane $\omega \cap \omega'$ of ω', and cannot be collinear with any further points of ω'. Since ξ is finite-dimensional and $\dim(\omega' \cap \xi) < \dim(\omega \cap \xi)$, we inductively end with a member of Ω meeting ξ precisely in R.

Now each of the four statements (i)–(iv) says that R is empty. □

2.2.4 Maximal singular subspaces

A *subspace* S of an embedded polar space (X, Ω) is a subset $S \subseteq X$ such that, if $x, y \in S$ are two distinct collinear points, then all points of the line $\langle x, y \rangle$ belong to S. A *singular* subspace S is a subspace containing no opposite pair of points.

Proposition 2.2.3 *In an embedded polar space (X, Ω), the maximal singular subspaces are precisely the elements of Ω.*

Proof. Certainly the elements of Ω are singular subspaces. Let S be a maximal singular subspace. Given $\omega \in \Omega$ not containing S, we find a neighbor ω' that meets S in a strictly larger subspace. But the intersection $\omega \cap S$ cannot have dimension larger than the rank n, so S is contained in, and therefore equals, some element of Ω. □

Corollary 2.2.4 (Buekenhout-Shult axiom) *Let L be a line containing two collinear points of an embedded polar space (X, Ω), and let $x \in X$. Then x is collinear to either exactly one or all points of L.*

Proof. Including L in a maximal singular subspace, which is a member of Ω by the foregoing proposition, yields the corollary. □

2.2.5 Order of an embedded polar space

For $x \in X$, let $x^\perp$ be the set of all points of X collinear with x.

Lemma 2.2.5 *Given an embedded polar space (X, Ω) with radical R, and a point $x \in X$,*
(i) for every $\omega \in \Omega$ with $x \in \omega$ there exists $\omega' \in \Omega$ with $\omega \cap \omega' = \langle x, R \rangle$;
(ii) for each point $y \in x^\perp \setminus \langle x, R \rangle$, there exists an opposite point $y' \in x^\perp$.

Proof. By Lemma 2.2.2 (b) we may assume $x \notin R$. Let $\omega \in \Omega$ with $x \in \omega$. By Lemma 2.2.2 (b) we find $\xi \in \Omega$ with $\omega \cap \xi = R$. By (EPS2), there exists $\omega' \in \Omega$ with $x \in \omega' \sim \xi$. A line in $\omega \cap \omega'$ intersects ξ nontrivially, hence intersects R. Consequently R is a hyperplane of $\omega \cap \omega'$ and so $\omega \cap \omega' = \langle x, R \rangle$. Take $y \in \omega \setminus \langle x, R \rangle$ arbitrarily (then y is arbitrary in $x^\perp \setminus \langle x, R \rangle$), then by (EPS2) there is a point $y' \in \omega'$ not collinear to y. □

Lemma 2.2.6 *Let (X, Ω) be an embedded polar space and $p \in X$. Let $H_p = \langle p^\perp \rangle$. Then $H_p \cap X = p^\perp$.*

Proof. Suppose not, and let $y \in H_p \cap X$ be opposite p such that the number, say m, of members $\omega_1, \ldots, \omega_m$ of Ω containing p needed to generate y is minimal. (Such an m exists since y is generated by a finite number of points of $p^\perp$.) Obviously $m > 1$. Let $\omega \in \Omega$ be such that $y \in \omega \sim \omega_m$ and put $S = \langle \omega_1, \ldots, \omega_{m-1} \rangle$. Since $\omega \subseteq \langle S, \omega_m \rangle$, so that $\dim\langle S, \omega \rangle \leq \dim\langle S, \omega_m \rangle$, we have $\dim(S \cap \omega) \geq \dim(S \cap \omega_m)$. And since $p \in \omega_m \setminus \omega$, there is at least one point z in $(S \cap \omega) \setminus \omega_m$. Now $z \in X$ since $\omega \subseteq X$, and $z \in p^\perp$ by minimality of m, so that z is collinear with all points of ω_m, contradicting (EPS2). $\square$

Proposition 2.2.7 *Let p be a point of the embedded polar space (X, Ω) of rank n and with radical R, where $p \notin R$. Put $X_p = p^\perp$ and $\Omega_p = \{\omega \in \Omega \mid p \in \omega\}$. Then (X_p, Ω_p) is an embedded polar space of rank n and radical $\langle p, R \rangle$ in H_p.*

Proof. Note that two points of X_p are collinear in (X_p, Ω_p) if and only if they are collinear in (X, Ω). $\square$

Lemma 2.2.8 *Let (X, Ω) be an embedded polar space of rank n with radical R, and let $p \in X \setminus R$. If $\dim R \leq n - 2$, then H_p is a hyperplane of $\mathsf{P}V$.*

Proof. Fix $z \in X \setminus p^\perp$ and let $z' \in X \setminus p^\perp$ be arbitrary. We show that $z' \in \langle H_p, z \rangle$, which completes the proof of the lemma. By Lemma 2.2.5 (*ii*) we can find noncollinear $y, y' \in p^\perp$ (since each ω on p has dimension n, while $\langle p, R \rangle$ has dimension at most $n - 1$). By Corollary 2.2.4 we may assume that $z \in y^\perp$ and $z' \in y'^\perp$ (by adapting the choices of y and y' in the lines $\langle p, y \rangle$ and $\langle p, y' \rangle$). By the same corollary, since y is not collinear to y', the point y is collinear to a unique point w' of the line $\langle z', y' \rangle$. Since lines contain at least three points, we can pick a point u' of $\langle z', y' \rangle \setminus \{y', w'\}$. The point u' is collinear to a unique point u of $\langle z, y \rangle \setminus \{y\}$. Again by Corollary 2.2.4, there is a unique point $v \in \langle u, u' \rangle \cap p^\perp$ and $v \notin \{u, u'\}$. We conclude $z' \in \langle H_p, u' \rangle = \langle H_p, u \rangle = \langle H_p, z \rangle$. $\square$

Two polar spaces (X, Ω) and (X', Ω') in the projective spaces $\mathsf{P}V$ and $\mathsf{P}V'$, respectively, are *isomorphic* if there is an isomorphism $\theta : \mathsf{P}V \to \mathsf{P}V'$ mapping X bijectively to X' and mapping Ω bijectively onto Ω'. Isomorphic embedded polar spaces have isomorphic collinearity graphs. An isomorphism from (X, Ω) to itself is called an *automorphism*, or a *collineation*.

Proposition 2.2.9 *Let x and y be two opposite points of an embedded polar space (X, Ω) of rank n and with radical R, where $\dim R \leq n - 2$. Set $\Omega_{x,y} = \{\omega \cap x^\perp \mid y \in \omega \in \Omega\}$. Then $\Omega_{x,y} = \Omega_{y,x}$, and $(x^\perp \cap y^\perp, \Omega_{x,y})$ is an embedded polar space of rank $n - 1$ in the subspace $\langle x^\perp \cap y^\perp \rangle$ of dimension $\dim V - 2$, with radical R.*

Moreover, if z is opposite x, then the embedded polar spaces $(x^\perp \cap y^\perp, \Omega_{x,y})$ and $(x^\perp \cap z^\perp, \Omega_{x,z})$ are isomorphic.

Proof. The fact that $\Omega_{x,y} = \Omega_{y,x}$ follows immediately from Axiom (EPS2). Since $\langle x^\perp \rangle = \langle x, x^\perp \cap y^\perp \rangle$, and hence $\langle x^\perp \cap y^\perp \rangle = \langle x^\perp \rangle \cap \langle y^\perp \rangle$, the fact that $\dim \langle x^\perp \cap y^\perp \rangle = \dim V - 2$ follows from Lemma 2.2.8. Now, (EPS1) is trivial for

$(x^\perp \cap y^\perp, \Omega_{x,y})$ and (EPS2) follows from noting that $x^\perp \cap y^\perp$ does not contain any element of Ω (hence the rank of $\Omega_{x,y}$ is $n-1$). The fact that R is the radical follows immediately from Proposition 2.2.7.

Finally, projection of $\langle x^\perp \cap y^\perp \rangle$ into $\langle x^\perp \cap z^\perp \rangle$ from x induces an isomorphism between $(x^\perp \cap y^\perp, \Omega_{x,y})$ and $(x^\perp \cap z^\perp, \Omega_{x,z})$. □

Let V be finite and defined over the field $\mathbb{F}_q$. We can show, in the nondegenerate case, that the graph $\Gamma = \Gamma(X, \Omega)$ is strongly regular, without explicitly calculating the precise parameters.

Theorem 2.2.10 *Let (X, Ω) be finite nondegenerate embedded polar space of rank at least 2. Then the associated collinearity graph Γ is strongly regular.*

Proof. Note that Γ is not complete and not edgeless. Proposition 2.2.9 implies that $|x^\perp| = 1 + q|x^\perp \cap y^\perp| = |y^\perp|$ for all pairs of opposite points $x, y \in X$. Now let $x, y \in X$ be collinear. Select $z \in \langle x, y\rangle \setminus \{x, y\}$. Then by Lemma 2.2.5(*ii*), there exists $u \in z^\perp$ opposite x and hence also opposite y by Corollary 2.2.4. Hence $k + 1 := |x^\perp| = |u^\perp| = |y^\perp|$ is a constant and Γ is k-regular. It also follows that the sets $x^\perp \cap y^\perp$, for x opposite y, have constant size k/q.

Given collinear points x, z, let $y \in z^\perp \setminus x^\perp$. Suppose w is a point in $x^\perp \cap z^\perp$ not on $\langle x, z\rangle$. Let v be the unique point collinear with y on $\langle x, w\rangle$. Then $v \in x^\perp \cap y^\perp \cap z^\perp$. Hence, if the valency of the graph $\Gamma(x^\perp \cap y^\perp, \Omega_{x,y})$ is k' (note that $k' = 0$ if $n = 2$), then the number of points of X collinear to both x and z is $q + 1 + qk'$. □

The last part of the proof also shows that there are a constant number of singular planes of (X, Ω) through a line of (X, Ω). Similarly, continuing that argument, we derive the following consequence, also valid in the infinite case.

Corollary 2.2.11 *If the rank of the not necessarily finite nondegenerate embedded polar space (X, Ω) is n, then there are a constant number of maximal singular subspaces of (X, Ω) which contain a given singular subspace of dimension $n - 1$.* □

Order

Let (X, Ω) be a finite nondegenerate embedded polar space of rank n. We set $t + 1$ equal to the number of maximal singular subspaces of (X, Ω) that contain a given singular subspace of dimension $n - 1$. We call (q, t) the *order* of (X, Ω). Note that $t \geq 0$ by Proposition 2.2.3. But in fact we have $t \geq 1$ by Lemma 2.2.5 (*ii*), as $t+1$ is the number of singular lines of a polar space of rank 2 through a point. Note that, by the definition of t, the order of $(x^\perp \cap y^\perp, \Omega_{x,y})$, for $x \in X$ opposite $y \in X$ is also (q, t). Polar spaces with $t = 1$ are called *non-thick*.

Clique and coclique extensions

If Γ is a graph with vertex set X, then its *m-coclique extension* is the graph with vertex set $M \times X$, where $|M| = m$, and adjacencies $(i, x) \sim (j, y)$ whenever $x \sim y$. Its *m-clique extension* is the graph with vertex set $M \times X$, and adjacencies $(i, x) \sim (j, y)$ whenever $x \sim y$ or $i \neq j, x = y$. If Γ has adjacency matrix A

then these graphs have adjacency matrices $J_m \otimes A$ and $(J_m \otimes (A + I_v)) - I_m \otimes I_v$, respectively, where $v = |X|$.

Now Proposition 2.2.9 implies: Let (X, Ω) be an embedded polar space of rank $n \geq 2$ and order (q, t). Then its collinearity graph Γ is locally the q-clique extension of a polar space of rank $n - 1$ and order (q, t).

2.2.6 Parameters and spectrum of the polar space strongly regular graphs

We are now ready to determine the parameters of Γ in terms of (q, t).

Theorem 2.2.12 *Let (X, Ω) be a finite embedded polar space of rank $n \geq 2$ and order (q, t). Then the strongly regular graph $\Gamma(X, \Omega)$ has parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ s^g$, where*

$$
\begin{aligned}
v &= \frac{q^n - 1}{q - 1}(tq^{n-1} + 1), & r &= q^{n-1} - 1, \\
k &= q\frac{q^{n-1} - 1}{q - 1}(tq^{n-2} + 1), & s &= -tq^{n-2} - 1, \\
\lambda &= q^2\frac{q^{n-2} - 1}{q - 1}(tq^{n-3} + 1) + q - 1, & f &= \frac{tq(q^n - 1)(tq^{n-2} + 1)}{(q - 1)(q + t)}, \\
\mu &= \frac{q^{n-1} - 1}{q - 1}(tq^{n-2} + 1) = \frac{k}{q}, & g &= \frac{q^2(q^{n-1} - 1)(tq^{n-1} + 1)}{(q - 1)(q + t)}.
\end{aligned}
$$

Note that the use of 'order' implies that (X, Ω) is nondegenerate.

Proof. We count the number $v = |X|$ of points as follows. Fix $\omega \in \Omega$. Every point x outside ω is contained in a unique $\xi \in \Omega$ with $x \in \xi$ and $\omega \cap \xi$ a hyperplane in ω. There are $(q^n - 1)/(q - 1)$ hyperplanes in ω each contained in t members of Ω distinct from ω itself. Each such member contains q^{n-1} points outside ω. This gives v.

Now μ is the number of points of a polar space of rank $n - 1$ and order (q, t) and $k = q\mu$. Finally, λ is q times the number of points in $x^\perp \cap y^\perp$ collinear to a given point $z \in x^\perp \cap y^\perp$ (including z), minus 1 (namely, excluding z itself). $\square$

Proposition 2.2.13 *Let (X, Ω) be a finite embedded polar space of rank $n \geq 2$ and order (q, t). Then $|\Omega| = \prod_{i=0}^{n-1}(tq^i + 1)$.* $\square$

2.2.7 Ovoids, spreads, m-systems, h-ovoids, hemisystems

Let (X, Ω) be a finite embedded polar space of rank $n \geq 2$ and order (q, t). Let $\Gamma = \Gamma(X, \Omega)$ be its collinearity graph.

Proposition 2.2.14 *In the strongly regular graph Γ the maximal cliques are precisely the elements of Ω (and have size $(q^n - 1)/(q - 1)$). Every coclique C satisfies $|C| \leq tq^{n-1} + 1$. Equivalent are: (i) $|C| = tq^{n-1} + 1$, (ii) $|C \cap \omega| = 1$ for each $\omega \in \Omega$, (iii) $|x^\perp \cap C| = tq^{n-2} + 1$ for each $x \notin C$.*

Proof. Since the span of a clique is a clique again, the maximal cliques of Γ are the maximal singular subspaces of (X, Ω), i.e., the elements of Ω. These have dimension n and size $(q^n - 1)/(q - 1)$, and hence attain the Hoffman bound (Proposition 1.1.7). For cocliques C, the upper bound on $|C|$, and the conclusions for equality follow from that same proposition. That (iii) implies (i) follows by counting edges between C and $X \setminus C$. Let $m_n(q, t)$ be the size of Ω as given in Proposition 2.2.13. That (ii) implies (i) follows from $|C| = m_n(q, t)/m_{n-1}(q, t)$. □

A set of points C that meets each $\omega \in \Omega$ in a single point is called an *ovoid* of the polar space. For a discussion of when ovoids exist, see the various subsections of §§2.5–2.7. A general result is that existence (nonexistence) of ovoids implies the existence (nonexistence) of ovoids in embedded polar spaces of smaller (larger) rank.

Proposition 2.2.15 *If $\Gamma(X, \Omega)$ has a coclique of size c, then there is a pair of opposite points $x, y \in X$ such that the embedded polar space $(x^\perp \cap y^\perp, \Omega_{x,y})$ of rank $n-1$ (with the notation of Proposition 2.2.9) has a coclique of size at least $\lceil 1 + \frac{c-1}{q} \rceil$.*

Proof. Let C be a coclique of $\Gamma(X, \Omega)$ and let L be a line of (X, Ω) containing a point $p \in C$. For each x in $L \setminus \{p\}$ we select $y \in X$ opposite x and construct a coclique C_x as follows. For every point $u \in C \cap x^\perp$, let u' be the point on the line xu collinear in (X, Ω) to y. Then the set of all such points u' is a coclique C_x in $\Gamma(x^\perp \cap y^\perp, \Omega_{x,y})$. Since every point of $C \setminus \{p\}$ is collinear to a unique point x of $L \setminus \{p\}$, the q cocliques thus obtained contain in total $q + |C| - 1$ points. Hence at least one among them contains at least $\lceil 1 + \frac{|C|-1}{q} \rceil$ points. □

In particular we have:

Corollary 2.2.16 *If the embedded polar spaces $(X', \Omega') = (x^\perp \cap y^\perp, \Omega_{x,y})$, for noncollinear $x, y \in X$, do not contain an ovoid, then neither does (X, Ω).* □

Spreads and m-systems

A *spread* of an embedded polar space (X, Ω) is a collection of members of Ω that partitions X. If (X, Ω) has rank $n \geq 2$ and order (q, t), then clearly $|S| \leq |X|/\frac{q^n-1}{q-1} = tq^{n-1} + 1$, the same bound we found for ovoids.

Shult & Thas [653] define more generally a *partial m-system* to be a collection $\{U_1, \ldots, U_r\}$ of singular subspaces U_i of projective dimension m such that $U_i^\perp \cap U_j = 0$ whenever $i \neq j$. They prove $r \leq tq^{n-1} + 1$, independent of m, and call the collection an *m-system* when equality holds. For $m = 0$ the m-systems are the ovoids, for $m = n - 1$ the spreads.

For an m-system, let $M = \bigcup_i U_i$ and $\omega \in \Omega$. Then $|M \cap \omega| = \frac{q^{m+1}-1}{q-1}$.

In the cases $\mathsf{Sp}_{2n}(q)$, $\mathsf{O}^-_{2n+2}(q)$ and $\mathsf{U}_{2n+1}(q)$ (cf. Theorem 2.3.6), the size of the intersection $H \cap M$ takes two values for hyperplanes H, so that in these spaces an m-system gives rise to a strongly regular graph (§7.1.1).

See also [654], [411], [658].

h-Ovoids, h-spreads and hemisystems

An *h-ovoid* of an embedded polar space (X, Ω) is a subset of X that meets each $\omega \in \Omega$ in precisely h points. Thus, a 1-ovoid is an ovoid. If O is an h-ovoid in (X, Ω), and H is a hyperplane spanned by $\Omega_H = \{\omega \in \Omega \mid \omega \subseteq H\}$, then $O \cap H$ is an h-ovoid in the induced embedded polar space $(X \cap H, \Omega_H)$. Every h-ovoid is a regular set of size $h(tq^{n-1}+1)$, degree $(h-1)(tq^{n-2}+1)$ and nexus $h(tq^{n-2}+1)$ in $\Gamma(X, \Omega)$.

An *h-spread* of an embedded polar space (X, Ω) is a collection of members of Ω such that each $x \in X$ is contained in precisely h of them. Thus, a 1-spread is a spread.

A *hemisystem* of an embedded polar space (X, Ω) is a subset of Ω that for each $x \in X$ contains precisely half of the members of Ω containing x. Thus, a hemisystem is an h-spread, where $h = \frac{1}{2}\prod_{i=0}^{n-2}(tq^i + 1)$. Hemisystems were introduced by Segre [640], who showed that a nontrivial h-spread of $\mathsf{U}(4, q)$ must be a hemisystem, and constructed such hemisystems in case $q = 3$. A *hemisystem of points* in an embedded polar space (X, Ω) of rank 2 is a $(q+1)/2$-ovoid, i.e., a subset of X meeting each line in exactly half of its points.

2.2.8 Intriguing or regular sets; i-tight sets

In the polar space literature the notion of 'intriguing set' is used to refer to a regular set of the underlying strongly regular graph; we shall use 'regular set'. Let Y be a regular set of the embedded polar space (X, Ω) of order (q, t), and let Y have degree d and nexus e. (Then $|Y|(k - d) = (v - |Y|)e$ implies $|Y| = \frac{ev}{k-d+e}$.) According to §1.1.13, there are two cases.

Case $d - e = r$. Since in this case $|Y| = \frac{ev}{k-r} = e \cdot \frac{q^n-1}{q^{n-1}-1}$, and $\gcd(q^n - 1, q^{n-1} - 1) = q - 1$, we deduce that e is a multiple of $\frac{q^{n-1}-1}{q-1}$, say $e = i \cdot \frac{q^{n-1}-1}{q-1}$. In this case, Y is called a *tight set* of (X, Ω), in particular an *i-tight set.* So, the size, degree, and nexus of i-tight sets is $|Y| = i \cdot \frac{q^n-1}{q-1}$, $d = q^{n-1} - 1 + i \cdot \frac{q^{n-1}-1}{q-1}$, and $e = i \cdot \frac{q^{n-1}-1}{q-1}$. The terminology 'tight set' is from [608].

An example of a 1-tight set is a maximal singular subspace.

Case $d - e = s$. Applying Proposition 1.1.3 to Y with Y' any maximal singular subspace, we see that each maximal singular subspace intersects Y in a constant number of points. Hence Y is an h-ovoid for some natural number h.

There are some results in the literature that classify i-tight sets for small i in various polar spaces. We will review some of these in the various subsections of §§2.5–2.7. For now, we content ourselves with mentioning some standard examples of i-tight sets.

Disjoint unions of maximal singular subspaces

Since disjoint unions of tight sets are tight again, in particular the disjoint union of i maximal singular subspaces is an i-tight set. For $i = 1$, this is the only possible example, as is easily seen from the value of the degree. The papers [42] and [565] contain for each finite polar space an upper bound b so that if $i \leq b$, then an i-tight set is automatically the union of disjoint maximal singular subspaces.

Polar subspaces of the same rank

Let (X, Ω) be an embedded polar space of rank n and order (q, t) and let (X', Ω') be an embedded polar space of rank n order (q, t'), with $X' \subseteq X$ and $\Omega' \subseteq \Omega$. If $t' < t$, then X' is an i-tight set with $i = t'q^{n-1} + 1$, that is the size of a putative ovoid or spread in (X', Ω'), which is not surprising as a spread in (X', Ω') gives rise to the tight set X' of (X, Ω) of type 'the disjoint union of maximal singular subspaces'.

2.2.9 Distance-regular graphs on singular subspaces

We show that the graphs $\Delta(X, \Omega)$ (on the maximal singular subspaces, adjacent when they meet in codimension 1) are distance-regular (cf. §1.2).

Theorem 2.2.17 *Let (X, Ω) be a finite embedded polar space of rank $n \geq 2$ and order (q, t). Then $\Delta(X, \Omega)$ is distance-regular of diameter n. The parameters are $c_i = (q^i - 1)/(q - 1)$ and $b_i = tq^i(q^{n-i} - 1)/(q - 1)$ for $0 \leq i \leq n$. The distance of two vertices ω, ω' in $\Delta(X, \Omega)$ is the codimension of $\omega \cap \omega'$ in both.*

Proof. Let $\omega, \omega' \in \Omega$, where $\dim \omega \cap \omega' = n - i$. If $\omega'' \sim \omega'$ and $\dim \omega \cap \omega'' > n - i$, then ω'' contains some $x \in \omega \setminus \omega'$ and then is uniquely determined by $x \in \omega'' \sim \omega'$. The number of such ω'' is $(q^i - 1)/(q - 1)$, which is nonzero for $i > 0$. This shows that ω and ω' have distance at most, and therefore precisely, i in $\Delta(X, \Omega)$, and that $c_i = (q^i - 1)/(q - 1)$. If $\omega'' \sim \omega'$ and $\dim \omega \cap \omega'' < n - i$, then $\omega' \cap \omega''$ is a hyperplane in ω' not containing $\omega \cap \omega'$. There are $(q^n - q^i)/(q - 1)$ such hyperplanes, each contributing t choices for ω''. This yields the stated value of b_i. □

These graphs have eigenvalues $\theta_i = t\dfrac{q^{n-i} - 1}{q - 1} - \dfrac{q^i - 1}{q - 1}$ $(0 \leq i \leq n)$.

For more details, see [123], §9.4.

The special case $n = 2$ yields strongly regular graphs. See Theorem 2.2.19 below.

2.2.10 Generalized quadrangles

A *generalized quadrangle* $(X, \mathscr{L})$ is a partial linear space such that given a point $x \in X$ and a line $L \in \mathscr{L}$ not incident with x, there is exactly one pair $(y, M) \in X \times \mathscr{L}$ with $x \in M \ni y \in L$.

If every point is contained in at least two lines, then the *dual* of a generalized quadrangle $(X, \mathscr{L})$ is the partial linear space with point set $\mathscr{L}$ and set of lines $\{\{L \in \mathscr{L} \mid x \in L\} \mid x \in X\}$ and is also a generalized quadrangle.

A finite generalized quadrangle is said to be *of order* (s, t) when $s > 0$, $t > 0$ and every line is incident with $s + 1$ points and every point is incident with $t + 1$ lines. Then its dual has order (t, s). An arbitrary generalized quadrangle of order (s, t) is often denoted by $\mathsf{GQ}(s, t)$.

Examples of generalized quadrangles of order (s, t) are known when $s = 1$ or $(s, t) = (q, q)$, (q, q^2), (q^2, q^3), or $(q - 1, q + 1)$ (where q is a prime power), and the duals of these. For constructions and properties, see [609], [687], [710].

Proposition 2.2.18 *The collinearity graph of a generalized quadrangle of order (s,t) is strongly regular with parameters (v,k,λ,μ) and spectrum $k^1\ r^f\ a^g$, where*

$$\begin{aligned} v &= (s+1)(st+1), & r &= s-1, \\ k &= s(t+1), & a &= -t-1, \\ \lambda &= s-1, & f &= \frac{s(s+1)t(t+1)}{s+t}, \\ \mu &= t+1, & g &= \frac{s^2(st+1)}{s+t}. \end{aligned}$$

(We named the negative eigenvalue here a instead of s to avoid a conflict with the s from the order.)

Since the multiplicities are integers, one has the divisibility condition that $(s+t) \mid s^2(s^2-1)$.

The 2nd Krein condition implies $s = 1$ or $t \le s^2$. Dually, one has $t = 1$ or $s \le t^2$.

2.2.11 Strongly regular graphs on the lines

An embedded polar space of rank 2 is a generalized quadrangle, and the above applies, and we have strongly regular graphs on the duals. Or, we might invoke Theorem 2.2.17 with $n = 2$.

Theorem 2.2.19 *Let (X,Ω) be a finite embedded polar space of rank 2 and order (q,t). Then the graph $\Delta = \Delta(X,\Omega)$ is strongly regular with parameters (v,k,λ,μ) and spectrum $k^1\ r^f\ s^g$, where*

$$\begin{aligned} v &= (1+t)(1+qt), & r &= t-1, \\ k &= t(q+1), & s &= -q-1, \\ \lambda &= t-1, & f &= \frac{tq(t+1)(q+1)}{q+t}, \\ \mu &= q+1 = \frac{k}{t}, & g &= \frac{t^2(tq+1)}{q+t}. \end{aligned}$$

2.2.12 Distance-regular graphs on half of the maximal singular subspaces

Let (X,Ω) be a finite embedded polar space of rank n and order $(q,1)$. We see from Theorem 2.2.17 that the graph $\Delta = \Delta(X,\Omega)$ has diameter n, and is bipartite (since $k = b_i + c_i$ for all i). Now Theorem 1.2.2 tells us that the halved graphs are distance-regular of diameter $\lfloor n/2 \rfloor$. In particular, they will be strongly regular for $n = 4, 5$.

Let $\Delta_{1/2}$ be one of the two connected components of the distance-2 graph of Δ. We will see later (see §3.2) that both components are isomorphic. It will also turn out that, if (X,Ω) has rank 4, then $\Gamma = \Gamma(X,\Omega)$ is isomorphic to $\Delta_{1/2}$. Hence, we only obtain a new strongly regular graph for rank 5. See §3.2 for more details.

Theorem 2.2.20 *Let (X, Ω) be a finite embedded polar space of rank 5 of order $(q, 1)$. Then the graph $\Delta_{1/2}$ is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ s^g$, where*

$$
\begin{aligned}
v &= (q^4+1)(q^3+1)(q^2+1)(q+1), & r &= q^5+q^4+q^3-1,\\
k &= q(q^2+1)\frac{q^5-1}{q-1} = q\begin{bmatrix}5\\2\end{bmatrix}_q, & s &= -q^2-1,\\
\lambda &= q-1+q^2(q+1)(q^2+q+1), & f &= q^7+q^5+q^4+q^3+q,\\
\mu &= (q^2+1)(q^2+q+1) = \begin{bmatrix}4\\2\end{bmatrix}_q, & g &= q^2(q^4+1)\frac{q^5-1}{q-1}.
\end{aligned}
$$

Proof. Immediate from Theorems 2.2.17 and 1.2.2 (i). □

2.3 Classification of finite embedded polar spaces

2.3.1 Residues

We first note the following straightforward generalization of Proposition 2.2.9. For a subset $A \subseteq X$ we use the notation $A^\perp$ to denote the set of points of X collinear to all points of A.

Proposition 2.3.1 *Let S be a singular subspace of dimension i, $0 \le i \le n-2$, of a nondegenerate embedded polar space (X, Ω) of rank n. Then there exists a singular subspace T of dimension i with the property that no point of $S \cup T$ is collinear to all points of $S \cup T$. Also, if $X_{S,T} = S^\perp \cap T^\perp$ and $\Omega_{S,T} = \{\omega \cap X_{S,T} \mid \omega \in \Omega\}$, then $(X_{S,T}, \Omega_{S,T})$ is a nondegenerate polar space in $\langle X_{S,T}\rangle$ of rank $n-i$ whose isomorphism type does not depend on T, i.e., if U is an arbitrary singular subspace of dimension i with the property that no point of $S \cup U$ is collinear to all points of $S \cup U$, then the embedded polar spaces $(X_{S,T}, \Omega_{S,T})$ and $(X_{S,U}, \Omega_{S,U})$ are isomorphic. We also have $\dim\langle X_{S,T}\rangle = \dim V - 2i$. In the finite case, (X, Ω) and $(X_{S,T}, \Omega_{S,T})$ have the same order.*

Proof. Proceeding by induction on $\dim S$, we note that for $\dim S = 1$ we can refer to Proposition 2.2.9. For $\dim S \ge 2$, we choose a pair of noncollinear points x, y with $x \in S$. Induction implies that we can find a singular subspace $T' \subseteq x^\perp \cap y^\perp$ with the property that no point of $(S \cap y^\perp) \cup T'$ is collinear to all points of $(S \cap y^\perp) \cup T'$. Set $T = \langle T', y\rangle$. Clearly, no point of $S \cup T$ is collinear to all points of $S \cup T$. Now we see that, inside $x^\perp \cap y^\perp$, the set of points collinear to all points of $(S \cap y^\perp) \cup T'$ is exactly $S^\perp \cap T^\perp$ and so induction yields that $(X_{S,T}, \Omega_{S,T})$ is a nondegenerate polar space in $\langle X_{S,T}\rangle$ of rank $n-i$.

The claim about the isomorphism type is proved exactly in the same way as the last assertion of Proposition 2.2.9. □

Since the isomorphism class of $(X_{S,T}, \Omega_{S,T})$ does not depend on T, we denote that polar space by $\operatorname{Res} S$ and call it the *residue of S*.

Subspaces S and T with the property that no point of $S \cup T$ is collinear to all points of $S \cup T$ are called *opposite*.

2.3.2 Reduction to rank 2

Let (X, Ω) be a nondegenerate embedded polar space of rank $n \geq 3$. Let $x \in X$. Then $\mathrm{Res}\, x$ is a nondegenerate embedded polar space of rank $n-1 \geq 2$ whose isomorphism type is independent of x. Hence, in order to classify all (finite) embedded polar spaces, it suffices to determine all nondegenerate embedded polar spaces of rank 2, and then determine all extensions to higher ranks. In this paragraph, we will show that a nondegenerate embedded polar space admits at most one extension to any given higher rank. More precisely:

Theorem 2.3.2 *Let (X, Ω) be a nondegenerate embedded polar space of rank $n \geq 2$. Then, up to isomorphism, there exists at most one nondegenerate embedded polar space (X', Ω') of rank $n+1$ such that for any pair of opposite points $x, y \in X'$ the embedded polar space $(X'_{x,y}, \Omega'_{x,y})$ is isomorphic to (X, Ω).*

Proof. Suppose two embedded polar spaces (X'_1, Ω'_1) and (X'_2, Ω'_2) as described in the theorem exist. Let $\perp_i$ denote the collinearity in (X'_i, Ω'_i), $i = 1, 2$, and let $\perp_X$ denote the collinearity in (X, Ω).

Let x_i, y_i be two opposite points of X'_i, $i = 1, 2$. Then $x_i^{\perp_i} \cap y_i^{\perp_i}$ is isomorphic to (X, Ω). By Proposition 2.3.1, $\dim\langle X'_1\rangle = \dim\langle X\rangle + 2 = \dim\langle X'_2\rangle$. For convenience, and without loss of generality, we can thus identify $x_i^{\perp_i} \cap y_i^{\perp_i}$ with X, $i = 1, 2$, and moreover assume that $x_1 = x_2 =: x$ and $y_1 = y_2 =: y$. Then $\langle X'_1\rangle = \langle X'_2\rangle$ and $x^{\perp_1} \cup y^{\perp_1}$ coincides with $x^{\perp_2} \cup y^{\perp_2}$.

Let $u, v \in X$ be two opposite points and choose arbitrarily $y' \in \langle u, y\rangle \setminus \{u, y\}$. By possibly applying a projective collineation fixing all points of $\langle y, X\rangle \cup \{x\}$, we may assume that the same point $x' \in \langle v, x\rangle \setminus \{v, x\}$ is collinear to y' in both (X'_1, Ω'_1) and (X'_2, Ω'_2). Then $\langle u^{\perp_X}, x'\rangle \subseteq \langle y'^{\perp_i}\rangle$. Since every line of (X'_1, Ω'_1), or equivalently, of (X'_2, Ω'_2), through x contains a unique point of $y'^{\perp_i}$, $i = 1, 2$, and also a unique point of $\langle u^{\perp_X}, x'\rangle$ (for dimension reasons: it is a hyperplane in $\langle x^{\perp_i}\rangle$), we see that $y'^{\perp_i} \cap x^{\perp_i} = x^{\perp_i} \cap \langle u^{\perp_X}, x'\rangle$, $i = 1, 2$. Since $x^{\perp_1} = x^{\perp_2}$, this shows that $y'^{\perp_1} = y'^{\perp_2}$. Now pick a maximal singular subspace $\omega \in \Omega$ through u and an opposite (disjoint) one, say ξ, containing v. Then $\omega' = \langle \omega, y\rangle$ and $\xi' = \langle \xi, x\rangle$ belong to $\Omega'_1 \cap \Omega'_2$. In (X'_i, Ω'_i), $i = 1, 2$, the mapping $\rho_i : \omega' \to \xi' : z \mapsto z^{\perp_i} \cap \xi'$ is an isomorphism from the n-dimensional projective space ω' to the dual of ξ'. The images of the points in ω under ρ_1 and ρ_2 coincide because if $z \in \omega$, then $\rho_i(z) = \langle z^{\perp_X} \cap \xi, x\rangle$, which is independent of i, $i \in \{1, 2\}$. Also, the images of y and y' under both ρ_1 and ρ_2 are ξ and $\langle y'^{\perp_X} \cap \xi, x'\rangle$, respectively. Hence ρ_1 and ρ_2 completely coincide. In particular, for each point y'' on $\langle y, u\rangle \setminus \{y, u\}$, we know that $y''^{\perp_1} \cap \langle x, v\rangle = y''^{\perp_2} \cap \langle x, v\rangle$. Then what we proved about y' also holds for y'', and in particular $y''^{\perp_1} = y''^{\perp_2}$. Since $u \in X$ was chosen arbitrarily, we conclude that $w^{\perp_1} = w^{\perp_2}$, for all $w \in (x^{\perp_1} \cup y^{\perp_1}) \setminus X$. Interchanging the roles of (v, y) and (x, y), this argument yields $v^{\perp_1} = v^{\perp_2}$. Since v in X was chosen arbitrarily, we conclude $w^{\perp_1} = w^{\perp_2}$ for all $w \in X$, too. Since every point of X' is $\perp_i$-collinear to at least one point of $\langle y, u\rangle$, this already implies $X'_1 = X'_2$.

The foregoing implies that collinearity in (X'_1, Ω'_1) coincides with collinearity in (X'_2, Ω'_2) as soon as there is a point of $x^{\perp_1} \cup y^{\perp_1}$ involved. Now let $w \in X'_1 \setminus (x^{\perp_1} \cup y^{\perp_1})$. Then the previous sentence implies $w^{\perp_1} \cap x^{\perp_1} = w^{\perp_2} \cap x^{\perp_2}$. This implies that collinearity in (X'_1, Ω'_1) coincides with collinearity in (X'_2, Ω'_2), globally. Hence $\Omega'_1 = \Omega'_2$. □

2.3.3 The finite rank 2 polar spaces in 3-space

Let an embedded generalized quadrangle be an embedded polar space of rank 2. Degenerate examples consist of a number of lines on a common point. We classify the finite nondegenerate examples. The classification is due to Buekenhout & Lefèvre [157].

Theorem 2.3.3 *Let $(X, \mathscr{L})$ be an embedded nondegenerate generalized quadrangle of order (q, t) with $\dim V = 4$. Then we have one of the following three possibilities.*

(*i*) *$t = 1$ and $\mathscr{L}$ is the set of lines of a nondegenerate ruled quadric.*

(*ii*) *$t = \sqrt{q}$ and $\mathscr{L}$ is the set of fixed lines under a unitary polarity, or equivalently, $(X, \mathscr{L})$ arises from a nondegenerate σ-Hermitian form on V, with $\sigma : \mathbb{F}_q \to \mathbb{F}_q : a \mapsto a^{\sqrt{q}}$.*

(*iii*) *$t = q$ and $\mathscr{L}$ is a linear complex, or equivalently, $\mathscr{L}$ is the set of fixed lines under a symplectic polarity, or equivalently, $(X, \mathscr{L})$ arises from a nondegenerate alternating form on V.*

Proof. We first note the following property $(*)$.

$(*)$ *All lines of $(X, \mathscr{L})$ through a point $x \in X$ are contained in a plane π_x and every line N of $\mathsf{P}V$ through x not in π_x contains $t + 1$ points of X.*

The first assertion of $(*)$ follows from Lemma 2.2.8.

For the second, let $L \in \mathscr{L}$ be such that $x \notin L$ and let $M \in \mathscr{L}$ with $x \in M$ and $M \cap L = \emptyset$. The plane $\pi = \langle M, N \rangle$ meets L in a point y not in π_x, and y is collinear with a point z on M distinct from x. Now $\pi = \pi_z$, and the $t + 1$ lines on z in $\mathscr{L}$ meet N in $t + 1$ points in X, and N cannot contain any further points in X. Property $(*)$ is proved.

Since $(X, \mathscr{L})$ is a generalized quadrangle, one has

$(**)$ *For $L \in \mathscr{L}$, let $\pi(L)$ be the set of planes of $\mathsf{P}V$ through L. The mapping $L \to \pi(L) : x \mapsto \pi_x$ is a bijection.*

Suppose first $t = 1$. Then clearly (i) holds.[1]

Suppose $t = q$. Then for every point $x \in X$ we have $\pi_x \subseteq X$ and $(**)$ implies that $X = \mathsf{P}V$. It is then routine to check that the mapping $x \mapsto \pi_x$ is a polarity all of whose points are incident with their image; hence the polarity is a symplectic one and (iii) follows.

Now suppose $1 < t < q$. Let π be a plane of $\mathsf{P}V$ containing some point $x \in X$, but not containing any line of $(X, \mathscr{L})$ through x (π exists since $t < q$). Then π does not contain any line of $(X, \mathscr{L})$, and every line through x in π except for $\pi \cap \pi_x$ contains $t+1$ points of X, while $\pi \cap \pi_x$ only contains x of X. It follows that $|\pi \cap X| = tq + 1$. Let L be any line of $\mathsf{P}V$ in π through x but not contained in π_x. Then $|L \cap X| = t + 1$. Pick $y_1, y_2 \in L \cap X \setminus \{x\}$, then each point of $y_1^\perp \cap y_2^\perp$ is collinear with y_1 and y_2, and hence with all points of $L \cap X$. It follows that π_x contains $\langle y_1^\perp \cap y_2^\perp \rangle$, which intersects π in a projective point x_L. Then each of the $t + 1$ lines joining x_L with a point of $L \cap X$ intersects X

[1] When $t = 1$ (two lines on each point), we have a grid consisting of $q + 1$ mutually skew lines, all intersected by $q + 1$ transversals. Any three mutually skew lines uniquely determine the $q + 1$ transversals, and then the remaining lines. The projective group is transitive on triples of mutually skew lines, so we may take them to be $X = Y = 0$, $Z = W = 0$, and $X = Z, Y = W$. Now all lines lie on the hyperbolic (ruled) quadric $XW = YZ$.

in exactly one point, whereas the other $q-t$ lines of π through x_L intersect X in at most $t+1$ points. It follows that

$$tq+1 \leq t+1+(t+1)(q-t),$$

which reduces to $t^2 \leq q$. Since also $q \leq t^2$ (by the Krein condition), we conclude $t = \sqrt{q}$. Then $U = \pi \cap X$ is a unital[2] in π with flat feet,[3] so, by THAS [686], it is a Hermitian unital arising from a unitary polarity.

Since all Hermitian unitals are projectively equivalent, we can fix one of them and so $\pi \cap X$ is projectively unique. Now we consider a line $K \in \mathscr{L}$ through x and two points $x_0, x_1 \in K \setminus \{x\}$. We also consider two lines M_0, M_1 in π through x distinct from $M := \pi \cap \pi_x$. Without loss of generality we may assume that $\pi_{x_i} = \langle x_i, M_i \rangle$, $i = 0, 1$. The substructure $U \cup x_0^\perp \cup x_1^\perp$ of X is projectively unique; we show that it determines the rest of X. Set $z = \langle x_0, x_{M_0} \rangle \cap \langle x_1, x_{M_1} \rangle$. Note that $x_{M_i} \in M$, $i = 0, 1$, and hence z is well defined. For any point $u \in U \cap (M_0 \cup M_1)$ we have $z \in \pi_u$. Hence, if $v \in U$ is a point on a block of U that intersects both $M_0 \cap U$ and $M_1 \cap U$ in points distinct from x, then $z \in \pi_v$ and hence $z \in \langle x_{\langle v,x \rangle}, v^\perp \cap K \rangle$. All points of U except for $q - 2\sqrt{q}$ points on $\sqrt{q} - 2$ blocks through x are on a block of U intersecting both $(M_0 \cap U) \setminus \{x\}$ and $(M_1 \cap U) \setminus \{x\}$. But then these $q - 2\sqrt{q}$ points are obtained by repeating the argument with M_1 replaced by another (appropriate) line of π through x. It follows that for an arbitrary point $x_2 \in K$, the plane π_{x_2} is determined by being spanned by x_2 and $\rho(\langle x_2, z \rangle \cap M)$, where ρ is the unitary polarity associated with U.

Hence the structure of $(X, \mathscr{L})$ is uniquely determined and it necessarily arises from a unitary polarity in $\mathsf{P}V$. $\square$

2.3.4 The finite embedded generalized quadrangles

Theorem 2.3.4 *Let $(X, \mathscr{L})$ be an embedded generalized quadrangle. Let $\langle X \rangle$ be a hyperplane of the projective space $\mathsf{P}V$. Suppose $(X', \mathscr{L}')$ is an embedded generalized quadrangle with $\langle X' \rangle = \mathsf{P}V$ and such that $X = X' \cap \langle X \rangle$. Then the isomorphism type of $(X', \mathscr{L}')$ only depends on the isomorphism type of $(X, \mathscr{L})$, and $X \neq \langle X \rangle$. In particular, $(X, \mathscr{L})$ is not a symplectic quadrangle.*

Proof. Let $x' \in X' \setminus X$ and let $L' \in \mathscr{L}'$ with $x' \in L'$. Then there is a unique point $x \in L' \cap X$. Set $O = x'^\perp \cap X$. Since $x'^\perp$ is a hyperplane of $\mathsf{P}V$, the set O spans a hyperplane of $\langle X \rangle$ and by Proposition 2.2.7 $O = x'^\perp \cap \langle X \rangle$. Clearly O is an ovoid of $(X, \mathscr{L})$. Hence $(X, \mathscr{L})$ is not symplectic as in this case every plane intersects the quadrangle in the perp of a point. Since in the other cases all hyperplanes intersecting $(X, \mathscr{L})$ in ovoids are projectively equivalent, the dataset $\{(X, \mathscr{L}), x', O\}$, with $x'^\perp \cap X = O$, is projectively unique. Let $y \in O \setminus \{x\}$, and let $u \in X$ be an arbitrary point distinct from y but collinear

[2] A *unital* is a Steiner system $S(2, a+1, a^3+1)$, that is, is a 2-$(a^3+1, a+1, 1)$ design. An *embedded unital* is such a design embedded in a projective plane $\mathsf{PG}(2, a^2)$, where the point set of the design is a subset P of the set of projective points, and the blocks of the design are the nontrivial intersections $P \cap L$ of P with projective lines L.

[3] A *tangent* of an embedded unital U is a projective line containing precisely one point of U. An embedded unital has a unique tangent at each point, and $a+1$ tangents through each point $p \notin U$. An embedded unital is said to have *flat feet* if for each point $p \notin U$ the tangents passing through p meet the unital in collinear points.

with y and not collinear with x. Let $v \in \langle x, x' \rangle$ be the unique point of $\langle x, x' \rangle$ collinear with u in X'. Up to a collineation fixing all points of $\langle X \rangle \cup \{x'\}$ the point v is unique; hence the dataset $\{(X, \mathscr{L}), x', O, u, v\}$ with the above relations is projectively unique. We now claim that it uniquely determines $(X', \mathscr{L}')$. In fact, since the symplectic quadrangle can never be an embedded subquadrangle of $(X', \mathscr{L}')$, the set X' determines $\mathscr{L}'$ and so we only need to show that X' is determined. Set $H = \langle x^{\perp_X} \rangle \cap \langle O \rangle = \langle x^{\perp} \rangle \cap \langle x'^{\perp} \rangle \cap \langle X \rangle$. Then H has codimension 2 in $\langle X \rangle$. Suppose $\langle v^{\perp} \rangle$ does not contain H. Then $\langle v^{\perp} \rangle \cap \langle x'^{\perp} \rangle$ contains a line M in $\langle X \rangle$ through x not contained in $x^{\perp}$. Hence M intersects X in a second point p. Then $p \in x'^{\perp} \cap v^{\perp}$ and so $p = x$, a contradiction. We have shown that $H \subseteq \langle v^{\perp} \rangle$. Now $H \cap \langle y^{\perp_X} \rangle$ and u belong to $v^{\perp}$, hence $H_v := \langle H \cap \langle y^{\perp_X} \rangle, u \rangle$ belongs to $v^{\perp}$. But H_v has codimension 1 in $\langle y^{\perp_X} \rangle$, hence H_v intersects every line of $(X, \mathscr{L})$ through y. Hence the set $v^{\perp} \cap y^{\perp_X}$ is uniquely determined.

Note that the previous argument shows that, in an embedded generalized quadrangle, the following property $(*)$ holds.

$(*)$ *If a line L is opposite two intersecting lines M_1, M_2, and L does not intersect the plane $\langle M_1, M_2 \rangle$, then all lines $\langle u_1, u_2 \rangle$, with $u_i \in M_i$, $i = 1, 2$, $u_1 \neq u_2$, collinear to the same point of L, contain a fixed point only depending on L and the plane $\langle M_1, M_2 \rangle$. (In the above argument, M_1 and M_2 are two lines of X through y, and L is the line $\langle x, x' \rangle$.)*

We can now interchange y with any other point of O collinear with any of $H_v \cap X$. The same argument then gives further points of $v^{\perp} \cap X$, and one point v' is enough to see that $v^{\perp} \cap X = \langle H_v, v' \rangle \cap X$ is determined. Hence all points of X' collinear with v are determined.

Set $L = \langle u, y \rangle$ and let M_1 be a line of X through x opposite L. Let w and w' be the unique point of X on M_1 collinear with u and y, respectively. Let v_0 be an arbitrary point on $\langle x, x' \rangle \setminus \{x'\}$. Let w_0 be the intersection of M_1 with the line $\langle v_0, \langle v, w \rangle \cap \langle x', w' \rangle \rangle$. By $(*)$ (with $M_2 = \langle x, x' \rangle$), the point v_0 is collinear with the unique point on L which is (in X) collinear with w_0. Hence we can interchange the roles of v and v_0 and by the foregoing, all points of $v_0^{\perp}$ are determined. Hence all points collinear with a point of $\langle x, x' \rangle \setminus \{x\}$ are determined. We can interchange x with any point $x_0 \in X$ collinear with X. But then all points of X' are determined as no point of $X' \setminus X$ is collinear with both x and x_0. □

2.3.5 Summary

We list the finite embedded polar spaces of rank at least 2 in a vector space V over the finite field $\mathbb{F}_q$, and give order and full collineation group G.

One also meets the notation $W_{2n-1}(q)$, $Q^+_{2n-1}(q)$, $Q_{2n}(q)$, $Q^-_{2n+1}(q)$, $H_{2n-1}(q^2)$, $H_{2n}(q^2)$ for the polar spaces $\mathsf{Sp}_{2n}(q)$, $\mathsf{O}^+_{2n}(q)$, $\mathsf{O}_{2n+1}(q)$, $\mathsf{O}^-_{2n+2}(q)$, $\mathsf{U}_{2n}(q)$, $\mathsf{U}_{2n+1}(q)$, respectively.

Theorem 2.3.5 *For $n = 2$, Table 2.1 is a list of all finite nondegenerate embedded generalized quadrangles.*

Proof. The case $\dim V = 4$ follows from Theorem 2.3.3. Theorem 2.3.4 implies that only $\mathsf{O}^+_4(q)$ and $\mathsf{U}_4(q)$ possibly extend to a quadrangle in dimension 4. And they do, in view of the existence of the appropriate quadrics and Hermitian

Name	Symbol	$\dim V$	Order	G
Symplectic polar space of rank n	$\mathsf{Sp}_{2n}(q)$	$2n$	(q,q)	$\mathsf{P\Gamma Sp}_{2n}(q)$
Hyperbolic orthogonal polar space of rank n	$\mathsf{O}^+_{2n}(q)$	$2n$	$(q,1)$	$\mathsf{P\Gamma O}^+_{2n}(q)$
Parabolic orthogonal polar space of rank n	$\mathsf{O}_{2n+1}(q)$	$2n+1$	(q,q)	$\mathsf{P\Gamma O}_{2n+1}(q)$
Elliptic orthogonal polar space of rank n	$\mathsf{O}^-_{2n+2}(q)$	$2n+2$	(q,q^2)	$\mathsf{P\Gamma O}^-_{2n+2}(q)$
Small unitary or Hermitian polar space of rank n	$\mathsf{U}_{2n}(\sqrt{q})$	$2n$	$(q,q^{1/2})$	$\mathsf{P\Gamma U}_{2n}(\sqrt{q})$
Large unitary or Hermitian polar space of rank n	$\mathsf{U}_{2n+1}(\sqrt{q})$	$2n+1$	$(q,q^{3/2})$	$\mathsf{P\Gamma U}_{2n+1}(\sqrt{q})$

Table 2.1: The finite nondegenerate embedded polar spaces.

forms, see §2.6–§2.7. Now let $\dim V = 6$. Note that, if Q' is a subquadrangle of order (s,t') of some generalized quadrangle Q of order (s,t), then through every point of Q' there are $t-t'$ lines of $Q \setminus Q'$. Hence $(1+t)(1+st) \geq (1+t')(1+st') + (1+s)(1+st')(t-t')$, which simplifies to $t \geq st'$. Hence, if $(s,t') = (q,q^{3/2})$, then $t \geq q^{5/2} > q^2$, a contradiction. Consequently, $\mathsf{U}_5(q)$ cannot be extended anymore. Since there exists a quadric with Witt index 2 in 6-dimensional space, it is the unique one extending $\mathsf{O}_5(q)$. Note that the above inequality yields $(q,t) = (q,q^2)$ for $\mathsf{O}^-_6(q)$ since $\mathsf{O}_5(q)$ has order (q,q). The latter follows from applying the Klein correspondence to the symplectic quadrangle (which shows that the quadrangle $\mathsf{Sp}_4(q)$ is the dual of $\mathsf{O}_5(q)$). The above inequality shows that $\mathsf{O}^-_6(q)$ does not extend to a quadrangle in dimension 7. □

Using Theorem 2.3.2 and the constructions in §2.5–§2.7, we have the following theorem.

Theorem 2.3.6 *Table 2.1 is a list of all finite nondegenerate embedded polar spaces.* □

The embedded polar space $\mathsf{O}^+_4(q)$ is a ruled quadric. The automorphism group of the embedded polar space (defined as the subgroup of the collineation group of $\mathsf{P}V$ preserving the embedded polar space) is $\mathsf{P\Gamma O}^+_4(q)$. If one forgets the embedding, this geometry is a grid, with automorphism group $\mathsf{S}_{q+1} \operatorname{wr} 2$. In all other cases (of rank at least 2) the corresponding two groups coincide.

2.3.6 Group orders

We define the various classical groups and give their orders. We follow the Atlas [215] where possible. Let $q = p^e$ and let V be a vector space over $\mathbb{F}_q$.

If G is the name of a group of semilinear transformations of V, then $\mathsf{P}G$ is the name of the corresponding projective group, that is, is $G/(G \cap Z)$ where $Z = \{cI \mid c \in \mathbb{F}_q\}$ is the group of scalars.

If G is the name of a group of linear transformations of V, then $\mathsf{S}G$ is the name of the subgroup of G consisting of the elements of determinant 1. The prefix SG is simplified to S.

Linear groups

The general linear group $\mathsf{GL}_n(q)$ is the group of nonsingular linear transformations of a vector space V of dimension n over $\mathbb{F}_q$. Its order is

$$N = \prod_{i=0}^{n-1}(q^n - q^i) = q^{\frac{1}{2}n(n-1)}\prod_{i=1}^{n}(q^i - 1).$$

We have $|\mathsf{SL}_n(q)| = |\mathsf{PGL}_n(q)| = N/(q-1)$ and $|\mathsf{PSL}_n(q)| = N/(d(q-1))$, where $d = (q-1, n)$. The group $\mathsf{PSL}_n(q)$ is also called $\mathsf{L}_n(q)$. The general semilinear group $\Gamma\mathsf{L}_n(q)$ consists of $\mathsf{GL}_n(q)$ extended by the field automorphisms. Its subgroup $\Sigma\mathsf{L}_n(q)$ consists of $\mathsf{SL}_n(q)$ extended by the field automorphisms. The size of $\Gamma\mathsf{L}_n(q)$, $\Sigma\mathsf{L}_n(q)$, $\mathsf{P}\Gamma\mathsf{L}_n(q)$, $\mathsf{P}\Sigma\mathsf{L}_n(q)$ is a factor e larger than that of $\mathsf{GL}_n(q)$, $\mathsf{SL}_n(q)$, $\mathsf{PGL}_n(q)$, $\mathsf{PSL}_n(q)$, respectively.

Unitary groups

The general unitary group $\mathsf{GU}_n(q)$ is the subgroup of $\mathsf{GL}_n(q^2)$ consisting of the elements that preserve a nondegenerate Hermitian form. Its order is

$$N = q^{\frac{1}{2}n(n-1)}\prod_{i=1}^{n}(q^i - (-1)^i).$$

We have $|\mathsf{SU}_n(q)| = |\mathsf{PGU}_n(q)| = N/(q+1)$ and $|\mathsf{PSU}_n(q)| = N/(d(q+1))$, where $d = (q+1, n)$. The group $\mathsf{PSU}_n(q)$ is also called $\mathsf{U}_n(q)$.

The Atlas [215] does not have separate names for groups larger than $\mathsf{GU}_n(q)$, but one has the group preserving the Hermitian form up to a scalar multiple, and the same group extended by the field automorphisms, and the projective versions of these two. We call the last group $\mathsf{P}\Gamma\mathsf{U}_n(q)$. Its size is $2eN/(q+1)$.

Symplectic groups

The symplectic group $\mathsf{Sp}_n(q)$, where $n = 2m$, is the subgroup of $\mathsf{GL}_n(q)$ consisting of the elements that preserve a nondegenerate symplectic form. (Such elements all have determinant 1.) Its order is

$$N = q^{m^2}\prod_{i=1}^{m}(q^{2i} - 1).$$

We have $|\mathsf{PSp}_n(q)| = N/d$ where $d = (2, q-1)$, and $|\mathsf{P}\Gamma\mathsf{Sp}_n(q)| = eN$. (The group $\mathsf{PSp}_n(q)$ is also called $\mathsf{S}_n(q)$.)

Orthogonal groups

The general orthogonal group $\mathsf{GO}_n(q)$ is the subgroup of $\mathsf{GL}_n(q)$ consisting of the elements that preserve a nondegenerate quadratic form. If $n = 2m+1$ is odd, its order is

$$N = dq^{m^2}\prod_{i=1}^{m}(q^{2i} - 1).$$

where $d = (2, q-1)$. We have $|\mathsf{SO}_n(q)| = |\mathsf{PGO}_n(q)| = |\mathsf{PSO}_n(q)| = N/d$. Also $|\Omega_n(q)| = |\mathsf{P}\Omega_n(q)| = |\mathsf{O}_n(q)| = N/d^2$ and $|\mathsf{P}\Gamma\mathsf{O}_n(q)| = eN/d$.

If $n = 2m$ is even, we distinguish $\mathsf{GO}_n^\varepsilon(q)$ with $\varepsilon = 1$ for hyperbolic and $\varepsilon = -1$ for elliptic forms. The order is

$$N = 2q^{m(m-1)}(q^m - \varepsilon) \prod_{i=1}^{m-1} (q^{2i} - 1).$$

We have $|\mathsf{SO}_n^\varepsilon(q)| = |\mathsf{PGO}_n^\varepsilon(q)| = N/d$, where $d = (2, q^m - \varepsilon)$, $|\mathsf{PSO}_n^\varepsilon(q)| = N/d^2$, $|\Omega_n^\varepsilon(q)| = N/(2d)$, $|\mathsf{P}\Omega_n^\varepsilon(q)| = |\mathsf{O}_n^\varepsilon(q)| = N/(2d')$, where $d' = (4, q^m - \varepsilon)$, and $|\mathsf{P\Gamma O}_n^\varepsilon(q)| = eN$.

In the above, if q is odd, then $\Omega_n^\varepsilon(q)$ is the subgroup of index 2 of $\mathsf{SO}_n^\varepsilon(q)$ consisting of the elements with spinor norm 1. If q is even and n is odd, then $\Omega_n(q) = \mathsf{SO}_n(q)$. For any q and even n, let the *quasideterminant* of an element be $(-1)^f$, where f is the dimension of the fixed space. If q is odd, this agrees with the determinant. If q is even, let $\Omega_n^\varepsilon(q)$ be the subgroup of $\mathsf{SO}_n^\varepsilon(q)$ of index 2 consisting of the elements with quasideterminant 1. Geometrically (for $\varepsilon = 1$) this is the subgroup preserving one of the two classes of maximal totally isotropic subspaces. (These are the Atlas [215] definitions.)

2.4 Witt's theorem

The spaces considered here have large groups of automorphisms, as follows from Witt's theorem. Witt's theorem concerns spaces with a reflexive form, and we first relate these to embedded polar spaces.

2.4.1 Reflexive forms

Let V be a vector space over the field F. A map $f\colon V \times V \to F$ is called *reflexive* when f is linear in the second coordinate, and $f(x, y) = 0 \Leftrightarrow f(y, x) = 0$ for all $x, y \in V$.

Two vectors x, y are called *orthogonal* (for a given reflexive f) when $f(x, y) = 0$. Orthogonality is a symmetric relation. If A is a set of vectors, then $A^\perp$ is the set of all vectors orthogonal to each element of A. This is a subspace of V. The pair (V, f) (or, when f or V is understood, just V or f) is called *nondegenerate* when $V^\perp = 0$.

If V is finite-dimensional and nondegenerate, and U is a subspace of V, then $U^{\perp\perp} = U$ and $\dim U + \dim U^\perp = \dim V$.

2.4.2 Reflexive forms and embedded polar spaces

Let V be a vector space over the field F, and let f be a reflexive form on V. A subspace U of V is called *totally isotropic* when the restriction of f to $U \times U$ vanishes identically.

Proposition 2.4.1 *Let V be finite-dimensional, and let Ω be the set of maximal totally isotropic subspaces of V, and let $X = \bigcup \Omega$. Then (X, Ω) is an embedded polar space.*

Proof. Indeed, first of all we have $f(x, x) = 0$ for all $x \in V$ with $\langle x \rangle \in X$, since such an x is contained in a totally isotropic subspace. Next, two points $\langle x \rangle$ and $\langle y \rangle$ are collinear if and only if they are orthogonal: If they are orthogonal, then the subspace $\langle x, y\rangle$ is totally isotropic and contained in a maximal totally isotropic subspace $\omega \in \Omega$. The converse is clear. Finally, axiom (EPS2) is satisfied: if $\langle x\rangle \in X$ and $\omega \in \Omega$ with $x \notin \omega$, then, since f is linear in the second coordinate, the set $\xi = \{y \in \omega \mid f(x, y) = 0\}$ is a codimension 1 subspace of ω, and $\eta = \langle x, \xi\rangle$ is totally isotropic. Since ω is maximal, and $x \notin \omega$, there is a $z \in \omega$ with $f(x, z) \neq 0$. Now $\omega = \langle z, \xi\rangle$. If η were not maximal, it would be properly contained in a totally isotropic η', and its subspace orthogonal to z would properly contain ξ, violating the maximality of ω. Hence $\eta \in \Omega$. □

2.4.3 Classification of sesquilinear reflexive forms

Let V be a vector space over a field F. A map $f: V \times V \to F$ is called *bilinear* if it is linear in each coordinate, and *sesquilinear*, more precisely σ-sesquilinear, where $\sigma: F \to F$ is a field automorphism, when it is additive in each coordinate, and $f(ax, by) = a^\sigma b f(x, y)$. Thus, the bilinear forms are the σ-sesquilinear forms where σ is the identity. A σ-sesquilinear form f is called *σ-Hermitian* when σ has order 2 and $f(y, x) = f(x, y)^\sigma$ for all $x, y \in V$.

A bilinear form f is called *symmetric* (resp. *skew-symmetric*) when $f(x, y) = f(y, x)$ (resp. $f(x, y) = -f(y, x)$) for all x, y. It is called *alternating* (or *symplectic*) when $f(x, x) = 0$ for all x. An alternating form is skew-symmetric since $0 = f(x+y, x+y) = f(x, x)+f(x, y)+f(y, x)+f(y, y) = f(x, y)+f(y, x)$. If F has characteristic different from 2, then a skew-symmetric form is alternating.

Clearly, symmetric and alternating and σ-Hermitian forms are reflexive, and we show that essentially there are no other sesquilinear reflexive forms.

Proposition 2.4.2 *A bilinear form f is reflexive if and only if it is either symmetric or alternating.*

Proof. Clearly, symmetric and alternating forms are reflexive. Now let f be reflexive and bilinear. Then for all x, y, z:

$$f(x, f(x, z)y - f(x, y)z) = f(x, z)f(x, y) - f(x, y)f(x, z) = 0,$$

and therefore

$$f(x, z)f(y, x) - f(x, y)f(z, x) = f(f(x, z)y - f(x, y)z, x) = 0. \tag{2.1}$$

Substituting $z = x$ yields $f(x, x)(f(y, x)-f(x, y)) = 0$ for all x, y. It follows that if $f(y, x) \neq f(x, y)$ then $f(x, x) = 0$. Suppose $f(y, x) \neq f(x, y)$ and $f(z, z) \neq 0$ for some x, y, z. Then $f(w, z) = f(z, w)$ for all w, and (2.1) implies $f(x, z) = 0$. By symmetry also $f(y, z) = 0$. Now $f(x + z, y) = f(x, y)$ and $f(y, x + z) = f(y, x)$, so that $f(x + z, y) \neq f(y, x + z)$ and therefore $f(x + z, x + z) = 0$. But $f(x+z, x+z) = f(x, x)+f(x, z)+f(z, x)+f(z, z) = f(z, z) \neq 0$, contradiction. □

Proposition 2.4.3 *Let f be a nondegenerate reflexive σ-sesquilinear form on V, where $\dim V \geq 2$. Then either $\sigma = 1$ and f is symmetric or alternating, or $\sigma \neq 1$, $\sigma^2 = 1$ and there is a nonzero constant $a \in F$ such that af is σ-Hermitian.*

Proof. For fixed x, the linear functionals $y \mapsto f(x,y)$ and $y \mapsto \sigma^{-1}f(y,x)$ have the same kernel, so differ by a constant. It follows that there are constants $c_x \in F$ such that $\sigma^{-1}f(y,x) = c_x f(x,y)$ for all $x, y \in V$. By linearity of f in the second argument, we have $c_{x+y}f(x+y,z) = \sigma^{-1}f(z,x+y) = c_x f(x,z) + c_y f(y,z)$, i.e., by additivity in the first coordinate, $f(d_{x+y}(x+y) - d_x x - d_y y, z) = 0$ for all x, y, z, where $c_x = \sigma(d_x)$ for all x. Since f is nondegenerate, it follows that $d_{x+y}(x+y) - d_x x - d_y y = 0$ for all x, y. If x, y are independent, $d_{x+y} = d_x = d_y$. If x, y are dependent, then we can pick z independent from x, y since $\dim V \geq 2$, and $d_x = d_z = d_y$. So, c_x and d_x do not depend on x, and we drop the index. From $f(y,x) = (cf(x,y))^\sigma = (c(cf(y,x))^\sigma)^\sigma$ it follows that $(cc^\sigma)^\sigma a^{\sigma^2} = a$ for all $a \in F$, so that $cc^\sigma = 1$ and $\sigma^2 = 1$. If $\sigma = 1$, then f is bilinear and the previous proposition applies. Otherwise, pick a constant a such that $c = a/a^\sigma$. Then af is σ-Hermitian. □

(If $cc^\sigma = 1$, does there exist an a with $c = a/a^\sigma$? Try $a = b + b^\sigma c$. Then $a^\sigma c = b^\sigma c + b = a$ as desired, and one only has to choose b so that $a \neq 0$.)

If $\dim V = 1$, then w.l.o.g. $V = F$, and up to a nonzero constant $f(a,b) = a^\sigma b$. There is no need for σ to have order 2.

2.4.4 Orthogonal direct sum decomposition

Let V be a finite-dimensional vector space provided with a reflexive form f. We write $V = V_1 \perp \ldots \perp V_r$ when V is the vector space direct sum of the V_i, and the V_i are mutually orthogonal, i.e., $f(x,y) = 0$ for $x \in V_i$, $y \in V_j$, $i \neq j$.

Conversely, let (V_i, f_i) $(1 \leq i \leq r)$ be finite-dimensional vector spaces provided with reflexive forms f_i. Put $V = \oplus_i V_i$ and define f by $f(x,y) = 0$ for $x \in V_i$, $y \in V_j$, $i \neq j$, and $f(x,y) = f_i(x,y)$ if $x, y \in V_i$. Then f is a reflexive form on V, and $V = V_1 \perp \ldots \perp V_r$ (for this f), and f is symmetric, or alternating, or σ-sesquilinear when each of the f_i is.

A point is called *isotropic* when it is totally isotropic. A *hyperbolic line* is a nondegenerate 2-space spanned by two isotropic points.

Symplectic spaces

Let f be a symplectic form (that is, f is bilinear and $f(x,x) = 0$ for all $x \in V$). Then V can be written

$$V = L_1 \perp \ldots \perp L_r \perp V^\perp$$

where the L_i are hyperbolic lines. (Indeed, if $x \notin V^\perp$, then there is a y with $f(x,y) \neq 0$ so that $L = \langle x, y \rangle$ is a hyperbolic line, and $V = L \perp L^\perp$. Now apply induction on $\dim V$.)

For a hyperbolic line $L = \langle x, y \rangle$ we may take $f(x,x) = f(y,y) = 0$, $f(x,y) = 1$, so that (V, f) is determined up to isomorphism by $\dim V$ and $\dim V^\perp$.

Orthogonal spaces

Let f be a symmetric bilinear form and assume $\operatorname{char} F \neq 2$. Then V can be written

$$V = P_1 \perp \ldots \perp P_r \perp V^\perp$$

where the P_i are nonisotropic points. (Indeed, if $f(x,x) = 0$ for all x, then f is skew-symmetric and hence identically zero since $\mathrm{char}\, F \neq 2$. If $f(x,x) \neq 0$ then $V = P \perp P^\perp$ where $P = \langle x \rangle$. Now apply induction on $\dim V$.)

Assume F is finite. If P is nonisotropic, then we can pick x with $P = \langle x \rangle$ so that either $f(x,x) = 1$ or $f(x,x) = a$, where a is a fixed nonsquare in F. If $V = P \perp Q$ is the sum of two nonisotropic points $P = \langle x \rangle$ and $Q = \langle y \rangle$, and both are of the second type, then $f(\lambda x + \mu y, \lambda x + \mu y) = (\lambda^2 + \mu^2)a$ and since $\mathrm{char}\, F \neq 2$ the squares do not form a field, and we may pick $R = \langle z \rangle$ with $z = \lambda x + \mu y$ such that $f(z,z) = 1$. (Now $P \perp Q = R \perp S$ for $S = \langle w \rangle$, $w = \lambda x - \mu y$, and $f(w,w) = 1$.) Thus, in the above orthogonal direct sum we may take all points P_i of the first type, with at most one exception.

A change of basis changes $\det f$ by a square, so the two types are really different, and (V,f) is determined up to isomorphism by $\dim V$ and $\dim V^\perp$ and the quadratic character of $\det f$.

Hermitian spaces

Let f be a σ-Hermitian form. Then V can be written

$$V = P_1 \perp \ldots \perp P_r \perp V^\perp$$

where the P_i are nonisotropic points. (Indeed, if $f(x,x) = 0$ for all x, then f is skew-symmetric and hence identically zero since $\sigma \neq 1$. If $f(x,x) \neq 0$ then $V = P \perp P^\perp$ where $P = \langle x \rangle$. Now apply induction on $\dim V$.)

Assume F is finite, and let F_0 be the subfield of F fixed by σ. Then $f(x,x) \in F_0$ for all x, and since aa^σ takes all values in F_0, we can write any nonisotropic point as $P = \langle x \rangle$ with $f(x,x) = 1$. Thus (V,f) is determined up to isomorphism by $\dim V$ and $\dim V^\perp$.

2.4.5 Witt's theorem

Let, just for this section, a *space* be a pair (V,f) where V is a finite-dimensional vector space and f a reflexive sesquilinear form on V, either symplectic, or orthogonal (with $\mathrm{char}\, F \neq 2$), or σ-Hermitian. Given two spaces (V,f) and (W,g), an injective linear map $\phi\colon V \to W$ is called an *isometry* when $f(x,y) = g(\phi(x), \phi(y))$ for all $x, y \in V$.

Theorem 2.4.4 (Witt's theorem) *Let V, V' be isometric nondegenerate spaces, and let $\phi\colon U \to V'$ be an isometry from a subspace U of V into V'. Then ϕ can be extended to an isometry from V onto V'.*

Proof. Induction on $\dim V$, and for fixed $\dim V$ on the codimension of U in V. Let $R = U \cap U^\perp$. If $R \neq 0$, then let r be a nonzero vector in R. Pick $s \in V$ with $f(r,s) \neq 0$. We may take $f(r,s) = 1$. Let $r' = \phi(r)$. We want to pick $s' \in V'$ with $g(r', s') = 1$ and $g(\phi(u), s') = 0$ whenever $f(u,s) = 0$. That is possible: let $Y = U \cap s^\perp$. Then Y is a hyperplane in U, and $\phi(Y)$ is a hyperplane in $\phi(U)$. Now $\phi(Y)^\perp$ strictly contains $\phi(U)^\perp$, and we can choose s' in $\phi(Y)^\perp \setminus \phi(U)^\perp$. Linearly extend ϕ to $\bar\phi$ by letting $\bar\phi(u) = \phi(u)$ for $u \in U$, and $\bar\phi(s) = s'$. Then $\bar\phi$ is an isometry defined on $\langle s, U \rangle$, and induction applies.

Now assume that $R = 0$, so that U is nondegenerate. Then $U^\perp$ is nondegenerate, and $V = U \perp U^\perp$. Since V and V' are isometric, also $U^\perp$ and

$\phi(U)^\perp$ are isometric, and given an isometry $\phi_1\colon U^\perp \to \phi(U)^\perp$ we can define $\bar{\phi}(u+u') = \phi(u) + \phi_1(u')$ for $u \in U$ and $u' \in U^\perp$. □

2.5 Symplectic polar spaces

We review some properties of the strongly regular graph defined by the points of a finite symplectic polar space, adjacent when collinear. We pay special attention to (maximal) cliques and cocliques, regular sets and geometric notions in the corresponding polar space such as h-ovoids and spreads.

2.5.1 Symplectic forms, polar spaces, and graphs

Symplectic forms

Let V be a vector space over a field F. A *symplectic form* f on V is a bilinear map $f\colon V \times V \to F$ such that $f(v,v) = 0$ for all $v \in V$. A symplectic form is *alternating*: since $f(v+w, v+w) = f(v,v) = f(w,w) = 0$ it follows that $f(w,v) = -f(v,w)$.

If S is a subset of V, then $S^\perp$ is the subspace of V consisting of all vectors v for which $f(s,v) = 0$ for all $s \in S$. The *radical* Rad V of (V,f) is $V^\perp$. The symplectic form is called *nondegenerate* if Rad $V = 0$. A subspace W of V is called *totally isotropic* when f vanishes identically on $W \times W$.

Symplectic polar spaces

Suppose V is finite-dimensional. Let X be the set of totally isotropic 1-spaces of V and let Ω be the set of maximal totally isotropic subspaces in V with respect to f. Then (X,Ω) is a polar space embedded in $\mathsf{P}V$, called a *symplectic polar space*. The radical of (X,Ω) coincides with the radical Rad V of (V,f), so that (X,Ω) is nondegenerate precisely when f is nondegenerate. If (X,Ω) is nondegenerate and $\dim V$ is finite, then $\dim V = 2n$, where n is the rank of (X,Ω), and this polar space is called $\mathsf{Sp}_{2n}(F)$. (In the literature one also finds $\mathsf{W}_{2n-1}(F)$.) If $F = \mathbb{F}_q$ then we also write $\mathsf{Sp}_{2n}(q)$.

Symplectic graphs

The *symplectic graph* of (V,f) is the collinearity graph $\Gamma = \Gamma(X,\Omega)$ of (X,Ω) and thus has as vertex set the set of points of $\mathsf{P}V$, where distinct vertices $\langle u\rangle$ and $\langle v\rangle$ are adjacent when $f(u,v) = 0$. Note that the condition $f(u,v) = 0$ does not depend on the choice of u and v in $\langle u\rangle$ and $\langle v\rangle$, and that it is symmetric: $f(u,v) = 0$ implies $f(v,u) = 0$.

2.5.2 Parameters

Let f be nondegenerate, let V have finite dimension $2n$, and let F be the finite field $\mathbb{F}_q$. We determine the parameters of Γ. For $n = 0$ the symplectic graph Γ has no vertices. For $n = 1$ the graph Γ is empty, a coclique of size $q+1$.

For $n > 1$ the graph Γ is strongly regular and the parameters are given as in Theorem 2.2.12 with $(q, t) = (q, q)$:

$$\begin{aligned} v &= (q^{2n} - 1)/(q - 1), & r &= q^{n-1} - 1, \\ k &= q(q^{2n-2} - 1)/(q - 1), & s &= -q^{n-1} - 1, \\ \lambda &= q^2(q^{2n-4} - 1)/(q - 1) + q - 1, & f &= \frac{1}{2}(\frac{q^{2n} - q}{q - 1} + q^n), \\ \mu &= (q^{2n-2} - 1)/(q - 1), & g &= \frac{1}{2}(\frac{q^{2n} - q}{q - 1} - q^n). \end{aligned}$$

so that $\lambda = \mu - 2$ and $\mu = k/q$.

2.5.3 Automorphism groups

The *symplectic group* $\mathsf{Sp}(V, f)$ is the group of all linear transformations of V that preserve the form f. The *general symplectic group* $\mathsf{GSp}(V, f)$ is the group of all linear transformations of V that preserve the form f up to a constant.

The subgroup D of $\mathsf{GL}(V)$ consisting of all multiples of the identity acts trivially on $\mathsf{P}V$, and $D \cap \mathsf{Sp}(V, f) = \{\pm I\}$. The *projective symplectic group* $\mathsf{PSp}(V, f)$ is the quotient $\mathsf{Sp}(V, f)/\{\pm I\}$. The *projective general symplectic group* $\mathsf{PGSp}(V, f)$ is $\mathsf{GSp}(V, f)/D$.

If f is nondegenerate and V has finite dimension $2n$ over the field F, we also write $\mathsf{Sp}_{2n}(F)$ etc. instead of $\mathsf{Sp}(V, f)$ etc.

The full automorphism group of Γ is $\mathsf{P\Gamma Sp}_{2n}(F)$, that is, $\mathsf{PGSp}_{2n}(F)$ extended by the field automorphisms of F. This group acts rank 3 on Γ, and already $\mathsf{PSp}_{2n}(F)$ acts rank 3.

For $n \geq 2$, the group $\mathsf{PSp}_{2n}(F)$ is simple if $(n, |F|) \neq (2, 2)$. The group $\mathsf{PSp}_4(2)$ is isomorphic to the symmetric group S_6 and has a simple subgroup of index 2 (isomorphic to the alternating group A_6), which also acts rank 3 on Γ.

2.5.4 Maximal cliques

As remarked in §2.2.7, the maximal cliques of Γ are the maximal totally isotropic subspaces of (V, f), i.e., the elements of Ω. In the finite nondegenerate case of $\mathsf{Sp}_{2n}(q)$ these have dimension n and size $(q^n - 1)/(q - 1)$. The maximal cliques form a single orbit under $\mathrm{Aut}\,\Gamma$. The polar space $\mathsf{Sp}_{2n}(q)$ has spreads (partitions into maximal cliques).

2.5.5 Ovoids

Recall that a (symplectic) *ovoid* in a nondegenerate symplectic polar space $\mathsf{Sp}_{2n}(F)$ is a set of points that meets every maximal totally isotropic subspace in precisely one point. Ovoids (when they exist) are maximal cocliques. In $\mathsf{Sp}_{2n}(q)$ one has $|C| \leq q^n + 1$ and $|O| = q^n + 1$ for each coclique C and ovoid O. Ovoids exist precisely when $n = 2$ and q is even. We first show the nonexistence part of that statement.

Proposition 2.5.1 *The generalized quadrangle* $\mathsf{Sp}_4(q)$ *has no ovoid when* q *is odd.*

Proof. Suppose O is an ovoid of $\mathsf{Sp}_4(q)$. Let L be a hyperbolic line of $\mathsf{P}V$. The $(q+1)^2$ lines of $\mathsf{P}V$ meeting both L and $L^\perp$ are totally isotropic. But every point of $\mathsf{P}V$ (remember V has dimension 4) is either on $L \cup L^\perp$, or on precisely one line meeting both L and $L^\perp$. Let a be the number of points of O on $L \cup L^\perp$; we may assume these a points are contained in L. Then there is a bijective correspondence between the points of $O \setminus L$ and the lines of $\mathsf{P}V$ joining a point of $L^\perp$ with a point of $L \setminus O$. Hence $q^2+1-a = (q+1)(q+1-a)$, implying $a = 2$. We conclude that every hyperbolic line contains an even number of points of O. Now let p be any point of O, select a totally isotropic line M through p and two points $x, y \in M \setminus O$, with $x \neq y$. In the plane $y^\perp$ all lines through x other than M are hyperbolic. Hence, by the foregoing, $y^\perp \setminus M$ contains an even number of points of O. But that number is q. Hence q is even. $\square$

Proposition 2.5.2 *The polar space $\mathsf{Sp}_{2n}(q)$ has no ovoid, for $n \geq 3$.*

Proof. By Corollary 2.2.16, it suffices to consider $n = 3$.

Let, for a contradiction, O be an ovoid of the polar space $\mathsf{Sp}_6(q)$. Then $|O| = q^3 + 1$. Every hyperplane of $\mathsf{P}V$ is the perp $p^\perp$ of some point p. If $p \in O$, then $|p^\perp \cap O| = 1$; if $p \notin O$, then $|p^\perp \cap O| = q^2+1$. Let W be a 4-space containing at least two points of O and set $|W \cap O| = t$. Every hyperplane containing W intersects O in $q^2 + 1$ points. Hence $q^3 + 1 = t + (q + 1)(q^2 + 1 - t)$, implying $t = q + 1$. Now let π be a plane in W containing at least 3 points ($q \geq 2$) of O and set $|\pi \cap O| = t'$. Then, similarly counting the number of points in O in some hyperplane H with $W \subseteq H$, we obtain $q^2 + 1 = t' + (q + 1)(q + 1 - t')$, implying $t' = 2$, a contradiction. $\square$

Proposition 2.5.3 *The generalized quadrangle $\mathsf{Sp}_4(q)$, q even, has an ovoid.*

Proof. Let V be 4-dimensional over $\mathbb{F}_q$, and let the alternating form f be given by

$$f((x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3)) = x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2.$$

Let $g(x) = x^2 + x + d$ be an irreducible quadratic polynomial over $\mathbb{F}_q$ (one can always find one of this form) and let $g(x, y) = x^2 + xy + dy^2$. Consider the set O of projective points $\{(x_0, x_1, 1, g(x_0, x_1)) : x_0, x_1 \in \mathbb{F}_q\} \cup \{(0, 0, 0, 1)\}$. We claim that no two points of O are conjugate with respect to the alternating form f. Denote $v_{x_0,x_1} = (x_0, x_1, 1, g(x_0, x_1))$ and $v_\infty = (0, 0, 0, 1)$. Then $f(v_{x_0,x_1}, v_\infty) = 1$. Also one computes

$$f(v_{x_0,x_1}, v_{y_0,y_1}) = g(x_0 + y_0, x_1 + y_1),$$

which implies our claim. Since $|O| = q^2 + 1$, the proposition is proved. $\square$

The Suzuki-Tits ovoids

For q divisible by 4, every known ovoid of $\mathsf{Sp}_4(q)$ is isomorphic to the example in the previous proposition (and we call that example the *classical ovoid*). However, for $q = 2^{2e-1}$, there is a unique second known example.

Let V and f be as above, and let $q = 2^{2e-1}$. Set $r = 2^e$ and define the following set of points of $\mathsf{P}V$, given by their coordinates:

$$O = \{(0,0,0,1)\} \cup \{(x_0, x_1, 1, x_0^{r+2} + x_0x_1 + x_1^r) \mid x_0, x_1 \in \mathbb{F}_q\}.$$

We claim that $\mathscr{P}$ is an ovoid of $\mathsf{Sp}_4(q)$. Since $|O| = q^2+1$, it suffices to show that no pair of points of O is collinear in $\mathsf{Sp}_4(q)$. Let p_{x_0,x_1} be the point with coordinates $(x_0, x_1, 1, x_0^{r+2} + x_0x_1 + x_1^r)$, $x_0, x_1 \in \mathbb{F}_q$, and $p_\infty = (0,0,0,1)$. Then clearly $p_\infty \not\perp p_{x_0,x_1}$. Also, $p_{x_0,x_1} \perp p_{y_0,y_1}$ if and only if

$$x_0y_1 + x_1y_0 + x_0^{r+2} + x_0x_1 + x_1^r + y_0^{r+2} + y_0y_1 + y_1^r = 0.$$

This is equivalent to

$$(x_0 + y_0)(x_1 + y_1) = x_0^{r+2} + y_0^{r+2} + (x_1 + y_1)^r.$$

If $x_0 = y_0$, then clearly also $x_1 = y_1$. Assume now $x_0 \neq y_0$. Then we can divide both sides of the equality by $(x_0 + y_0)^{r+2}$ and obtain, after some elementary manipulation,

$$\frac{x_1 + y_1}{(x_0 + y_0)^{r+1}} = 1 + (\frac{x_0}{x_0 + y_0})^r + (\frac{x_0}{x_0 + y_0})^2 + \frac{(x_1 + y_1)^r}{(x_0 + y_0)^{r+2}},$$

which we can rewrite, setting

$$z_0 = (\frac{x_0}{x_0 + y_0})^r, \qquad z_1 = \frac{x_1 + y_1}{(x_0 + y_0)^{r+1}},$$

as

$$z_1 + z_1^r = 1 + z_0 + z_0^r,$$

so that $w = z_0 + z_1$ satisfies $w^r + w + 1 = 0$. Raising this equality to the power r, we obtain $w^2 + w^r + 1 = 0$, which combines to $w + w^2 = 0$. However, then $w \in \{0, 1\}$ and clearly $w^r + w + 1 = 1 \neq 0$. The claim is proved.

This ovoid O is called the *Suzuki-Tits ovoid* (also sometimes the Suzuki ovoid, or the Tits ovoid). When q is even and $q \leq 32$, then it is known that the only ovoids (up to a collineation) of $\mathsf{Sp}_4(q)$ are the classical and Suzuki-Tits ovoids (only appearing for $q = 8, 32$; for $q = 2$, the classical ovoid and the Suzuki-Tits ovoid are equivalent). For $q = 2, 4$, this is folklore; for $q = 8$ this was first proved by FELLEGARA [317]; for $q = 16$, see O'KEEFE & PENTTILA [594]; for $q = 32$, see O'KEEFE, PENTTILA & ROYLE [595].

The Suzuki-Tits ovoid of $\mathsf{Sp}_4(2^{2e-1})$ can also be constructed as follows. It is well-known that, as a(n abstract) rank 2 geometry, $\mathsf{Sp}_4(2^{2e-1})$ is a self-dual geometry which even admits a (unique up to conjugacy) *polarity*, i.e., a permutation of order 2 of the union of the point and the line set interchanging the points with the lines, and preserving incidence. The set of points which are incident with their image is precisely a Suzuki-Tits ovoid, see TITS [692, 693].

2.5.6 Maximal cocliques

In $\mathsf{Sp}_{2n}(q)$, the smallest maximal cocliques are the hyperbolic lines (of size $q+1$).

In $\mathsf{Sp}_4(q)$, q odd, a coclique (partial ovoid) has size at most q^2-q+1 (TALLINI [675]). For $q = 3, 5, 7$ the largest partial ovoids have size 7, 18, 33 (CIMRÁKOVÁ & FACK [197]). Upper bounds for the size of cocliques in $\mathsf{Sp}_{2n}(q)$, $n \geq 3$, have been given by THAS [685], DYE [301], and DE BEULE et al. [255] (Theorem 6.1).

The following proposition, derived from Section 6 of DE BEULE et al. [255], summarizes the best bounds at present.

Proposition 2.5.4 *A coclique of* $\Gamma(\mathsf{Sp}_{2n}(q))$, $n \geq 3$, *has at most*

- $2n+1$ *vertices,* $q = 2$;
- $15 \cdot 2^{n-3} - 2$ *vertices,* $q = 3$;
- $\dfrac{q(q-1)^{n-3}-2}{q-2} + \dfrac{1}{2}q(q-1)^{n-3}(\sqrt{5q^4+6q^3+7q^2+6q+1} - q^2 - q - 1)$ *vertices,* $q \geq 4$.

The first bound of the previous proposition is sharp. Indeed, we inductively define a coclique of size $2n+1$ in $\mathsf{Sp}_{2n}(2)$, $n \geq 2$, as follows. For $n = 2$, it is just an ovoid (as for instance constructed in Proposition 2.5.3). Now let $n \geq 3$. Let $\{x_0, x_1, x_2\}$ be a hyperbolic line in $\mathsf{P}V$. The definition of embedded polar space yields $x_1^\perp \cap x_0^\perp = x_2^\perp \cap x_0^\perp$. Hence on each line L through x_0 there is a unique point $x_L \notin (x_1^\perp \cup x_2^\perp)$. Let $\mathscr{C}'$ be a coclique of size $2n-1$ of the symplectic polar space $x_1^\perp \cap x_2^\perp$, then $\mathscr{C}' \subseteq x_0^\perp$, and

$$\mathscr{C} = \{x_1, x_2\} \cup \{x_L \mid x_0 \in L \text{ and } L \cap \mathscr{C}' \neq \emptyset\}$$

is a coclique of $\mathsf{Sp}_{2n}(2)$ of size $2n+1$.

The case $(n, q) = (3, 3)$ of the second bound is exact (namely 13). It is not known whether the other bounds are sharp, but presumably they are not.

2.5.7 h-Ovoids

As we saw, $\mathsf{Sp}_{2n}(q)$ has 1-ovoids (i.e., ovoids) if and only if $n = 2$ and q is even. For odd q there is a partition of $\mathsf{Sp}_4(q)$ into 2-ovoids ([43], Cor. 5.2), and there exist many $\frac{1}{2}(q+1)$-ovoids ([227]). For even q and $n = 2$ there are h-ovoids for all h, $1 \leq h \leq q$ ([227]). See also [321] for examples in spaces of larger rank. Note that since h-ovoids are regular sets in $\Gamma(\mathsf{Sp}_{2n}(q))$ and the point neighborhoods are the hyperplanes, these h-ovoids are also two-character sets in $\mathsf{PG}(2n-1, q)$. By Theorem 13 of [42], no h-ovoids of $\mathsf{Sp}_{2n}(q)$, $n \geq 3$, exist for $1 \leq h \leq (-3 + \sqrt{9+4q^n})/(2q-2)$.

2.5.8 Spreads

Dye [300] showed that the $\mathsf{Sp}_{2n}(q)$ polar space has spreads, partitions of the point set into $q^n + 1$ pairwise disjoint t.i. subspaces. For $\Gamma = \Gamma(\mathsf{Sp}_{2n}(q))$ this means that its complement has chromatic number $\chi(\overline{\Gamma}) = q^n + 1$.

2.5.9 Tight sets

Since $\mathsf{Sp}_{2n}(q)$ has spreads, it has i-tight sets for all $i \in \{1, 2, \ldots, q^n\}$.

It is possible to prove for small i that an i-tight set must be the union of some specific examples. If q is a square and $i < 1 + q^{5/8}/\sqrt{2}$ (De Beule et al. [254]), or $q \geq 81$ is an odd square and $i < (q^{2/3} - 1)/2$ (Nakić & Storme [584]), an i-tight subset X of $\mathsf{Sp}_{2n}(q)$ must be the disjoint union of pairwise disjoint subspaces $\mathsf{PG}(n-1, q)$ and Baer subgeometries $\mathsf{PG}(2n-1, \sqrt{q})$. More precisely, X must be a disjoint union of some of the examples (i)-(iv) below.

(i) A maximal t.i. subspace ($i = 1$).

(ii) The union $W \cup W^\perp$ of a conjugate pair of nondegenerate n-spaces ($i = 2$).

(iii) The point set Z of a Baer subgeometry $\mathsf{Sp}_{2n}(\sqrt{q})$ invariant for the symplectic polarity ($i = \sqrt{q}+1$).

(iv) The union $Z_1 \cup Z_2$ of two disjoint Baer subgeometries $\mathsf{Sp}_{2n}(\sqrt{q})$ conjugate under the symplectic polarity ($i = 2\sqrt{q}+2$).

Two Baer subgeometries Z and Z' are called conjugate under the symplectic polarity when for each $z \in Z$ the hyperplane $z^\perp$ meets Z' in a $\mathsf{PG}(2n-2, \sqrt{q})$.

Further examples can be constructed as follows. Let $\mathsf{Sp}_{2n}(\sqrt{q})$ be naturally embedded in $\mathsf{Sp}_{2n}(q) = (X, \Omega)$. Let X' be the set of points of $\mathsf{Sp}_{2n}(\sqrt{q})$. Each point of $X \setminus X'$ is contained in a unique line meeting X' in $\sqrt{q}+1$ points. Let X'' (resp. X''') be the set of points of $X \setminus X'$ where this line is totally isotropic (resp. hyperbolic). Then Theorem 8 of [42] asserts that each of X', X'', X''' is tight. (They are i-tight for $i = \sqrt{q}+1$, $i = \sqrt{q}(q^{n-1}-1)$, and $i = q^{n-1}(q-\sqrt{q})$, respectively.)

Example 4 of Section 8 of [43] yields i-tight sets in $\mathsf{Sp}_4(7)$, with $i = 5, 15$ (and $i = 35, 45$ for the complementary set), containing no singular line. With the notation of §10.89C, the 15-tight set is $\mathscr{P}_0$. The 5-tight set is obtained as the union of the four vertices of 10 quadrangles; one typical quadrangle has sides corresponding to the points $(1,2,4,0,0,0,0)$, $(1,4,2,0,0,0,0)$, $(0,0,0,1,2,4,0)$ and $(0,0,0,1,4,2,0)$ (still using the notation of §10.89C), the other quadrangles are obtained by letting S_6 act on the first six coordinates.

In the same vein, §10.89B provides a 10-tight set in $\mathsf{Sp}_4(7)$ not containing any singular line by taking the orbit of size 80 in $\mathsf{PG}(3,7)$ of the group $2^4 : \mathsf{S}_5$. The fact that this is a tight set in $\mathsf{Sp}_4(7)$ follows, with the notation of §10.89B, from the fact that $2^4 : \mathsf{S}_5$ fixes the point $(00000; 1)$ and hence also its perp, which defines $\mathsf{Sp}_4(7)$.

Other constructions of tight sets in symplectic polar spaces are contained in [231]. For q even, we also refer to the tight sets mentioned on p. 71 for parabolic polar spaces.

2.5.10 Local graph

Suppose f is nondegenerate. If U is totally isotropic then f induces a nondegenerate symplectic form f_U on $U^\perp/U$ given by $f_U(v+U, w+U) = f(v,w)$. In particular, for $\mathsf{Sp}_{2n}(q)$, if x is a point of PV, then $x^\perp/x$ carries the structure of $\mathsf{Sp}_{2n-2}(q)$. This means that $\Gamma(\mathsf{Sp}_{2n}(q))$ is locally the q-clique extension of $\Gamma(\mathsf{Sp}_{2n-2}(q))$.

2.6 Orthogonal polar spaces

We review some properties of the strongly regular graph defined by the points of a finite orthogonal polar space, adjacent when collinear. We pay special attention to (maximal) cliques and cocliques, regular sets and geometric notions in the corresponding polar space such as h-ovoids and spreads.

2.6.1 Quadratic forms and orthogonal polar spaces

Quadratic forms

Let V be a vector space over a field F. A quadratic form is a map $Q\colon V \to F$ satisfying the two conditions

- $Q(\lambda v) = \lambda^2 Q(v)$, for all $v \in V$ and all $\lambda \in F$;
- the (symmetric) form $f_Q\colon V \times V \to F$ defined by $(v, w) \mapsto f_Q(v, w) = Q(v + w) - Q(v) - Q(w)$ is bilinear.

If Q is a quadratic form, we call (V, Q) a *quadratic space*. The quadratic form Q is called *anisotropic* on a subspace U of V if $Q(u) = 0$ for $u \in U$ only if $u = 0$. If S is a subset of V, then we denote by $S^\perp$ the subspace of V consisting of all vectors v for which $f_Q(s, v) = 0$ for all $s \in S$. The *radical* Rad V of Q is the subspace $V^\perp$. The quadratic form is called *nondegenerate* if Q is anisotropic on the radical Rad V. We then say that (V, Q) is a *nondegenerate* quadratic space.

A subspace W of V is called *totally singular* when Q vanishes identically on W. The *Witt index* is the dimension of a maximum totally singular subspace. The set of totally singular 1-spaces, also called the null set of Q, is a *quadric* in $\mathsf{P}V$.

Orthogonal polar spaces

Let V be a vector space over F and let (V, Q) be a quadratic space. Let X be the set of totally singular 1-spaces of V and let Ω be the set of maximal totally singular subspaces in V with respect to Q. If X spans V, then (X, Ω) is a polar space embedded in $\mathsf{P}V$, called an *orthogonal* polar space. Moreover, the radical R of (X, Ω) coincides with the intersection $X \cap \text{Rad } V$ of the radical of Q with the set of totally singular 1-spaces. Hence (X, Ω) is nondegenerate precisely when Q is nondegenerate. We have $\langle X \rangle = V$ when either $\Omega \neq \{R\}$, or $V = R$. Two points $\langle v \rangle$ and $\langle w \rangle$ of (X, Ω) are collinear if and only if $f_Q(v, w) = 0$.

We have the following reduction theorem for nondegenerate quadratic forms.

Theorem 2.6.1 *Let (V, Q) be a nondegenerate quadratic space with finite Witt index n. Then V admits a direct sum decomposition $V = V_0 \oplus V_1$ such that $\dim V_0 = 2n$, and there exists a basis $E = \{e_{-n}, e_{-n+1}, \ldots, e_{-1}, e_1, e_2, \ldots, e_n\}$ of V_0 such that Q is given by*

$$Q(\sum_{i=1}^{n}(x_{-i}e_{-i} + x_i e_i) + v_1) = x_{-n}x_n + \cdots + x_{-2}x_2 + x_{-1}x_1 + Q(v_1),$$

with Q anisotropic on V_1.

Proof. The 1-spaces $\langle e_i \rangle$, $i \in \{-n, -n+1, \ldots, -1, 1, 2, \ldots, n\}$ correspond to points in two disjoint maximal singular subspaces, chosen in such a way that $e_i \in e_j^\perp$ if and only if $i + j \neq 0$. Set $V_0 = \langle E \rangle$, and set $V_1 = E^\perp$. It is routine to check that $V = V_0 \oplus V_1$ and an elementary calculation proves the last assertion. $\square$

2.6.2 Finite orthogonal polar spaces and graphs

By Theorem 2.6.1, the nondegenerate orthogonal polar spaces of rank n over a field F are classified by anisotropic quadratic forms over F. There are always two standard anisotropic quadratic forms which exist over any field: the trivial one (in a 0-dimensional vector space), and the form $Q : F \to F : x \mapsto x^2$.

If $F = \mathbb{F}_q$, then the fact that all quadratic field extensions are isomorphic yields that the embedded polar spaces arising from anisotropic quadratic forms in 2-dimensional vector spaces are isomorphic to each other. Since a quadratic field extension always exists, there exists, for any rank n, a nondegenerate orthogonal polar space in a $(2n+1)$-dimensional projective space over $\mathbb{F}_q$.

There is no anisotropic quadratic form Q on a vector space V of dimension at least 3 over $\mathbb{F}_q$. Indeed, let x, y, z be three vectors in V that are mutually orthogonal for f_Q. Then $Q(x + \lambda y + \mu z) = Q(x) + \lambda^2 Q(y) + \mu^2 Q(z)$. Now each of λ^2 and μ^2 takes at least $(q+1)/2$ values, so $Q(x) + \lambda^2 Q(y)$ and $-\mu^2 Q(z)$ have a common value, and for this λ, μ the point $x + \lambda y + \mu z$ is isotropic. (This also follows from Theorem 2.3.4.)

Hence there are exactly three cases: The trivial anisotropic quadratic form (*hyperbolic orthogonal polar spaces*, also said to be of type $+1$), the 1-dimensional one (*parabolic orthogonal polar spaces*), and the 2-dimensional one (*elliptic orthogonal polar spaces*, also said to be of type -1).

The *orthogonal graph* of (V, Q) is the collinearity graph $\Gamma = \Gamma(X, \Omega)$ of (X, Ω) and thus has as vertex set the set X of points, where distinct vertices $\langle u \rangle$ and $\langle v \rangle$ are adjacent when $f(u, v) = 0$. Note that, as before, the condition $f(u, v) = 0$ does not depend on the choice of u and v in $\langle u \rangle$ and $\langle v \rangle$, and it is obviously symmetric since $f(u, v) = f(v, u)$. For Witt index at most 1, the graph Γ has no edges.

2.6.3 Parameters

If (V, Q) has finite Witt index n, and V is defined over $\mathbb{F}_q$, and Q is nondegenerate, then $\dim V \in \{2n, 2n+1, 2n+2\}$ and the corresponding quadric is denoted by $\mathsf{O}^+_{2n}(q)$, $\mathsf{O}_{2n+1}(q)$ and $\mathsf{O}^-_{2n+2}(q)$, respectively. (In the literature one also finds $\mathsf{Q}^+_{2n-1}(q)$, $\mathsf{Q}_{2n}(q)$ and $\mathsf{Q}^-_{2n+1}(q)$.) If the Witt index is at least 2, then the corresponding embedded polar space has order $(q, 1)$, (q, q), (q, q^2), respectively.

For $n \geq 2$, the orthogonal graphs are strongly regular and the parameters are given as in Theorem 2.2.12 with $(q, t) \in \{(q, 1), (q, q), (q, q^2)\}$:

- The hyperbolic orthogonal graph $\Gamma(\mathsf{O}^+_{2n}(q))$.

$$
\begin{aligned}
v &= (q^n - 1)(q^{n-1} + 1)/(q - 1), \\
k &= q(q^{n-1} - 1)(q^{n-2} + 1)/(q - 1), \\
\lambda &= q^2(q^{n-2} - 1)(q^{n-3} + 1)/(q - 1) + q - 1, \\
\mu &= (q^{n-1} - 1)(q^{n-2} + 1)/(q - 1),
\end{aligned}
$$

  so that $\mu = k/q$. The eigenvalues are k, $-1 + q^{n-1}$ and $-1 - q^{n-2}$ with multiplicities 1, $f = \frac{q(q^n-1)(q^{n-2}+1)}{q^2-1}$ and $g = \frac{q^2(q^{2n-2}-1)}{q^2-1}$, respectively.

- The parabolic orthogonal graph $\Gamma(\mathsf{O}_{2n+1}(q))$. Here we find the same parameters as for $\Gamma(\mathsf{Sp}_{2n}(q))$ but the graphs are not isomorphic in characteristic different from 2; they are isomorphic in characteristic 2.

$$\begin{aligned} v &= (q^{2n}-1)/(q-1), \\ k &= q(q^{2n-2}-1)/(q-1), \\ \lambda &= q^2(q^{2n-4}-1)/(q-1)+q-1, \\ \mu &= (q^{2n-2}-1)/(q-1), \end{aligned}$$

  so that $v-k-1=q^{2n-1}$, $\lambda=\mu-2$ and $\mu=k/q$. The eigenvalues are k and $-1\pm q^{n-1}$ with multiplicities 1, $f=\frac{1}{2}(\frac{q^{2n}-q}{q-1}+q^n)$, $g=\frac{1}{2}(\frac{q^{2n}-q}{q-1}-q^n)$.

- The elliptic orthogonal graph $\Gamma(\mathsf{O}^-_{2n+2}(q))$.

$$\begin{aligned} v &= (q^n-1)(q^{n+1}+1)/(q-1), \\ k &= q(q^{n-1}-1)(q^n+1)/(q-1), \\ \lambda &= q^2(q^{n-2}-1)(q^{n-1}+1)/(q-1)+q-1, \\ \mu &= (q^{n-1}-1)(q^n+1)/(q-1), \end{aligned}$$

  so that $\mu=k/q$. The eigenvalues are k, $-1+q^{n-1}$ and $-1-q^n$ with multiplicities 1, $f=\frac{q^2(q^{2n}-1)}{q^2-1}$ and $g=\frac{q(q^{n-1}-1)(q^{n+1}+1)}{q^2-1}$, respectively.

- For convenience, we give combined expressions for $\Gamma(\mathsf{O}^\varepsilon_{2m}(q))$.

$$\begin{aligned} v &= (q^m-\varepsilon)(q^{m-1}+\varepsilon)/(q-1), \\ k &= q(q^{m-1}-\varepsilon)(q^{m-2}+\varepsilon)/(q-1), \\ \lambda &= q^2(q^{m-2}-\varepsilon)(q^{m-3}+\varepsilon)/(q-1)+q-1, \\ \mu &= (q^{m-1}-\varepsilon)(q^{m-2}+\varepsilon)/(q-1), \end{aligned}$$

  so that $v-k-1=q^{2m-2}$ and $\mu=k/q$. The eigenvalues are k, $-1+\varepsilon q^{m-1}$ and $-1-\varepsilon q^{m-2}$ with multiplicities 1, $\frac{q(q^m-\varepsilon)(q^{m-2}+\varepsilon)}{q^2-1}$ and $\frac{q^2(q^{2m-2}-1)}{q^2-1}$, respectively.

2.6.4 Isomorphisms

- As already mentioned, the graph $\Gamma(\mathsf{O}_{2n+1}(q))$ is isomorphic to $\Gamma(\mathsf{Sp}_{2n}(q))$ when q is even.
- The graph $\Gamma(\mathsf{O}_5(q))$ is isomorphic to the graph $\Delta(\mathsf{Sp}_4(q))$ of maximal singular subspaces of $\mathsf{Sp}_4(q)$, see §2.2.11.
- The graph $\Gamma(\mathsf{O}^-_6(q))$ is isomorphic to the graph $\Delta(\mathsf{U}_4(q))$ of maximal singular subspaces of $\mathsf{U}_4(q)$, see §2.2.11.
- The graph $\Gamma(\mathsf{O}^+_4(q))$ is isomorphic to the Hamming graph $H(2,q+1)$.
- The graph $\Gamma(\mathsf{O}_5(2))\cong\Gamma(\mathsf{Sp}_4(2))$ is isomorphic to the complement of the Johnson graph $J(6,2)$.
- The graph $\Gamma(\mathsf{O}^-_6(2))$ is isomorphic to the complement of the Schläfli graph (§10.10), or the complement of the collinearity graph of $\mathsf{E}_{6,1}(1)$ (§4.9).
- Since $\mathsf{O}^+_6(q)$ is the so-called *Klein quadric*, the graph $\Gamma(\mathsf{O}^+_6(q))$ is isomorphic to the Grassmann graph $J_q(4,2)$.

2.6.5 Automorphism groups

Let the *general orthogonal group* $\mathsf{GO}(V,Q)$ be the group of all linear transformations of V that preserve the nondegenerate quadratic form Q. Let (just here) $\mathsf{GGO}(V,Q)$ be the group of all linear transformations of V that preserve Q up to a constant.

The subgroup D of $\mathsf{GL}(V)$ consisting of all multiples of the identity acts trivially on $\mathsf{P}V$, and $D \cap \mathsf{GO}(V,Q) = \{\pm I\}$. Let the *projective general orthogonal group* $\mathsf{PGO}(V,Q)$ be the quotient $\mathsf{GO}(V,Q)/\{\pm I\}$, and let (just here) $\mathsf{PGGO}(V,Q) = \mathsf{GGO}(V,Q)/D$.

The full automorphism group of Γ and of the corresponding embedded polar space (X,Ω) is $\mathsf{P\Gamma O}(V,Q)$, that is, $\mathsf{PGGO}(V,Q)$ extended by the field automorphisms of the underlying field F, except if the corresponding embedded polar space is $\mathsf{O}^+(4,q)$, in which case $\mathrm{Aut}\,\Gamma$ is $\mathsf{S}_{q+1}\,\mathrm{wr}\,2$ (see §1.1.8).

If V and F are finite, say V is n-dimensional over $F = \mathbb{F}_q$, then we denote $\mathsf{GO}(V,Q)$ and $\mathsf{PGO}(V,Q)$ by

$$\begin{cases} \mathsf{GO}_n(q) \text{ and } \mathsf{PGO}_n(q) & \text{if } n \text{ is odd (and hence } (V,Q) \text{ is parabolic)},\\ \mathsf{GO}^-_n(q) \text{ and } \mathsf{PGO}^-_n(q) & \text{if } n \text{ is even and } (V,Q) \text{ is elliptic},\\ \mathsf{GO}^+_n(q) \text{ and } \mathsf{PGO}^+_n(q) & \text{if } n \text{ is even and } (V,Q) \text{ is hyperbolic.} \end{cases}$$

Unlike the symplectic case, the group $\mathsf{PGO}(V,Q)$ is in general not simple. One reason is that the determinant of an element of $\mathsf{GO}(V,Q)$ can be equal to -1. So let $\mathsf{SO}(V,Q)$ be the (normal) subgroup of $\mathsf{GO}(V,Q)$ of matrices with determinant 1, and let $\mathsf{PSO}(V,Q)$ be its quotient with the subgroup of scalar matrices it contains. In the finite case we also use the corresponding more specific (self-explaining) notation $\mathsf{SO}_n(q)$, $\mathsf{PSO}_n(q)$ (n odd), $\mathsf{SO}^-_n(q)$, $\mathsf{PSO}^-_n(q)$, $\mathsf{SO}^+_n(q)$ and $\mathsf{PSO}^+_n(q)$ (n even).

Now $\mathsf{PSO}_n(q)$ is simple if $(n,q) \neq (5,2)$, and it is denoted by $\mathsf{O}_n(q)$. If $n = (5,2)$, then $\mathsf{PSO}_5(2)$ is isomorphic to the symmetric group S_6. However, in the elliptic and hyperbolic cases, the groups $\mathsf{PSO}^-_n(q)$ and $\mathsf{PSO}^+_n(q)$ are generally not simple. Here, the reason is that hyperbolic polar spaces contain two systems of maximal singular subspaces, and the stabilizer of these systems in $\mathsf{PSO}^+_n(q)$ is a normal subgroup of index at most 2 (could be 1), which we denote by $\mathsf{O}^+_n(q)$ if $n \geq 6$. The latter is always simple; if $n = 4$, that normal subgroup is the direct product of two copies of $\mathsf{PSL}_2(q)$. In the elliptic polar space $\mathsf{O}^-_{2m}(q)$, there are likewise two systems of imaginary maximal singular subspaces (of dimension m) obtained after quadratically extending the field to $\mathbb{F}_{q^2}$. Again, the stabilizer of these systems in $\mathsf{PSO}^-_n(q)$, $n = 2m$, is a normal subgroup of index at most 2 (could be 1), which we denote by $\mathsf{O}^-_n(q)$. The latter is again always simple.

In all cases, except the case $\mathsf{O}^+_4(q)$, the simple group is the intersection of all groups acting rank 3 on the graph Γ, and hence that group and all larger groups in the full automorphism group act rank 3.

2.6.6 Maximal cliques

As remarked in §2.2.7, the maximal cliques of Γ are the maximal totally singular subspaces of (V,Q), i.e., the elements of Ω. In the finite nondegenerate cases $\mathsf{O}^+_{2n}(q)$, $\mathsf{O}_{2n+1}(q)$, and $\mathsf{O}^-_{2n+2}(q)$, these have dimension n and size $(q^n-1)/(q-1)$. The maximal cliques form a single orbit under $\mathrm{Aut}\,\Gamma$.

2.6.7 Ovoids and maximal cocliques

Recall that an *ovoid* in a nondegenerate orthogonal polar space is a set of points that meets every maximal totally singular subspace in precisely one point. Also recall that ovoids (when they exist) are maximal cocliques of the corresponding graph. For $\dim V = n+m+1$, where n is the Witt index and $m \in \{n-1, n, n+1\}$, one has $|C| \le q^m + 1$ and $|O| = q^m + 1$ for each coclique C and ovoid O. There are ovoids in $\mathsf{O}_4^+(q)$, in $\mathsf{O}_6^+(q)$, in $\mathsf{O}_8^+(q)$ for q even, q an odd prime, or $q \equiv 2 \pmod 3$), in $\mathsf{O}_5(q)$, and in $\mathsf{O}_7(q)$, $q = 3^h$. Not in $\mathsf{O}_{2n+1}(q)$, $n > 2$, q even or prime $\ne 3$, or $n > 3$, $q = 5^e$; not in $\mathsf{O}_{2n+2}^-(q)$, $n > 1$; not in $\mathsf{O}_{2n}^+(q)$, $n > 4$, $q = 2^e, 3^e$ or $n > 5$, $q = 5^e, 7^e$. (Thas [683], Kantor [478], Conway et al. [216], Shult [652], Blokhuis & Moorhouse [82].)

No ovoids are known in finite embedded polar spaces of rank at least 5, and we can conjecture there are none. It seems hard to prove this conjecture, but many partial results exist. Below we discuss some details.

A bound on the size of caps

Let $q = p^e$ where p is prime, and let A be the point-hyperplane incidence matrix of $\mathsf{PG}(d,q)$. Then $\mathrm{rk}_p A = \binom{p+d-1}{d}^e + 1$. Let Q be a nondegenerate quadratic form on this projective space. It induces a partition $A = \left(\begin{smallmatrix} B & C \\ C^\top & D \end{smallmatrix}\right)$ with the rows partitioned into those for singular and nonsingular points and columns ordered like the rows, with column $x^\perp$ corresponding to row x, so that A is a symmetric matrix, and B is the collinearity matrix of the polar space on the quadric defined by Q. Blokhuis & Moorhouse [82] show that $\mathrm{rk}_p(B\ C) = (\binom{p+d-1}{d} - \binom{p+d-3}{d})^e + 1$. Let a *cap* or *partial ovoid* be a coclique in the collinearity graph on Q. If K is a cap, then the corresponding rows and columns induce an identity submatrix of the matrix B, so that $|K| \le \mathrm{rk}_p(B\ C)$. Subsequently, Arslan & Sin [11] determined the precise value of $\mathrm{rk}_p B$. Often this equals $\mathrm{rk}_p(B\ C)$, sometimes it is slightly smaller.

Theorem 2.6.2 (Blokhuis & Moorhouse [82]) *Let K be a coclique in the collinearity graph of the polar space on $\mathsf{PG}(d,q)$ provided with nondegenerate quadric. Let $q = p^e$. Then $|K| \le (\binom{p+d-1}{d} - \binom{p+d-3}{d})^e + 1$.* □

Theorem 2.6.3 (Arslan & Sin [11]) *In the above situation, let $n = d+1$.*

(i) Let $p = 2$. If n is even, $\mathrm{rk}_p B = n^e + 1$. *If n is odd,* $\mathrm{rk}_p B = (n-1)^e + 1$.

(ii) Let $p > 2$. If there exists a positive integer a such that $a+1 \equiv n \pmod 2$ and $n-3 \le ap \le n+p-5$, then $\mathrm{rk}_p B = (\binom{p+n-2}{n-1} - \binom{p+n-4}{n-1} - \binom{ap+2}{n-1} + \binom{ap}{n-1})^e + 1$, *otherwise* $\mathrm{rk}_p B = (\binom{p+n-2}{n-1} - \binom{p+n-4}{n-1})^e + 1$.

In particular, if an ovoid would have size $q^m + 1$, then an ovoid can exist only when $p^m \le \binom{p+d-1}{d} - \binom{p+d-3}{d}$. For example, this shows that $\mathsf{O}_9(5^e)$ does not have ovoids. This bound on the size of caps is sometimes tight for $q = 2$, see §3.6.

Ovoids in elliptic polar spaces

Proposition 2.6.4 *The elliptic polar space $\mathsf{O}_{2n+2}^-(q)$, $n \ge 2$, q an arbitrary prime power, does not have an ovoid.*

Proof. By Corollary 2.2.16, it suffices to show that the generalized quadrangle $\mathsf{O}_6^-(q)$ (of order (q, q^2)) does not admit an ovoid. But that is a special case of the following proposition. □

Proposition 2.6.5 *Suppose a generalized quadrangle* $\mathsf{GQ}(s,t)$ *has an ovoid (i.e., a coclique of size* $1+st$*). Then* $t \le s(s-1)$ *or* $s = 1$*.*

Proof. The collinearity graph has eigenvalues $s(t+1)$, $s-1$ and $-t-1$ with multiplicities 1, $s(s+1)t(t+1)/(s+t)$ and $s^2(st+1)/(s+t)$, respectively. By the Cvetković bound, the size of a coclique is at most the number of nonpositive eigenvalues, so if $s > 1$ then $st+1 \le s^2(st+1)/(s+t)$, i.e., $t \le s^2 - s$. □

Partial ovoids in elliptic polar spaces

Let $x_{n,q}$ be the maximal size of a partial ovoid of $\mathsf{O}_{2n+2}^-(q)$, and let $q = p^e$. Klein [491] showed that if $n \ge 2$ then $x_{n,q} - 2 \le \frac{q^n-1}{q^{n-1}-1}(x_{n-1,q} - 2)$. De Beule et al. [256] showed that $x_{2,q} \le \frac{1}{2}q(q^2+1)+1$. The bound from Theorem 2.6.3 is (for odd p) $x_{2,q} \le (\frac{1}{3}p(2p^2+1))^e + 1$, which is better for $e \ge 2$, and (for $p = 2$) $x_{2,q} \le 6^e + 1$, which is better for $e \ge 3$. Also for larger n and small p Theorem 2.6.3 is sometimes better.

For $(n,q) \in \{(2,2),(2,3)\}$ the bound is sharp: $x_{2,2} = 6$ and $x_{2,3} = 16$.

If $(n,q) = (2,2)$, then the graph $\Gamma(\mathsf{O}_6^-(2))$ is the complement of the Schläfli graph (§10.10) and a maximum coclique has 6 vertices.

If $(n,q) = (2,3)$, let $Q(x) = \sum_{i=1}^6 x_i^2$. The set of 16 isotropic points without zero coordinate and with an even number of 2's is a coclique. See also Ebert & Hirschfeld [303].

The value of $x_{n,q}$ is known exactly for $q = 2$, see §3.6.

Ovoids in parabolic polar spaces

For finite parabolic polar spaces, the situation concerning existence of ovoids is still satisfying, although not as straight as for the elliptic case. We start with some constructions.

Proposition 2.6.6 *The generalized quadrangle* $\mathsf{O}_5(q)$ *has ovoids for any prime power* q*.*

Proof. The intersection of $\mathsf{O}_5(q)$ with a hyperplane that contains no lines of $\mathsf{O}_5(q)$ is an ovoid. More concretely, let $\mathsf{O}_5(q)$ be given in $\mathsf{P}V$, with V 5-dimensional over $\mathbb{F}_q$, by the equation $X_1X_2 + X_3X_4 = X_0^2$. Let $x^2 - tx + n$ be an irreducible quadratic polynomial over $\mathbb{F}_q$. Then the hyperplane of $\mathsf{P}V$ given by the equation $X_4 = tX_0 - nX_3$ intersects $\mathsf{O}_5(q)$ in an elliptic quadric (with equation $X_1X_2 = X_0^2 - tX_0X_3 + nX_3^2$), which contains q^2+1 points and does not contain any pair of collinear points. □

For q a power of 2, the quadrangles $\mathsf{O}_5(q)$ and $\mathsf{Sp}_4(q)$ are isomorphic and the above construction is equivalent to the one in Proposition 2.5.3. Also, because of that isomorphism, $\mathsf{O}_5(q)$ is self dual. It is self polar (meaning it admits a polarity) if and only if $q = 2^{2e-1}$, for some positive integer e (see Tits [693]). Hence $\mathsf{O}_5(2^{2e-1})$ admits a second isomorphism class of ovoids, $e \ge 2$, namely the Suzuki-Tits ovoids.

Ball, Govaerts & Storme [36] prove that for q a prime, the ovoids constructed in Proposition 2.6.6 are unique, up to a collineation.

Proposition 2.6.7 *The parabolic polar space* $\mathsf{O}_7(3^e)$ *has ovoids for any integer* $e \geq 1$.

Proof. For $r = 3^e$, let $\mathsf{O}_7(r)$ be given in projective 6-space by the standard equation $x_0x_4 + x_1x_5 + x_2x_6 = x_3^2$, so that the corresponding bilinear form f on V is given by

$$f((x_0, x_1, \ldots, x_6), (y_0, y_1, \ldots, y_6)) = x_0y_4+x_1y_5+x_2y_6+x_3y_3+x_4y_0+x_5y_1+x_6y_2.$$

Let $\gamma \in \mathbb{F}_r$ be an arbitrary nonsquare. Let $P(x, y, z)$ denote the point with coordinates

$$(z^2 - \gamma^{-1}(\gamma x^2 - y^2)^2,\ x,\ y,\ z,\ 1,\ \gamma x^3 - xy^2 - yz,\ \gamma^{-1}y^3 + xz - x^2y),$$

$x, y, z \in \mathbb{F}_r$. Set $P(\infty) = (1, 0, 0, 0, 0, 0, 0)$. We claim that

$$O_\gamma = \{P(\infty)\} \cup \{P(x, y, z) \mid x, y, z \in \mathbb{F}_r\}$$

is an ovoid of $\mathsf{O}_7(r)$.

Clearly $P(\infty)$ is not collinear to any other point of O_γ. Now assume for a contradiction that $P(x, y, z)$ and $P(u, v, w)$ are collinear points of $\mathsf{O}_7(r)$. Then, using the bilinear form f given above, one calculates that

$$-\gamma^{-1}(\gamma(x - u)^2 - (y - v)^2)^2 + (z - w - xv + yu)^2 = 0,$$

which contradicts γ being a nonsquare in $\mathbb{F}_r$.

Hence O is a coclique, and since $|O| = 1 + r^3$, it is an ovoid. □

One might wonder where the algebraic construction in the above proof comes from. It has to do with the existence of a generalized hexagon, called the split Cayley hexagon $\mathsf{G}_2(q)$, whose points are all the points of $\mathsf{O}_7(q)$ and whose lines are some lines on the quadric (see Tits [691] and §4.8). The lines of $\mathsf{G}_2(q)$ in an elliptic hyperplane of $\mathsf{O}_7(q)$ constitute a *spread* of $\mathsf{G}_2(q)$, which is a set of $q^3 + 1$ lines, pairwise opposite in both $\mathsf{G}_2(q)$ and $\mathsf{O}_7(q)$. (This spread is called a Hermitian spread.) If q is a power of 3, then $\mathsf{G}_2(q)$ is a self-dual geometry and a duality takes this spread to an ovoid of both $\mathsf{G}_2(q)$ and $\mathsf{O}_7(q)$ (where an ovoid in a generalized hexagon is defined to be a set of points such that every point is equal to or collinear with exactly one point of the ovoid).

Just like $\mathsf{O}_5(2^{2e-1})$, $e \geq 2$, admits Suzuki-Tits ovoids, $\mathsf{O}_7(3^{2e-1})$, $e \geq 2$, admits a second isomorphism type of ovoids, called the *Ree-Tits ovoids*. These arise as the set of points incident with their image under a polarity of the generalized hexagon $\mathsf{G}_2(q)$ mentioned in the previous paragraph and embedded in $\mathsf{O}_7(3^{2e-1})$. An explicit coordinate description of a Ree-Tits ovoid can be found in Section 9.2.4 of Thas, Thas & Van Maldeghem [687].

From the isomorphism $\Gamma(\mathsf{O}_{2n+1}(q)) \cong \Gamma(\mathsf{Sp}_{2n}(q))$ we deduce with Proposition 2.5.2:

Proposition 2.6.8 *The parabolic polar space* $\mathsf{O}_{2n+1}(2^e)$, $n \geq 3$, *has no ovoids for any integer* $e \geq 1$. □

For odd q not a prime power of 3, the situation is that no ovoids are known, but only for q a prime there is a nonexistence proof. We will not reproduce that proof here; it goes in two steps. First, O'KEEFE & THAS [596] show that, if $\mathsf{O}_5(q)$ only admits ovoids equivalent to the classical one (given in the proof of Proposition 2.6.6), then $\mathsf{O}_7(q)$ does not admit any ovoid at all. Then we can use the result by BALL, GOVAERTS & STORME [36] mentioned above to conclude the following proposition.

Proposition 2.6.9 *The parabolic polar space $\mathsf{O}_{2n+1}(p)$, $n \geq 3$, has no ovoids for any prime $p > 3$.* □

Now we consider the case of rank at least 4. If q is even, then no ovoids exist by Proposition 2.6.8. But also for odd q, $\mathsf{O}_{2n+1}(q)$ does not admit any ovoid if $n \geq 4$.

Proposition 2.6.10 (GUNAWARDENA & MOORHOUSE [371]) *For $n \geq 4$ and any prime power q, the parabolic polar space $\mathsf{O}_{2n+1}(q)$ has no ovoids.*

Proof. By Corollary 2.2.16 it suffices to prove this for $n = 4$ and by Proposition 2.6.8 we may assume q odd. Assume for a contradiction that O is an ovoid of the polar space $\mathsf{O}_9(q)$. Pick a point $p \in O$. Let $X = O \setminus \{p\}$ and define a symmetric relation $\sim$ on X by $x \sim x'$ if $p^\perp \cap x^\perp \cap x'^\perp$ is a hyperbolic quadric (the only alternative is an elliptic quadric since p, x, x' are pairwise noncollinear). We show that $(X, \sim)$ is a strongly regular graph.

First, we claim that $(X, \sim)$ is regular with degree $\frac{1}{2}(q^3+1)(q-1)$. Indeed, fix a point $x \in X$ and let k be its degree. We count the pairs (u, y), with $u \perp y$, $u \in p^\perp \cap x^\perp$, and $y \in X \setminus \{x\}$. For u we have $q^5+q^4+q^3+q^2+q+1$ choices (the number of points of $\mathsf{O}_7(q)$), while for given u, there are q^3+1 members of O collinear to u (among which p and x). Hence there are $(q^6-1)(q^2+q+1)$ pairs (u, y) as described. Now, there are k choices for $y \sim x$ and q^4-1-k for $y \in X$ not adjacent to x. If $y \sim x$, then there are $(q^2+1)(q^2+q+1)$ points $u \in p^\perp \cap x^\perp \cap y^\perp$; otherwise this number is $(q+1)(q^3+1)$. Hence

$$(q^6-1)(q^2+q+1) = k(q^2+1)(q^2+q+1) + (q^4-1-k)(q+1)(q^3+1).$$

It follows that $k = \frac{1}{2}(q^3+1)(q-1)$. Hence $(X, \sim)$ is regular.

Now let (V, f) be the associated orthogonal space of $\mathsf{O}_9(q)$. If $v_1, v_2, v_3 \in V$ are three pairwise non-conjugate isotropic vectors, and if p_1, p_2, p_3 are the associated points in $\mathsf{P}V$, then the type of $p_1^\perp \cap p_2^\perp \cap p_3^\perp$ only depends on the quadratic residue class of $n(v_1, v_2, v_3) := f(v_1, v_2)f(v_2, v_3)f(v_3, v_1)$. If $p_1, p_2, p_3, p_4 \in O$, and $p_4 = \langle v_4 \rangle$, then

$$n(v_1, v_2, v_3)n(v_1, v_2, v_4)n(v_1, v_3, v_4)n(v_2, v_3, v_4) = \left[\prod_{1 \leq i < j \leq 4} f(v_i, v_j)\right]^2$$

is a square and hence an even number of triples from $\{p_1, p_2, p_3, p_4\}$ are orthogonal to a hyperbolic quadric. So we obtain a two-graph Γ with vertex set O and triples defining hyperbolic quadrics. The descendant Γ_p is precisely $(X, \sim)$. Since $(X, \sim)$ is regular, so is Γ, and hence, by §1.1.12, $(X, \sim)$ is strongly regular with parameters

$$(q^4,\ \frac{1}{2}(q^3+1)(q-1),\ \frac{1}{4}(q^4-3q^3+3q-5),\ \frac{1}{4}(q^3+1)(q-1)).$$

One now easily calculates $r = \frac{1}{2}(q-1)$ and $s = -\frac{1}{2}(q^3+1)$. It follows that $g = q^2(q^2-1)/(q^2+1)$, which is never an integer. □

Partial ovoids in parabolic polar spaces

In the cases where it is known that no ovoid exist, there are usually better upper bounds for the coclique number of the corresponding graph than just the ovoid number minus one. Note first that, if q is even, then $\Gamma(\mathsf{O}_{2n+1}(q))$ is isomorphic to $\Gamma(\mathsf{Sp}_{2n}(q))$, and hence Proposition 2.5.4 applies.

Now let q be odd. Then we know that there are no ovoids if either $n \geq 4$, or if $n = 3$ and $q > 3$, q prime. The following two results are proved by De Beule et al. [255].

Proposition 2.6.11 *A coclique of the parabolic polar space graph* $\Gamma(\mathsf{O}_{2n+1}(q))$*, $n \geq 4$, q odd, q not a prime, has at most*

$$q^n - q^{n-\frac{5}{2}} - q^{n-4} + 1$$

vertices.

For primes q there is a better bound, at least, when $q \geq 17$:

Proposition 2.6.12 *A coclique of the parabolic polar space graph* $\Gamma(\mathsf{O}_{2n+1}(q))$*, $n \geq 3$, $q \geq 17$ a prime, has at most*

$$q^n - 2q^{n-2} + 1$$

vertices.

For $q \in \{5, 7, 11, 13\}$, we only have the trivial upper bound q^3 for the size of a partial ovoid of $\mathsf{O}_7(q)$.

From Theorem 2.6.2 we obtain a bound for $\mathsf{O}_9(q)$ when q is a power of 5.

Proposition 2.6.13 *Let K be a coclique of the parabolic polar space graph $\Gamma(\mathsf{O}_{2n+1}(q))$, where $q = 5^e$, e a natural number, and $n \geq 4$. Then*
(i) $|K| \leq q^n \cdot \left(\frac{18}{25}\right)^e + 1$,
(ii) $|K| \leq \left(\frac{1}{6}\, n(n+1)(2n+1)(2n+7)\right)^e + 1$.

Proof. (i) Use Theorem 2.6.2 for $n = 4$ and apply Proposition 2.2.15. (ii) Use Theorem 2.6.2 directly. □

h-Ovoids in $\mathsf{O}_5(q)$

For each odd prime power q there is an h-ovoid in $\mathsf{O}_5(q)$ with $h = (q-1)/2$, see [42], [319], [320].

Ovoids in hyperbolic polar spaces

We start with ranks 2 and 3, where ovoids always exist.

Proposition 2.6.14 *The hyperbolic polar spaces $\mathsf{O}_4^+(q)$ and $\mathsf{O}_6^+(q)$ always admit ovoids, for each prime power q.*

Proof. Since the hyperbolic polar space $\mathsf{O}_4^+(q)$ is just the $(q+1)\times(q+1)$ grid, it has precisely $(q+1)!$ ovoids, namely all grid transversals.

There exists a solid (4-space) Σ intersecting $\mathsf{O}_6^+(q)$ in an elliptic quadric Q; then Q is an ovoid since every plane of $\mathsf{P}V$ intersects Σ nontrivially by a dimension argument. In fact, every spread of projective 3-space becomes under the Klein correspondence an ovoid of $\mathsf{O}_6^+(q)$ and vice versa (ovoids of $\mathsf{O}_6^+(q)$ and spreads of projective 3-space over $\mathbb{F}_q$ are equivalent objects). □

The construction in the beginning of the last paragraph of the previous proof can obviously be generalized as follows: If O is an ovoid of $\mathsf{O}_{2n+1}(q)$, $n \geq 2$, and we see $\mathsf{O}_{2n+1}(q)$ as a hyperplane section of $\mathsf{O}_{2n+2}^+(q)$, then O is an ovoid of $\mathsf{O}_{2n+2}^+(q)$. Applied to the case $n=3$, this gives us the following result.

Proposition 2.6.15 *The hyperbolic polar space $\mathsf{O}_8^+(3^e)$ has ovoids for each integer $e \geq 1$.*

Proof. This follows from the previous discussion and Proposition 2.6.7. □

However, there is no argument to make the converse of the preceding argument work, i.e., the fact that $\mathsf{O}_{2n+1}(q)$, $n \geq 2$, does not admit an ovoid does not guarantee that $\mathsf{O}_{2n+2}^+(q)$ has no ovoid. Here are some counterexamples.

Proposition 2.6.16 *The hyperbolic polar space $\mathsf{O}_8^+(q)$ has ovoids (i) for $q = p$, a prime, (ii) for prime powers q such that $q \equiv 2 \bmod 3$, (iii) for $q = 2^e$, $e \geq 1$.*

Proof. For the construction of ovoids of $\mathsf{O}_8^+(p)$, with p prime, see Conway, Kleidman & Wilson [216]. The construction uses the E_8 root lattice modulo 2 and 3, and the set of vectors of that lattice with norm p and $2p$, respectively (these two constructions are referred to as the *binary* and the *ternary* construction). Moorhouse [573] generalized this to a construction modulo r for arbitrary primes r (instead of 2 and 3). See also [572].

Let $q \equiv 2 \bmod 3$. We present an explicit construction of an ovoid in $\mathsf{O}_8^+(q)$.

First note that the condition $q \equiv 2 \bmod 3$ implies that $\mathbb{F}_q$ has no nontrivial cubic roots of unity. In particular, the quadratic polynomial $x^2 - x + 1$ is irreducible over $\mathbb{F}_q$. Let η and $\overline{\eta} := \eta^q$ be the roots of $x^2 - x + 1 = 0$ in $\mathbb{F}_{q^2}$. We shall from now on write x^q more compactly as $\overline{x}$, $x \in \mathbb{F}_{q^2}$.

Let the quadric $\mathsf{O}_8^+(q)$ be given by the equation $X_0X_1 + X_2X_3 + X_4X_5 + X_6X_7 = 0$. Consider the set of points $O = \{1,0,0,0,0,0,0,0)\} \cup \{P(a,b) \mid a \in \mathbb{F}_q, b \in \mathbb{F}_{q^2}\}$, where $P(a,b)$ is the point with coordinates

$$\begin{aligned}(9ab\overline{b} - 9a^2 - 3(b\overline{b})^2, 1, b+\overline{b}, -3a(\overline{\eta}b + \eta\overline{b}) + b\overline{b}(b + \overline{b} + \overline{\eta}b + \eta\overline{b}),\\ \overline{\eta}b + \eta\overline{b}, 3a(b+\overline{b}) + b\overline{b}(\overline{\eta}b + \eta\overline{b} - 2(b+\overline{b})), 3a, 3a - 3b\overline{b}).\end{aligned}$$

With an elementary calculation one verifies that $O \subseteq \mathsf{O}_8^+(q)$. Now, $|O| = q^3 + 1$ for if $P(a,b) = P(a',b')$, then the second last coordinate implies $a = a'$, and the third and fifth imply $b = b'$ since

$$\begin{vmatrix} 1 & 1 \\ \overline{\eta} & \eta \end{vmatrix} \neq 0.$$

Clearly $(1,0,0,0,0,0,0,0)$ is not collinear to $P(a,b)$, for any $a \in \mathbb{F}_q$ and any $b \in \mathbb{F}_{q^2}$. Now assume for a contradiction that $P(a,b)$ is collinear to $P(a',b')$,

$a, a' \in \mathbb{F}_q$, $b, b' \in \mathbb{F}_{q^2}$, $(a, b) \neq (a', b')$. After simplification, the algebraic condition expressing this is

$$\begin{aligned} 3(a-a')^2 - (a-a')[3(b\bar{b} - b'\bar{b}') + (\eta - \bar{\eta})(b\bar{b}' - \bar{b}b')] \\ -b\bar{b}(\bar{\eta}b\bar{b}' + \eta\bar{b}b' - b\bar{b}) - b'\bar{b}'(\bar{\eta}b'\bar{b} + \eta\bar{b}'b - b'\bar{b}') &= 0. \end{aligned}$$

The discriminant of this equation (viewing $a - a'$ as the unknown) is, after simplification, equal to $-3[(b-b')(\bar{b}-\bar{b}')]^2$. Since -3 is the discriminant of the equation $x^2 - x + 1 = 0$, which has no solution in $\mathbb{F}_q$, we see that -3 is not a square in $\mathbb{F}_q$. Hence $b = b'$. It then easily follows that $a = a'$, a contradiction. Hence O is an ovoid.

Finally, let $q = 2^e$, $e \geq 1$. We construct an ovoid in $\mathsf{O}_8^+(q)$, cf. Kantor [480].
For $x \in \mathbb{F}_{q^3}$, let $\bar{x} = x^q$ and $T(x) = x + \bar{x} + \bar{\bar{x}}$ and $N(x) = x\bar{x}\bar{\bar{x}}$.
Choose[4] $r \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$ so that $T(r) \neq 0$ and $T(r\bar{r}) = 0$.
For each $x \in \mathbb{F}_{q^3}$, define the following point $P(x)$:

$$(T(r), T(r)N(x), T(rx), T(r\bar{x}\bar{\bar{x}}), T(\bar{r}x), T(\bar{r}\bar{x}\bar{\bar{x}}), T(\bar{\bar{r}}x), T(\bar{\bar{r}}\bar{x}\bar{\bar{x}})).$$

An elementary calculation shows that $P(x)$ belongs to $\mathsf{O}_8^+(q)$. Now let $x, y \in \mathbb{F}_{q^3}$ with $x \neq y$. Then we show that $P(x)$ and $P(y)$ are noncollinear on $\mathsf{O}_8^+(q)$. In view of the equation of the quadric $\mathsf{O}_8^+(q)$ given above, $P(x)$ and $P(y)$ are noncollinear if and only if

$$\begin{aligned} 0 \neq\ & T(r)^2(N(x)+N(y)) + T(rx)T(r\bar{y}\bar{\bar{y}}) + T(ry)T(r\bar{x}\bar{\bar{x}}) \\ & + T(\bar{r}x)T(\bar{r}\bar{y}\bar{\bar{y}}) + T(\bar{r}y)T(\bar{r}\bar{x}\bar{\bar{x}}) + T(\bar{\bar{r}}x)T(\bar{\bar{r}}\bar{y}\bar{\bar{y}}) + T(\bar{\bar{r}}y)T(\bar{\bar{r}}\bar{x}\bar{\bar{x}}), \end{aligned}$$

which is easily seen to be equivalent to

$$0 \neq T(r)^2 N(x+y),$$

which is true. It follows that the set $\{P(x) \mid x \in \mathbb{F}_{q^3}\} \cup \{(0,1,0,0,0,0)\}$ is an ovoid of $\mathsf{O}_8^+(q)$. □

From Theorem 2.6.2 we get bounds for q a power of $2, 3, 5$ and 7.

Proposition 2.6.17 *Let $q = p^e$, where p is prime and e a natural number. Let K be a coclique of the hyperbolic polar space graph $\Gamma(\mathsf{O}_{2n}^+(q))$.*

(i) If $p = 2$, $n \geq 5$, then $|K| \leq (2n)^e + 1$.

(ii) If $p = 3$, $n \geq 5$, then $|K| \leq ((n+1)(2n-1))^e + 1$.

(iii) If $p = 5$, $n \geq 6$, then $|K| \leq \left(\frac{n+3}{2}\binom{2n+1}{3}\right)^e + 1$.

---

[4]Suppose for a contradiction that $T(x\bar{x}) = 0$ implies $T(x) = 0$. Let $y \in \mathbb{F}_{q^3}$ be such that $T(y) = 0$ and set $y' = y + \sqrt{T(y\bar{y})}$. Then $T(y\bar{y}) = 0$ and $T(y') = \sqrt{T(y\bar{y})}$, hence $\sqrt{T(y\bar{y})} = 0$ and the conditions $T(x\bar{x}) = 0$ and $T(x) = 0$ are equivalent. Choose a basis $\{1, u, v\}$ of $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$ so that $T(u) = T(v) = 0$ (obtained by possibly replacing u by $u + T(u)$ and v by $v + T(v)$). Let $a, b \in \mathbb{F}_q$ be arbitrary and set $w = au + bv$. Then $T(w) = 0$ and hence $T(w\bar{w}) = 0$. This easily implies $w^2 + w\bar{w} + \bar{w}^2 = 0$, hence $\bar{w} = \epsilon w$, with ϵ a nontrivial third root of unity (which must necessarily belong to $\mathbb{F}_q$). Then $w^3 = w(\epsilon w)(\epsilon^2 w) = N(w) \in \mathbb{F}_q$. Hence there are at least $q^2 - 1$ solutions of an equation $x^3 = c$, with $c \in \mathbb{F}_q \setminus \{0\}$. Since also 1 is such a solution, we have $q^2 \leq 3(q-1)$, a contradiction.

(iv) If $p = 7$, $n \geq 6$, then $|K| \leq \left(\frac{n+5}{3}\binom{2n+3}{5}\right)^e + 1$. □

For $q = 2$ the sizes of cocliques are given in §3.6. The previous proposition implies that $\mathsf{O}^+_{2n}(q)$ has no ovoids if $n \geq 5$ and $p \in \{2, 3\}$, and if $n \geq 6$ and $p \in \{5, 7\}$. Whilst Proposition 2.6.17 relies on an algebraic argument (p-ranks of matrices), BAMBERG, DE BEULE & IHRINGER [38] produce a particularly nice geometric argument to disprove the existence of ovoids in $\mathsf{O}^+_{10}(q)$ for q even. Note that, however, the proof of their Lemma 4.3 is incorrect; they will present a corrected version on arXiv. We here present another argument to bypass their Lemma 4.3.

Proposition 2.6.18 *No ovoids exist in* $\mathsf{O}^+_{10}(2^e)$, $e \geq 1$.

Proof. Let, for a contradiction, O be an ovoid of $\mathsf{O}^+_{10}(2^e)$. Select two points $x_1, x_2 \in O$ and consider the polar space with point set $X = x_1^\perp \cap x_2^\perp$. (This is isomorphic to $\mathsf{O}^+_8(2^e)$.) Consider three pairwise disjoint maximal t.s. subspaces W_1, W_2, W_3 in X. The map ρ taking each point $p_1 \in W_1$ to the hyperplane $p_2^\perp \cap W_1$, where $\{p_2\} = W_2 \cap (p_1^\perp \cap W_3)^\perp = W_2 \cap \langle p_1, p_1^\perp \cap W_3\rangle$, is easily checked to be a symplectic polarity of W_1. (It is obviously a duality every point of which is contained in its image; then by Lemma 3.2 of [680], it is a polarity). Also, the map taking p_1 to p_2 (defined as above) is an isomorphism $\beta : W_1 \to W_2$.

Let x be an arbitrary point of $O \setminus \{p_1, p_2\}$. Then $x^\perp \cap W_i$ is a plane π_i, $i = 1, 2$. If $\beta(\pi_1) = \pi_2$, then $\rho(\pi_1) \perp \pi_2$ and $x \in \langle \rho(\pi_1), \pi_2, x_1\rangle$, for some $i \in \{1, 2\}$, a contradiction. Hence $\beta(\pi_1)$ intersects π_2 in a line L_2; set $L_1 = \beta^{-1}(L_2)$. Then x is collinear to each line $\langle z, \beta(z)\rangle$, for $z \in L_1$, and not collinear to any line $\langle u, \beta(u)\rangle$, for $u \in W_1 \setminus L_1$. Note that L_1 is not fixed by ρ.

Now let Q be an elliptic quadric $\mathsf{O}^-_4(2^e)$ in W_1 such that Q is an ovoid of the symplectic polar space (generalized quadrangle) defined by ρ. If x is collinear to a line $\langle y, \beta(y)\rangle$, with $y \in Q$, then the previous paragraph (in particular the fact that L_1 is not fixed under ρ and hence is not tangent to Q) implies that there is precisely one other such line $\langle y', \beta(y')\rangle$, $y' \in Q \setminus \{y\}$, collinear to x. So $x^\perp$ contains an even number of lines $\langle y, \beta(y)\rangle$ with $y \in Q$. Since $|Q|$ is odd and since each line $\langle z, \beta(z)\rangle$, $z \in W_1$, is collinear to an odd number $2^{2e} - 1$ of points of $O \setminus \{x_1, x_2\}$, this leads to a contradiction. □

2.6.8 Tight sets, spreads, and h-ovoids

Elliptic case The natural inclusions $\mathsf{O}^+_{2n}(q) \subseteq \mathsf{O}_{2n+1}(q) \subseteq \mathsf{O}^-_{2n+2}(q)$ give rise to $(q^{n-1} + 1)$-tight and $(q^n + 1)$-tight sets, respectively, of $\mathsf{O}^-_{2n+2}(q)$.

If U is a nondegenerate n-space in $\mathsf{O}^-_{2n}(q)$, then $U \cup U^\perp$ is 2-tight. For odd n one can choose U such that U and $U^\perp$ induce $\mathsf{O}_n(q)$. For even n one can choose U such that U and $U^\perp$ induce $\mathsf{O}^-_n(q)$ and $\mathsf{O}^+_n(q)$, respectively (DE BRUYN [262]).

For other values of i, the only known examples are partial spreads and disjoint unions of maximal singular subspaces and (disjoint) copies of the 2-tight sets of the previous paragraph. METSCH [565] shows that, as soon as $i^3 - 3i + 6 \leq q$, every i-tight set is like that. Not much is known about partial spreads, except for q even or n small.

Indeed, if q is even, then $\mathsf{O}_{2n+1}(q) \cong \mathsf{Sp}_{2n}(q)$ has a spread. Intersecting with a nondegenerate hyperplane yields a spread of $\mathsf{O}^-_{2n}(q)$. Also, nondegenerate hyperplane sections of $\mathsf{U}_4(q)$ yield ovoids of $\mathsf{U}_4(q)$ and hence spreads of $\mathsf{O}_6^-(q)$, for all prime powers q.

Except for the cases $n = 3, 4$, not much is known about the existence of h-ovoids in $\mathsf{O}_{2n}^-(q)$, $n \geq 3$. SEGRE [640] showed that any h-ovoid of $\mathsf{O}_6^-(q)$ is a hemisystem of points, and that these do not exist for q even. COSSIDENTE & PENTTILA [233] construct hemisystems of points of $\mathsf{O}_6^-(q)$, for every odd q, admitting the group $\mathsf{P\Omega}_4(q)$.

By Theorem 13 of [42], no h-ovoids of $\mathsf{O}_{2n}^-(q)$, $n \geq 3$, exist for $1 \leq h \leq (-3 + \sqrt{9 + 4q^n})/(2q - 2)$.

Parabolic case The natural inclusions $\mathsf{O}_{2n}^-(q) \subseteq \mathsf{O}_{2n+1}(q)$ and $\mathsf{O}_{2n}^+(q) \subseteq \mathsf{O}_{2n+1}(q)$ give rise to $(q^{n-1} - 1)/(q - 1)$-ovoids and $(q^{n-1} + 1)$-tight sets, respectively. For n odd, one can also take away two disjoint maximal singular subspaces from $\mathsf{O}_{2n}^+(q)$, which produces a $(q^{n-1} - 1)$-tight set of $\mathsf{O}_{2n+1}(q)$ which is neither the union of maximal singular subspaces, nor the complement of such union.

Like in the symplectic case, other examples of tight sets can be constructed as follows. Let $\mathsf{O}_{2n+1}(\sqrt{q})$ be naturally embedded in $\mathsf{O}_{2n+1}(q) = (X, \Omega)$. Let X' be the set of points of $\mathsf{O}_{2n+1}(\sqrt{q})$, let X'' be the set of points of $X \setminus X'$ contained in a line of $\mathsf{O}_{2n+1}(\sqrt{q})$, and set $X''' = X \setminus (X' \cup X'')$. Then Theorem 8 of [42] asserts that each of X', X'', X''' is tight. (They are i-tight for $i = \sqrt{q} + 1$, $i = \sqrt{q}(q^{n-1} - 1)$, and $i = q^{n-1}(q - \sqrt{q})$, respectively.)

By Theorem 14 of [42], no 2-ovoids of $\mathsf{O}_{2n+1}(q)$, $n \geq 5$, exist.

For q even, $\mathsf{O}_{2n+1}(q)$ is isomorphic to $\mathsf{Sp}_{2n}(q)$. For q odd not much additionally to the previous constructions and nonexistence is known, neither about spreads, except for the cases $n = 2, 3$.

For $n = 2$, METSCH [565] shows that, if an i-tight set of $\mathsf{O}_5(q)$ is not the union of pairwise disjoint lines, then $i \geq \sqrt{q} + 1$, with equality if and only if the tight set is an embedded $\mathsf{O}_5(\sqrt{q})$.

For $n = 3$, there exist some computer results for $q \in \{3, 5\}$, see Section 7.3 of [42]. In general, one explores the link with the split Cayley generalized hexagon $\mathsf{G}_2(q)$, see §4.8, whose point set is exactly the point set of $\mathsf{O}_7(q)$, as follows.

(i) A *distance-2-ovoid* O of $\mathsf{G}_2(q)$ is a set of points meeting every line exactly once. Then O is a $(q^2 - q + 1)$-tight set of $\mathsf{O}_7(q)$. There exist examples for $q = 2, 3, 4$, see [288], [286].

(ii) The point set of a (non-thick) subhexagon H of $\mathsf{G}_2(q)$ of order $(q, 1)$ is a $(q + 1)$-tight set of $\mathsf{O}_7(q)$. Such subhexagons exist if and only if q is a power of 3, see [710].

(iii) A *spread* S of $\mathsf{G}_2(q)$ is a set of $q^3 + 1$ lines which are pairwise opposite, that is, the only pairs of collinear points on the union of all members of S are contained in the members of S. Then the union of all members of S is a $(q^{n-1} - 1)/(q - 1)$-ovoid. There is always the so-called *Hermitian spread* S_H in $\mathsf{G}_2(q)$, which is obtained by taking the lines of $\mathsf{G}_2(q)$ in an elliptic hyperplane of $\mathsf{O}_7(q)$. It follows that the union of all members of S_H is the point set of $\mathsf{O}_6^-(q)$, and so the corresponding $(q^{n-1} - 1)/(q - 1)$-ovoids are not new. If $q \not\equiv 2 \bmod 3$, then nonisomorphic spreads exist, see [71], and these yield new $(q^{n-1} - 1)/(q - 1)$-ovoids of $\mathsf{O}_7(q)$.

Hyperbolic case The natural inclusion $\mathsf{O}_{2n-1}(q) \subseteq \mathsf{O}^+_{2n}(q)$ yields a $(q^{n-1}-1)/(q-1)$-ovoid. Cardinali & De Bruyn [185] construct (q^3+1)-tight sets of $\mathsf{O}^+_8(q^2)$ as follows. Let $X_0^2 + aX_0X_1 + bX_1^2$ be a quadratic form with $a, b \in \mathbb{F}_q$, which is irreducible over $\mathbb{F}_q$ but reducible over $\mathbb{F}_{q^2}$. Let $\mathsf{O}^+_8(q^2)$ be defined by the equation

$$X_0^2 + aX_0X_1 + bX_1^2 + X_2X_3 + X_4X_5 + X_6X_8 = 0$$

over $\mathbb{F}_{q^2}$, and let $\mathsf{O}^-_8(q)$ be defined by the same equation, but then considered over $\mathbb{F}_q$. This way, $\mathsf{O}^-_8(q) \subseteq \mathsf{O}^+_8(q^2)$. Recall that the graph on maximal singular subspaces (which are 4-spaces) of $\mathsf{O}^+_8(q^2)$, adjacent when intersecting in a plane (a 3-space) is bipartite. Let Φ be one of the corresponding bipartition classes. Let Φ' be the subset of Φ consisting of the members containing a singular plane of $\mathsf{O}^-_8(q)$. Let τ be a triality (cf. §3.2.2) of $\mathsf{O}^+_8(q^2)$ mapping Φ to the set of points of the polar space $\mathsf{O}^+_8(q^2)$. Then $\tau(\Phi')$ is a (q^3+1)-tight set of $\mathsf{O}^+_8(q^2)$.

For $n \geq 4$, there are no known examples of tight sets other than the ones in the previous paragraph and disjoint unions of maximal singular subspaces. Upper bounds b on i such that an i-tight set of $\mathsf{O}^+_{2n}(q)$, with $i \leq b$, is automatically the union of maximal singular subspaces are given in [42], [63] and [564]. Gavrilyuk [334] provides other restrictions on i for i-tight sets that are not the union of maximal singular subspaces.

Through the Klein correspondence, tight sets of $\mathsf{O}^+_6(q)$ are equivalent to so-called *Cameron-Liebler line classes* (first studied by Cameron & Liebler [180]), for which many examples and nonexistence results exist. See, e.g., [615], [295], [151], [563], [360], [339], [318], [257], [338], [232].

2.7 Hermitian or unitary polar spaces

We review some properties of the strongly regular graph defined by the points of a finite unitary polar space, adjacent when collinear. We pay special attention to (maximal) cocliques and regular sets (which translate to the geometric notions of partial ovoids and tight sets, respectively, in the corresponding polar space). We also mention a result on hemisystems.

2.7.1 Hermitian forms

Let V be a vector space over a field F, and let $\sigma\colon F \to F$ be an involutive field automorphism. Recall that a map $f\colon V \times V \to F$ is called a (σ-)*Hermitian form* if it is additive in each coordinate, semi-linear in the first and linear in the second component, i.e., $f(ax, by) = a^\sigma b f(x, y)$, for all $a, b \in F$ and all $x, y \in V$, and σ-symmetric, i.e., $f(y, x) = f(x, y)^\sigma$ for all $x, y \in V$.

If f is a σ-Hermitian form, then we call (V, f) a *Hermitian space*, and σ is called the *companion field automorphism* of f. The fixed point set of σ is a subfield of F which we denote by F_σ. A σ-Hermitian form is *nondegenerate* if for all $x \in V$, $x = 0$ as soon as $f(x, y) = 0$, for all $y \in V$. The set $\{x \in V : f(x, y) = 0, \forall y \in V\}$ is again called the *radical* of f and denoted $\mathrm{Rad}(f)$ (and then f is nondegenerate precisely when $\mathrm{Rad}(f)$ is trivial). The Hermitian form f is called *anisotropic* if $f(x, x) = 0$ implies $x = 0$, for all $x \in V$.

We adopt the same notation as in §2.6. Given a σ-Hermitian form f and a subset S of V, put $S^\perp = \{x \in V : f(s, x) = 0, \forall s \in S\}$ (then the radical is just $V^\perp$ again). A subspace W of V is called *totally isotropic* when f vanishes identically on W. The *Witt index* is the dimension of a maximal isotropic subspace. The set of totally isotropic 1-spaces, also called the null set of f, is a *Hermitian variety* in $\mathsf{P}V$, sometimes also called a *σ-quadric.*

2.7.2 Hermitian or unitary polar spaces

Suppose V is a vector space over F and let (V, f) be a Hermitian space. Let X be the set of totally isotropic 1-spaces of V and let Ω be the set of maximal totally isotropic subspaces in V with respect to Q. Then it is easy to check that, if the Witt index is at least 2, (X, Ω) is a polar space embedded in $\mathsf{P}V$, called a *Hermitian* or *unitary polar space*. The singular subspaces of (X, Ω) coincide with the totally isotropic subspaces of (V, f). Moreover, the radical of (X, Ω) coincides with Rad (f). Hence (X, Ω) is nondegenerate precisely when f is nondegenerate. Moreover, one checks that two points $\langle v\rangle$ and $\langle w\rangle$ of (X, Ω) are collinear if and only if $f(v, w) = 0$.

Similarly to Theorem 2.6.1, we have the following reduction theorem for nondegenerate σ-Hermitian forms. The proof is also similar and is omitted.

Theorem 2.7.1 *Let (V, f) be a nondegenerate unitary space with finite Witt index n. Then V admits a direct sum decomposition $V = V_0 \oplus V_1$ such that $\dim V_0 = 2n$, and there exists a basis $E = \{e_{-n}, e_{-n+1}, \ldots, e_{-1}, e_1, e_2, \ldots, e_n\}$ of V_0 such that f is given by*

$$f(\sum_{i=1}^{n}(x_{-i}e_{-i} + x_i e_i) + v_1,\ \sum_{i=1}^{n}(y_{-i}e_{-i} + y_i e_i) + w_1)$$
$$= x_{-n}^\sigma y_n + \cdots + x_{-2}^\sigma y_2 + x_{-1}^\sigma y_1 + x_1^\sigma y_{-1} + \cdots + x_n^\sigma y_{-n} + f_1(v_1, w_1),$$

with $f_1\colon V_1 \times V_1 \to F : (v_1, w_1) \mapsto f(v_1, w_1)$ anisotropic.

2.7.3 Finite unitary polar spaces and graphs

By Theorem 2.7.1, the nondegenerate unitary polar spaces of rank n over a field F are classified by anisotropic Hermitian forms over F. There are always two standard anisotropic Hermitian forms which exist over any field F admitting an involutary field automorphism σ: the trivial one (in a 0-dimensional vector space), and the form $f\colon F \times F \to F : (x, y) \mapsto x^\sigma y$.

For finite fields F no 2-dimensional anisotropic Hermitian form over F exists. Indeed, let F be arbitrary, with involutive field automorphism σ, and let f be such a form. Its null set is given by an equation (in the unknowns x, y) of shape

$$axx^\sigma + bx^\sigma y + b^\sigma xy^\sigma + cyy^\sigma = 0,$$

with $a, c \in F_\sigma$, $b \in F$.

Since f is anisotropic, $(x, y) = (1, 0)$ is not a solution, so $a \neq 0$ and we may assume $a = 1$. Substituting x by $x - by$, the equation reduces to $xx^\sigma = (bb^\sigma - c)yy^\sigma$, which has a solution if and only if $bb^\sigma - c = zz^\sigma$ for some $z \in F$. Hence no 2-dimensional anisotropic σ-Hermitian form exists over F if and only if every element of F_σ can be written as xx^σ, $x \in F$.

A finite field admits an involutive automorphism if and only if its order is a square. So let $F = \mathbb{F}_{q^2}$, then $\sigma\colon x \mapsto x^q$ is the unique involutive field automorphism. Now $F_\sigma = \mathbb{F}_q$ and since the polynomial x^{q+1}, that maps the $q^2 - 1$ nonzero elements of $\mathbb{F}_{q^2}$ to the $q-1$ nonzero elements of $\mathbb{F}_q$, can take any value at most $q+1$ times, it must take each value precisely $q+1$ times, and in particular at least once. Hence there are no 2-dimensional anisotropic Hermitian forms over a finite field.

We find that if F is finite, there are exactly two cases: The trivial anisotropic Hermitian form (*small unitary polar spaces*) and the unique 1-dimensional one *large unitary polar spaces*). This means that in every finite-dimensional vector space over a given finite field of square order, there exists a unique nondegenerate Hermitian form. Hence in every projective space of dimension at least 3 over a field of square order a unique unitary embedded polar space (X, Ω) exists.

The *unitary graph* of a Hermitian space (V, f) is the collinearity graph $\Gamma = \Gamma(X, \Omega)$ of the corresponding embedded polar space (X, Ω) and thus has as vertex set the set X of points of Δ, where distinct vertices $\langle u\rangle$ and $\langle v\rangle$ are adjacent when $f(u, v) = 0$. Note that, as before, the condition $f(u, v) = 0$ does not depend on the choice of u and v in $\langle u\rangle$ and $\langle v\rangle$, and it is obviously symmetric since $f(u, v) = f(v, u)^\sigma$. The graph Γ can similarly also be defined for Witt index ≤ 1, but then it is has no edges.

2.7.4 Parameters

If V has finite dimension m over the field $\mathbb{F}_{q^2}$, and f is nondegenerate, then (V, f) has Witt index $n = \lfloor m/2 \rfloor$ and the corresponding Hermitian variety is denoted by $\mathsf{U}_m(q)$. (In the literature one also finds $\mathsf{H}_{m-1}(q^2)$.) If the Witt index is at least 2, then the corresponding embedded polar space has order (q^2, q) if m is even, and (q^2, q^3) if m is odd. The collinearity graph of this polar space is called $\Gamma(\mathsf{U}_m(q))$.

For $m \geq 4$, the unitary graphs are strongly regular and the parameters are given as in Theorem 2.2.12 with (q, t) replaced by (q^2, q) or (q^2, q^3). Let $\varepsilon = (-1)^m$. The unitary graph $\Gamma(\mathsf{U}_m(q))$, has the following parameters.

$$\begin{aligned}
v &= (q^m - \varepsilon)(q^{m-1} + \varepsilon)/(q^2 - 1), \\
k &= q^2(q^{m-2} - \varepsilon)(q^{m-3} + \varepsilon)/(q^2 - 1), \\
\lambda &= q^4(q^{m-4} - \varepsilon)(q^{m-5} + \varepsilon)/(q^2 - 1) + q^2 - 1, \\
\mu &= (q^{m-2} - \varepsilon)(q^{m-3} + \varepsilon)/(q^2 - 1),
\end{aligned}$$

so that $\mu = k/q^2$. The eigenvalues are k, $-1 + \varepsilon q^{m-2}$ and $-1 - \varepsilon q^{m-3}$ with multiplicities 1, $\frac{q^2(q^m-\varepsilon)(q^{m-3}+\varepsilon)}{(q^2-1)(q+1)}$ and $\frac{q^3(q^{m-2}-\varepsilon)(q^{m-1}+\varepsilon)}{(q^2-1)(q+1)}$, respectively.

2.7.5 Isomorphisms

The generalized quadrangle $\mathsf{U}_4(q)$ is dual to the orthogonal quadrangle $\mathsf{O}_6^-(q)$. Hence the graph $\Gamma(\mathsf{U}_4(q))$ is isomorphic to the graph $\Delta(\mathsf{O}_6^-(q))$ on the maximal singular subspaces of $\mathsf{O}_6^-(q)$ and the graph $\Gamma(\mathsf{O}_6^-(q))$ is isomorphic to the graph $\Delta(\mathsf{U}_4(q))$ on the maximal singular subspaces of $\mathsf{U}_4(q)$.

2.7.6 Automorphism groups

Let the *general unitary group* $\mathsf{GU}(V, f)$ be the group of all linear transformations of V that preserve the nondegenerate σ-Hermitian form f. The subgroup D

of $\mathsf{GL}(V)$ consisting of all multiples of the identity acts trivially on $\mathsf{P}V$, and $D \cap \mathsf{GU}(V, f) = \{aI : aa^\sigma = 1\}$. Let the *projective general unitary group* $\mathsf{PGU}(V, f)$ be the quotient $\mathsf{GU}(V, Q)/\{aI : aa^\sigma = 1\}$. The automorphism group $\mathrm{Aut}\,\Gamma$ contains $\mathsf{PGU}(V, f)$. The full automorphism group of Γ is $\mathsf{P\Gamma}_\sigma\mathsf{U}(V, f)$, that is, $\mathsf{PGU}(V, f)$ extended by the field automorphisms of the underlying field F commuting with σ. It is also the full automorphism group of the embedded polar space $\mathsf{U}_n(q)$.

If V and F are finite, say V is n-dimensional over $F = \mathbb{F}_{q^2}$, then we denote $\mathsf{GU}(V, f)$ and $\mathsf{PGU}(V, f)$ by $\mathsf{GU}_n(q)$ and $\mathsf{PGU}_n(q)$, respectively. Also, in this case, the automorphism group of the field is abelian and so the subscript σ in the notation of the full automorphism group is redundant and is omitted; hence we denote $\mathsf{P\Gamma U}_n(q)$.

The group $\mathsf{PGU}(V, f)$ is in general not simple. Let $\mathsf{SU}(V, f)$ be the (normal) subgroup of $\mathsf{GU}(V, f)$ of all its matrices with determinant 1, and let $\mathsf{PSU}(V, f)$ be its quotient with the subgroup of scalar matrices it contains. In the finite case we also use the corresponding more specific (self-explaining) notation $\mathsf{SU}_n(q)$ and $\mathsf{PSU}_n(q)$, respectively. The latter is also denoted by $\mathsf{U}_n(q)$ (there will be no confusion with the polar space) and is simple (remember we have $n \geq 4$).

The group $\mathsf{U}_n(q)$ is the intersection of all groups acting rank 3 on the graph $\Gamma(\mathsf{U}_n(q))$. Hence that group, and all overgroups in $\mathsf{P\Gamma U}_n(q)$, act rank 3 on the graph Γ.

2.7.7 Maximal cliques

Again, the maximal cliques of Γ are the maximal totally isotropic subspaces of (V, f). In the finite nondegenerate cases $\mathsf{U}_{2n}(q)$ and $\mathsf{U}_{2n+1}(q)$, these have dimension n and size $(q^{2n} - 1)/(q^2 - 1)$. The maximal cliques form a single orbit under $\mathrm{Aut}\,\Gamma$.

2.7.8 Maximal cocliques

Recall that an *ovoid* in a nondegenerate unitary polar space is a set of points that meets every maximal totally singular subspace in precisely one point. Also, ovoids (when they exist) are maximal cocliques. For $\dim V = n+m+1$, where n is the Witt index and $m \in \{n-1, n\}$, one has $|C| \leq q^{2m+1}+1$ and $|O| = q^{2m+1}+1$ for each coclique C and ovoid O.

There are ovoids in $\mathsf{U}_4(q)$ and no ovoids in $\mathsf{U}_{2n+1}(q)$ for $n \geq 2$. There is no ovoid in $\mathsf{U}_6(2)$ ([258]; see also §10.74). For $\mathsf{U}_{2n+2}(q)$, $n \geq 2$, the best result is due to Moorhouse [574], see Proposition 2.7.8 below.

Proposition 2.7.2 *The generalized quadrangle $\mathsf{U}_4(q)$ has ovoids for each prime power q.*

Proof. Any plane of $\mathsf{P}V$ that does not have any line in common with $\mathsf{U}_4(q)$ intersects the latter in an ovoid. Such planes exist in abundance. Algebraically, if $\mathsf{U}_4(q)$ is given by the equation $x_0x_1^q + x_1x_0^q + x_2x_3^q + x_3x_2^q = 0$, then pick $a \in \mathbb{F}_{q^2}$ so that $a + a^q = 1$ and the plane with equation $x_3 = ax_2$ intersects $\mathsf{U}_4(q)$ in the Hermitian curve O with (more or less) standard equations

$$\begin{cases} 0 = x_3 - ax_2, \\ 0 = x_0x_1^q + x_1x_0^q + x_2^{q+1}, \end{cases}$$

which clearly contains no lines. □

Let $\mathsf{U}_4(q)$ have point set X. Let a *plane ovoid* of $\mathsf{U}_4(q)$ be the intersection $X \cap \pi$ of X with a nontangent plane π. If H is any hyperbolic line, then H and $H^\perp$ meet precisely the same totally isotropic lines. Thus, if O is any ovoid, and $H \cap X \subseteq O$ then $(O \setminus H) \cup (H^\perp \cap X)$ is again an ovoid. In particular this applies to plane ovoids and produces nonisomorphic ovoids. This can be done multiple times (at least $q^2 - q + 1$ times) to produce many nonisomorphic classes of ovoids.

No partition of X into ovoids can consist of plane ovoids only. But nevertheless we can construct such partitions.

Proposition 2.7.3 (Brouwer & Wilbrink [144]) *The generalized quadrangle* $\mathsf{U}_4(q)$ *admits partitions into ovoids.*

Proof. Fix a nonisotropic point p, an isotropic point $x \in p^\perp$, and a tangent T on x in $p^\perp$. Put $O_x = X \cap p^\perp$, and $O_y = ((y^\perp \cap X) \setminus p^\perp) \cup (\langle y, p\rangle \cap X)$ for each $y \in T \setminus \{x\}$. Then each O_u is an ovoid, and $\{O_u \mid u \in T\}$ is a partition of X into ovoids. □

Corollary 2.7.4 *The graph* $\Gamma(\mathsf{U}_4(q))$ *has chromatic number* $q^2 + 1$. □

Proposition 2.7.5 *The unitary polar space* $\mathsf{U}_{2n+1}(q)$, $n \geq 2$, *has no ovoids for any prime power* q.

Proof. As before, due to Corollary 2.2.16, it suffices to show the assertion for $n = 2$.

In this case, an ovoid O has q^5+1 points. A hyperplane (4-space) H contains $q^3 + 1$ points of O if the hyperplane intersects $\mathsf{U}_5(q)$ in a nondegenerate unitary space (isomorphic to $\mathsf{U}_4(q)$; in this case $H \cap O$ is an ovoid of $\mathsf{U}_4(q)$), or if it intersects $\mathsf{U}_5(q)$ in a degenerate unitary space and the radical p does not correspond to a point of O (then every line of $\mathsf{U}_5(q)$ through p contains a unique point of O and there are precisely $q^3 + 1$ such lines). Otherwise the hyperplane contains a unique point of O, and we can pick a plane π disjoint from O. If exactly k hyperplanes through π intersect O in $q^3 + 1$ points, then $k(q^3 + 1) + (q^2 + 1 - k) = q^5 + 1$. This is impossible for integer k. □

We mention without proof the following upper bound, proved by De Beule et al. [256].

Proposition 2.7.6 *The maximum size of a coclique of the graph* $\Gamma(\mathsf{U}_{2n+1}(q))$ *is*

$$1 + q^{2(n-3)}(q^7 - q^6 + q^5 + 1) - q^3 \cdot \frac{q^{2(n-2)} - 1}{q^2 - 1}.$$

Finally, we mention a result due to Blokhuis & Moorhouse [82].

Proposition 2.7.7 *The unitary polar space* $\mathsf{U}_{2n}(q)$, *has no ovoids for* $n \geq 4$ *and* q *a power of* 2 *or* 3. *Also, it has no ovoids for* $n \geq 5$ *and* q *a power of* 5 *or* 7.

The latter proposition is also a consequence of the following stronger result due to Moorhouse [574].

Proposition 2.7.8 *Let $q = p^e$, p prime and e a positive integer. If C is a coclique of $\Gamma(\mathsf{U}_m(q))$, then*

$$|C| \leq \left[\binom{p+m-2}{m-1}^2 - \binom{p+m-3}{m-1}^2 \right]^e + 1.$$

If $\mathsf{U}_{2n}(q)$ contains an ovoid, then

$$p^{2n-1} \leq \binom{p+2n-2}{2n-1}^2 - \binom{p+2n-3}{2n-1}^2.$$

The bounds presented in Propositions 2.7.6 and 2.7.8 are complementary; none of them is always better than the other.

2.7.9 Tight sets

The natural inclusions $\mathsf{U}_{2n}(q) \subseteq \mathsf{U}_{2n+1}(q) \subseteq \mathsf{U}_{2n+1}(q)$ give rise to standard $(q^{2n-1}+1)$-tight sets and $(q^{2n}-1)/(q^2-1)$-ovoids of $U_{2n+1}(q)$ and $U_{2n+2}(q)$, respectively. Every i-tight set of $\mathsf{U}_{2n}(q)$ is by natural inclusion also an i-tight set of $\mathsf{U}_{2n+1}(q)$. There are two other generic examples, which we now describe. Let $\mathsf{U}_{2n+1}(q)$ be defined by the Hermitian form

$$f : (x_0, x_1, \ldots, x_{2n}) \mapsto \sum_{i=0}^{2n} (-1)^i x_i x_i^q.$$

Its restriction to $\mathbb{F}_q$ defines a polar space $\mathsf{O}_{2n+1}(q)$ contained in $\mathsf{U}_{2n+1}(q)$, and this is a $(q+1)$-tight set (as shown in [259]). Also, let $\mathsf{U}_{2n}(q)$ be defined by the Hermitian form

$$f : (x_1, x_2, \ldots, x_{2n}) \mapsto \sum_{i=1}^{n} x_{2i-1} x_{2i}^q - x_{2i} x_{2i-1}^q.$$

Its restriction to $\mathbb{F}_q$ defines a polar space $\mathsf{Sp}_{2n}(q)$ contained in $\mathsf{U}_{2n}(q)$ and this is a $(q+1)$-tight set of both $\mathsf{U}_{2n}(q)$ and $\mathsf{U}_{2n+1}(q)$. Except for some sporadic examples in small cases, these are essentially the only known tight sets (up to disjoint unions of these) in finite Hermitian polar spaces.

Nakić & Storme [584] prove that every i-tight set of $\mathsf{U}_{2n}(q)$, with $q \geq 9$ odd, and $i < q^{4/3}-1)/2$, is the disjoint union of a number of the above examples.

Metsch & Werner [566] prove that every i-tight set of $\mathsf{U}_{2n+1}(q)$, with $i \leq (q+1)/2$, is the disjoint union of a number of maximal singular subspaces. The ultimate conjecture is that this is true as soon as $i < q+1$. This conjecture is proved for $\mathsf{U}_5(q)$ by De Beule & Metsch [259]. The latter paper also contains an improvement for the above bound when $n = 3$: Every i-tight set of $\mathsf{U}_7(q)$, with $i \leq q + 1 - \sqrt{2q}$, is the disjoint union of a number of maximal singular subspaces.

There is not much hope of constructing large tight sets using disjoint maximal singular subspaces as it is known that $\mathsf{U}_{2n}(q)$ and $\mathsf{U}_5(2)$ have no spreads (see [683] and §10.63, respectively), and for the other Hermitian polar spaces, nothing is known about the existence of spreads.

2.7.10 Partial spreads

Luyckx [530] constructed partial spreads of size $q^{2n+1}+1$ in $\mathsf{U}_{4n+2}(q)$, and Vanhove [708] shows that there are no larger partial spreads.

2.7.11 Hemisystems

A *hemisystem* in $\mathsf{U}_4(q)$, where q is odd, is a system of lines covering each point $(q+1)/2$ times. Equivalently, a *hemisystem of points* in $\mathsf{O}_6^-(q)$ is a set of isotropic points that meets every t.i. line in $(q+1)/2$ points.

Proposition 2.7.9 (Cameron et al. [177]) *Let S be a hemisystem of points in $\mathsf{O}_6^-(q)$. Then S induces in $\Gamma(\mathsf{O}_6^-(q))$ a strongly regular graph with parameters $(v,k,\lambda,\mu) = (\frac{1}{2}(q+1)(q^3+1), \frac{1}{2}(q-1)(q^2+1), \frac{1}{2}(q-3), \frac{1}{2}(q-1)^2)$ and eigenvalues $r = q-1$, $s = -\frac{1}{2}q(q-1)-1$.*

This is a special case of the following lemma.

Lemma 2.7.10 *Let Γ be a strongly regular graph with spectrum $k^1\ r^f\ s^g$, and let C be a regular subset of $\mathsf{V}\Gamma$ of size c, degree d, and nexus e. If $d-e=s$ and $cd-d^2-(c-g)r^2-\frac{(d+(c-g)r)^2}{g-1}=0$, then C induces a strongly regular graph with eigenvalues d, r, and $-\frac{d+(c-g)r}{g-1}$.*

Proof. Use that the sum of the eigenvalues of a graph with adjacency matrix A is $\mathrm{tr}A = 0$, and the sum of the squares of the eigenvalues is $\mathrm{tr}A^2$ which is twice the number of edges. Apply this to the graph $\Gamma[C]$ induced by Γ on C. It has eigenvalues d, and r with multiplicity at least $c-g$, and certain other eigenvalues θ_i, say. (Let U be the space of vectors indexed by $\mathsf{V}\Gamma$ that vanish outside C and are orthogonal to the s-eigenspace W of Γ. Then $\dim U \ge c-g$ and all $u \in U$ restrict to r-eigenvectors of $\Gamma[C]$ since W contains a nonzero vector constant on C.) We find $\sum_i 1 = g-1$, $\sum_i \theta_i = -d-(c-g)r$ and $\sum_i \theta_i^2 = cd-d^2-(c-g)r^2$. If $\bar\theta$ is the average of the θ_i, then $\bar\theta = -\frac{d+(c-g)r}{g-1}$ and our condition says $\sum_i(\theta_i-\bar\theta)^2 = 0$, so that all θ_i are equal. □

Other examples of this situation are subgraphs $4K_2$ in the complement of the Clebsch graph, and Hoffman-Singleton subgraphs of the Higman-Sims graph.

Chapter 3

Graphs related to polar spaces

The previous chapter discussed the collinearity graphs of embedded polar spaces. Here we discuss other strongly regular graphs that are found in the same setting, such as graphs on the nonisotropic points or on the maximal singular subspaces.

3.1 Graphs on the nonsingular or nonisotropic points

3.1.1 Association scheme in even characteristic

Let q be a power of 2, and $n \geq 3$. Let V be an n-dimensional vector space over $\mathbb{F}_q$ provided with a nondegenerate quadratic form. If n is odd, there will be a nucleus $N = V^{\perp}$. Let X be the set of nonsingular points other than N.

Consider the following relations on X.

$$\begin{aligned}
R_0 &= \{(x,x) \mid x \in X\}, \\
R_1 &= \{(x,y) \mid \langle x,y \rangle \text{ is a hyperbolic line (secant)}\}, \\
R_2 &= \{(x,y) \mid \langle x,y \rangle \text{ is an elliptic line (exterior line)}\}, \\
R_3 &= \{(x,y) \mid \langle x,y \rangle \text{ is a tangent}\}, \\
R_{3a} &= \{(x,y) \mid \langle x,y \rangle \text{ is a tangent not on } N\}, \\
R_{3n} &= \{(x,y) \mid \langle x,y \rangle \text{ is a tangent on } N\}.
\end{aligned}$$

Note that every line on N is a tangent, and that for $n = 3$ there are no other tangents, so that R_{3a} is empty. For $q = 2$ a hyperbolic line contains only one nonsingular point, and a tangent on N contains only one nonsingular point distinct from N, so that R_1 and R_{3n} are empty.

Theorem 3.1.1 (Vanhove [709])
(i) If n is even or $n = 3$, then $(X, \{R_0, R_1, R_2, R_3\})$ is an association scheme.
(ii) If n is odd and $n \geq 5$, then $(X, \{R_0, R_1, R_2, R_{3a}, R_{3n}\})$ is an association scheme.

(Part (ii) corrects [123], Theorem 12.1.1.)

All parameters p^i_{jk} are given in *loc. cit.* For $n = 2m$ the graph (X, R_3) has parameters $v = q^{2m-1} - \varepsilon q^{m-1}$, $k = n_3 = q^{2m-2} - 1$, $\lambda = p^3_{33} = q^{2m-3} - 2$. It is strongly regular only when $q = 2$.

The eigenvalue matrix and multiplicities are for $\mathsf{O}^{\varepsilon}_{2m}(q)$:

$$P = \begin{pmatrix} 1 & \frac{1}{2}q^{m-1}(q^{m-1}+\varepsilon)(q-2) & \frac{1}{2}q^m(q^{m-1}-\varepsilon) & q^{2m-2}-1 \\ 1 & \frac{1}{2}\varepsilon q^{m-2}(q+1)(q-2) & -\frac{1}{2}\varepsilon q^{m-1}(q-1) & \varepsilon q^{m-2}-1 \\ 1 & 0 & \varepsilon q^{m-1} & -\varepsilon q^{m-1}-1 \\ 1 & -\varepsilon q^{m-1} & 0 & \varepsilon q^{m-1}-1 \end{pmatrix}$$

with multiplicities (in the order of the rows of P) 1, $q^2(q^{2m-2}-1)/(q^2-1)$, $\frac{1}{2}q(q^{m-1}-\varepsilon)(q^m-\varepsilon)/(q+1)$, $\frac{1}{2}(q-2)(q^{m-1}+\varepsilon)(q^m-\varepsilon)/(q-1)$.

For $\mathsf{O}_{2m+1}(q)$:

$$P = \begin{pmatrix} 1 & \frac{1}{2}q^{2m-1}(q-2) & \frac{1}{2}q^{2m} & q(q^{2m-2}-1) & q-2 \\ 1 & \frac{1}{2}q^{m-1}(q-2) & \frac{1}{2}q^m & -(q^{m-1}+1)(q-1) & q-2 \\ 1 & -\frac{1}{2}q^{m-1}(q-2) & -\frac{1}{2}q^m & (q^{m-1}-1)(q-1) & q-2 \\ 1 & \frac{1}{2}q^m & -\frac{1}{2}q^m & 0 & -1 \\ 1 & -\frac{1}{2}q^m & \frac{1}{2}q^m & 0 & -1 \end{pmatrix}$$

with multiplicities (in the order of the rows of P) 1, $\frac{1}{2}q(q^m+1)(q^{m-1}-1)/(q-1)$, $\frac{1}{2}q(q^m-1)(q^{m-1}+1)/(q-1)$, $\frac{1}{2}(q-2)(q^{2m}-1)/(q-1)$ (twice).

For $q = 2$ and $n = 2m+1$ the graph (X, R_3) is isomorphic to $\Gamma(O_{2m+1}(2))$. The graph obtained for $q = 2$ and $n = 2m$ is discussed below.

3.1.2 Nonsingular points over $\mathbb{F}_2$

Let V be a vector space of dimension $2m$ over $\mathbb{F}_2$, provided with a nondegenerate quadratic form of type ε, $\varepsilon = \pm 1$. The corresponding quadric has $2^{2m-1} + \varepsilon 2^{m-1} - 1$ points, so that V has $2^{2m-1} - \varepsilon 2^{m-1}$ nonsingular points. Let Γ be the graph on these nonsingular points, adjacent when they are orthogonal, i.e., when the connecting line is a tangent. If $m \geq 2$, then Γ is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$, where

$$\begin{aligned} v &= 2^{2m-1} - \varepsilon\, 2^{m-1}, & \theta_1 &= \varepsilon\, 2^{m-2} - 1, \\ k &= 2^{2m-2} - 1, & \theta_2 &= -\varepsilon\, 2^{m-1} - 1, \\ \lambda &= 2^{2m-3} - 2, & m_1 &= \frac{4}{3}(2^{2m-2} - 1), \\ \mu &= 2^{2m-3} + \varepsilon\, 2^{m-2}, & m_2 &= \frac{1}{3}(2^{m-1} - \varepsilon)(2^m - \varepsilon). \end{aligned}$$

(The identification of $\theta_1^{m_1}\ \theta_2^{m_2}$ with $r^f\ s^g$ depends on the sign of ε.)

We shall denote this graph by $NO^{\varepsilon}_{2m}(2)$.

The group $\mathsf{O}^{\varepsilon}_{2m}(2)$ acts as a group of automorphisms.

For $m = 1, 2$ one finds $NO^+_2(2) = K_1$, $NO^-_2(2) = 3K_1$, $NO^+_4(2) = K_{3,3}$, and $NO^-_4(2) = \overline{T(5)}$.

Details on cliques and cocliques are given in §3.6.

3.1.3 Nonsingular points of one type over $\mathbb{F}_3$ in dimension $2m$

Let V be a vector space of dimension $2m$ over $\mathbb{F}_3$, provided with a nondegenerate quadratic form Q of type ε, $\varepsilon = \pm 1$. The corresponding quadric has $\frac{1}{2}(3^{2m-1} + \varepsilon 3^{m-1} - 1)$ points, and the set of nonsingular points is split into two parts of equal size by considering the value of Q. Let Γ be the graph on one part, where two points are adjacent when they are orthogonal (i.e., when the connecting line is elliptic). If $m \geq 2$, then Γ is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$, where

$$\begin{aligned} v &= \frac{1}{2}3^{m-1}(3^m - \varepsilon), & \theta_1 &= \varepsilon 3^{m-1}, \\ k &= \frac{1}{2}3^{m-1}(3^{m-1} - \varepsilon), & \theta_2 &= -\varepsilon 3^{m-2}, \\ \lambda &= \frac{1}{2}3^{m-2}(3^{m-1} + \varepsilon), & m_1 &= \frac{1}{8}(3^m - \varepsilon)(3^{m-1} - \varepsilon), \\ \mu &= \frac{1}{2}3^{m-1}(3^{m-2} - \varepsilon), & m_2 &= \frac{9}{8}(3^{2m-2} - 1). \end{aligned}$$

We shall denote this graph by $NO^{\varepsilon}_{2m}(3)$.

The group $\mathsf{O}^{\varepsilon}_{2m}(3)$ acts as a group of automorphisms.

For $m = 1, 2$ one finds $NO^{+}_{2}(3) = K_1$, $NO^{-}_{2}(3) = K_2$, $NO^{+}_{4}(3) = 3K_4$, and $NO^{-}_{4}(3) = \mathsf{Sp}_4(2)$.

Cocliques and chromatic number of $NO^{+}_{2m}(3)$

The Hoffman bound for cocliques in $NO^{+}_{2m}(3)$ is 3^{m-1}. Cocliques of this size are for example the sets C_U of vertices in a perp $U^{\perp}$ where U is a t.s. $(m-1)$-space. If U runs through all hyperplanes of a t.s. m-space, then the sets C_U partition $\mathsf{V}\Gamma$, so that this graph has chromatic number $\frac{1}{2}(3^m - 1)$.

If $m \geq 2$, and W is a t.s. $(m-2)$-space, then $W^{\perp}/W$ is a 4-space in which $NO^{+}_{4}(3) = 3K_4$. That means that $W^{\perp}$ meets the vertex set in the 3^{m-2}-coclique extension of $3K_4$ and we find 4^3 cocliques of size 3^{m-1} in $W^{\perp}$. Of these, 4^2 are of the form C_U for some $U \supset W$. In particular, we found two types of cocliques meeting the Hoffman bound.

3.1.4 Nonsingular points of one type in dimension $2m+1$

Let V be a vector space of dimension $2m+1$ over $\mathbb{F}_q$, where $m \geq 1$, provided with a nondegenerate quadratic form Q. The set of nonsingular hyperplanes is split into two parts of sizes $\frac{1}{2}q^m(q^m + \varepsilon)$ $(\varepsilon = \pm 1)$, with $\varepsilon = +1$ (resp. -1) for hyperbolic (resp. elliptic) hyperplanes. Let Γ be the graph on one part, where two hyperplanes x, y are adjacent when $Q \cap x \cap y$ is degenerate. Then Γ is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$, where

$$\begin{aligned} v &= \frac{1}{2}q^m(q^m + \varepsilon), & \theta_1 &= -\varepsilon q^{m-1} - 1, \\ k &= (q^{m-1} + \varepsilon)(q^m - \varepsilon), & \theta_2 &= \varepsilon(q-2)q^{m-1} - 1, \\ \lambda &= 2(q^{2m-2} - 1) + \varepsilon q^{m-1}(q-1), & m_1 &= \frac{1}{2}q^{2m} - 1 - \frac{q(q^{2m-1} - 1)}{2(q-1)}, \\ \mu &= 2q^{m-1}(q^{m-1} + \varepsilon), & m_2 &= \frac{q(q^{2m-1} - 1)}{2(q-1)} + \frac{1}{2}\varepsilon q^m. \end{aligned}$$

(To be more precise: this graph is complete if $q = 2$, edgeless if $(m, \varepsilon) = (1, -1)$, and strongly regular otherwise.)

This construction is due to Wilbrink (cf. [137]).

We shall denote this graph by $NO^{\varepsilon}_{2m+1}(q)$.

The group $\mathsf{O}_{2m+1}(q)$ acts as a group of automorphisms, see below for the cases of a rank 3 action.

For odd q, this description is equivalent to: Let V be a vector space of dimension $2m + 1$ over $\mathbb{F}_q$, where $m \geq 1$, provided with a nondegenerate quadratic form Q. The set of nonsingular points is split into two parts of sizes $\frac{1}{2}q^m(q^m + \varepsilon)$ $(\varepsilon = \pm 1)$, where the points x are distinguished by the type ε $(= \pm 1)$ of the hyperplane $x^{\perp}$. Let Γ be the graph on one part, where two points are adjacent when the line joining them is a tangent. Then Γ is strongly regular with parameters as given above. For even q this second description fails because of the nucleus.

For $\varepsilon = +1$, the maximum cliques have size q^m, and reach the Hoffman bound.

For $\varepsilon = +1$, $m = 2$ and odd q, the maximum cocliques have size $(q^2 + 1)/2$ and reach the Hoffman bound.

The complementary graph $\overline{\Gamma}$ is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$, where

$$\begin{aligned}
v &= \frac{1}{2}q^m(q^m + \varepsilon), & \theta_1 &= -\varepsilon(q-2)q^{m-1},\\
k &= \frac{1}{2}(q-2)q^{m-1}(q^m - \varepsilon), & \theta_2 &= \varepsilon q^{m-1},\\
\lambda &= \frac{1}{2}(q-2)^2q^{2m-2} - \frac{1}{2}\varepsilon(3q-8)q^{m-1}, & m_1 &= \frac{q(q^{2m-1}-1)}{2(q-1)} + \frac{1}{2}\varepsilon q^m,\\
\mu &= \frac{1}{2}(q-2)^2q^{2m-2} - \frac{1}{2}\varepsilon(q-2)q^{m-1}, & m_2 &= \frac{1}{2}q^{2m} - 1 - \frac{q(q^{2m-1}-1)}{2(q-1)}.
\end{aligned}$$

In the special case $q = 3$ this graph $\overline{\Gamma}$ has parameters

$$\begin{aligned}
v &= \frac{1}{2}3^m(3^m + \varepsilon), & \theta_1 &= -\varepsilon 3^{m-1},\\
k &= \frac{1}{2}3^{m-1}(3^m - \varepsilon), & \theta_2 &= \varepsilon 3^{m-1},\\
\lambda &= \frac{1}{2}3^{m-1}(3^{m-1} - \varepsilon), & m_1 &= \frac{3(3^{2m-1}-1)}{4} + \frac{1}{2}\varepsilon 3^m,\\
\mu &= \frac{1}{2}3^{m-1}(3^{m-1} - \varepsilon), & m_2 &= \frac{1}{2}3^{2m} - 1 - \frac{3(3^{2m-1}-1)}{4}.
\end{aligned}$$

and vertices (in the second description) are adjacent when they are orthogonal. We shall also call this latter graph $NO^{\varepsilon\perp}_{2m+1}(3)$, so that $NO^{\varepsilon\perp}_{2m+1}(3)$ is the same as $\overline{NO^{\varepsilon}_{2m+1}(3)}$. One has $NO^{+\perp}_{3}(3) = 3K_2$ and $NO^{-\perp}_{3}(3) = K_3$.

Rank 3 graphs

The graph $NO^{\varepsilon}_{2m+1}(q)$ is rank 3 for $q \in \{3, 4, 8\}$ and $(m, \varepsilon) \neq (1, -1)$, and for $(m, \varepsilon) = (1, 1)$ and any q. In the latter case this graph is the triangular graph $T(q + 1)$. The groups $\mathsf{O}_3(3).2 \simeq \mathsf{PGL}_2(3)$, $\mathsf{O}_3(4) \simeq \mathsf{PSL}_2(4)$ and $\mathsf{O}_3(8) : 3 \simeq$

$\mathsf{P\Gamma L}_2(8)$ act as a rank 3 permutation group on $NO_3^+(3), NO_3^+(4)$ and $NO_3^+(8)$, respectively. For $q = 8$, we always need to extend the group by the nontrivial field automorphisms to obtain a rank 3 action on $NO_{2m+1}^{\varepsilon}(8)$, $(m, \varepsilon) \neq (1, -1)$.

For $m = 3$, the graph $NO_7^{\varepsilon}(q)$ admits a description using the generalized hexagons $\mathsf{G}_2(q)$, see §4.8. Indeed, a hyperbolic hyperplane of V intersects $\mathsf{G}_2(q)$ in a subhexagon of order $(1, q)$ (and all such subhexagons arise this way) and an elliptic hyperplane of V intersects $\mathsf{G}_2(q)$ in a Hermitian spread (and all Hermitian spreads arise this way). Then $NO_7^+(q)$ is the graph with vertices the subhexagons of order $(1, q)$ of $\mathsf{G}_2(q)$, adjacent when they share at least one point (in which case they share exactly 2 or $q+2$ points), and $NO_7^-(q)$ is the graph with vertices the Hermitian spreads of $\mathsf{G}_2(q)$, adjacent when they share exactly one line (the only alternative is that they share a regulus of a hyperbolic quadric in the intersection of Q and a solid).

For $q = 3, 4, 8$, the group $\mathsf{G}_2(q)$ (extended by the field automorphisms if $q = 8$) acts rank 3 on $NO_7^-(q)$; it cannot act rank 3 on $NO_7^+(q)$ as there are always three possibilities for the number of points in the intersection of two subhexagons of order $(1, q)$ of the split Cayley hexagon $\mathsf{G}_2(q)$, namely 0, 2 and $q + 2$. However, it is rank 4 precisely when the stabilizer in the group $\mathsf{G}_2(q)$ of a given subhexagon H of order $(1, q)$ acts transitively on the subhexagons sharing exactly a given set of $q + 1$ (mutually opposite) lines with H (and no points). Since such hexagons have two points on either such line, this is equivalent to $\mathsf{PGL}_2(q)$ acting rank 3 on the triangular graph $T(q + 1)$ obtained from the projective line $\mathsf{PG}(1, q)$ by taking pairs of points, adjacent when sharing a point. Hence $\mathsf{G}_2(q)$ acts rank 4 on $NO_7^+(q)$ precisely when $q \in \{3, 4\}$, and $\mathsf{G}_2(8) : 3$ acts rank 4 on $NO_7^+(8)$ (as one can easily check; it also follows from Theorem 11.3.3(ii)).

Tower and clique sizes

The $NO^*(3)$ graphs form a tower: the graph $NO_{2n+2}^{-\varepsilon}(3)$ is locally $NO_{2n+1}^{\varepsilon\perp}(3)$, and the graph $NO_{2n+1}^{\varepsilon\perp}(3)$ is locally $NO_{2n}^{\varepsilon}(3)$. Conversely, Pasechnik [599] shows that for $n \geq 3$ the only locally $NO_{2n}^{\varepsilon}(3)$ graph is $NO_{2n+1}^{\varepsilon\perp}(3)$, and the only locally $NO_{2n+1}^{\varepsilon\perp}(3)$ graph is $NO_{2n+2}^{-\varepsilon}(3)$.

It follows that maximum cliques in $NO_{2m}^{\varepsilon}(3)$ have size $2m$ if $\varepsilon = (-1)^m$, and $2m - 1$ otherwise, and that maximum cliques in $NO_{2m+1}^{\varepsilon\perp}(3)$ have size $2m + 1$ if $\varepsilon = (-1)^m$, and $2m$ otherwise.

3.1.5 Nonsingular points of one type over $\mathbb{F}_5$ in dimension $2m + 1$

Let V be a vector space of dimension $2m + 1$ over $\mathbb{F}_5$, provided with a nondegenerate quadratic form Q. The set of nonsingular points is split into two parts, depending on the type ε $(= \pm 1)$ of the hyperplane $x^\perp$. Let Γ be the graph on one part, where two points are adjacent when they are orthogonal. Then Γ is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$, where

$$\begin{aligned}
v &= 5^m(5^m + \varepsilon)/2, & \theta_1 &= 2\varepsilon\, 5^{m-1}, \\
k &= 5^{m-1}(5^m - \varepsilon)/2, & \theta_2 &= -\varepsilon\, 5^{m-1}, \\
\lambda &= 5^{m-1}(5^{m-1} + \varepsilon)/2, & m_1 &= \frac{1}{6}(5^{2m} - 1), \\
\mu &= 5^{m-1}(5^{m-1} - \varepsilon)/2, & m_2 &= \frac{5}{6}(5^m - \varepsilon)(2 \cdot 5^{m-1} + \varepsilon).
\end{aligned}$$

This construction is due to Wilbrink (cf. [137]).

We shall call this graph $NO^{\varepsilon\perp}_{2m+1}(5)$.

The group $\mathsf{O}_{2m+1}(5)$ acts as a group of automorphisms.

3.1.6 Nonisotropic points for a Hermitian form

Let V be a vector space of dimension n over $\mathbb{F}_{q^2}$, provided with a nondegenerate Hermitian form. Let $n \geq 3$ and $\varepsilon = (-1)^n$. Let Γ be the graph on the nonisotropic points, adjacent when joined by a tangent. Then Γ is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$, where

$$v = q^{n-1}(q^n - \varepsilon)/(q+1), \qquad \theta_1 = \varepsilon q^{n-2} - 1,$$
$$k = (q^{n-1} + \varepsilon)(q^{n-2} - \varepsilon), \qquad \theta_2 = -\varepsilon(q^2 - q - 1)q^{n-3} - 1,$$
$$\lambda = q^{2n-5}(q+1) - \varepsilon q^{n-2}(q-1) - 2, \qquad m_1 = \frac{(q^2-q-1)(q^n-\varepsilon)(q^{n-1}+\varepsilon)}{(q+1)(q^2-1)},$$
$$\mu = q^{n-3}(q+1)(q^{n-2} - \varepsilon), \qquad m_2 = \frac{q^3(q^{n-2}-\varepsilon)(q^{n-1}+\varepsilon)}{(q+1)(q^2-1)}.$$

We shall denote this graph by $NU_n(q)$. The group $\mathsf{U}_n(q)$ acts as a group of automorphisms.

If n is odd, the Hoffman bound for cliques is q^{n-1}. Cliques meeting this bound are obtained as $Z^\perp \setminus Z$ where Z is a maximal totally isotropic subspace.

If $n = 3$, the Hoffman bound for cocliques is $q^2 - q + 1$. Cocliques meeting this bound are obtained as the sets C_x of vertices in $\{x\} \cup x^\perp$ for nonisotropic x. The collection of sets C_x where x varies on a fixed tangent line is a partition of the vertex set, so that $NU_3(q)$ has chromatic number q^2. (See §10.22 and §10.52 for $q = 3, 4$).

The complementary graph $\overline{NU_n(q)}$ is the graph on the nonisotropic points, adjacent when on a secant. It is strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$, where

$$v = \frac{q^{n-1}(q^n-\varepsilon)}{q+1}, \qquad \theta_1 = \varepsilon q^{n-3} r,$$
$$k = \frac{q^{n-2} r (q^{n-1}+\varepsilon)}{q+1}, \qquad \theta_2 = -\varepsilon q^{n-2},$$
$$\lambda = \mu + \varepsilon q^{n-3} r - \varepsilon q^{n-2}, \qquad m_1 = \frac{q^3(q^{n-2}-\varepsilon)(q^{n-1}+\varepsilon)}{(q+1)(q^2-1)},$$
$$\mu = \frac{q^{n-2} r (q^{n-3} r + \varepsilon)}{q+1}, \qquad m_2 = \frac{r(q^n-\varepsilon)(q^{n-1}+\varepsilon)}{(q+1)(q^2-1)},$$

where $r = q^2 - q - 1$.

Rank 3 tower

The case $q = 2$ is special. The group action is rank 3 for $q = 2$. The graph $\overline{NU_n(2)}$ is the graph on the nonisotropic points, adjacent when orthogonal. It is locally $\overline{NU_{n-1}(2)}$.

Hyperplanes

The graph $\overline{NU_n(q)}$ can also be seen as the set of nondegenerate hyperplanes of a nondegenerate hermitian form on a vector space V of dimension n over $\mathbb{F}_{q^2}$, adjacent if they intersect in a nondegenerate *subhyperplane* (i.e., a subspace of codimension 2). For $n = 3$, $\overline{NU_3(q)}$ is hence the graph on the blocks of a Hermitian unital, adjacent if they are disjoint.

Orthogonality in the plane

Let V be a vector space of dimension 3 over $\mathbb{F}_{q^2}$, provided with a nondegenerate Hermitian form. Let Γ be the graph on the nonisotropic points, adjacent when orthogonal. Then Γ has $v = q^2(q^2 - q + 1)$ vertices. If $q = 2$, then $\Gamma \simeq 4K_3$, the disjoint union of four triangles. If $q > 2$ the Γ is distance-regular with intersection array $\{q^2 - q, q^2 - q - 2, q + 1; 1, 1, q^2 - 2q\}$ and spectrum $(q^2 - q)^1\ q^f\ (-1)^{q^3}\ (-q)^g$, where $f = \frac{1}{2}(q^2 - q)(q^2 - q + 1)$ and $g = \frac{1}{2}(q^2 - q - 2)(q^2 - q + 1)$. See [123], Theorem 12.4.1.

Unitals and O'Nan configurations

A *unital* (of order q) is a Steiner system $S(2, q + 1, q^3 + 1)$, that is, a 2-$(q^3 + 1, q + 1, 1)$ design. The order q need not be a prime power; examples for $q = 6$ were constructed in [547] and [32]. An *embedded unital* is a unital of order q of which the point set X is a subset of the set of points of a(n arbitrary) projective plane $\mathsf{PG}(2, q^2)$, and the blocks are the nontrivial intersections of X with lines. For example, the Hermitian unitals (where the point set is the set of absolute points for a unitary polarity) are embedded unitals, and the name comes from this example. Embedded unitals are two-character sets: each line meets X in either 1 or $q + 1$ points. A monograph on embedded unitals is [51].

An *O'Nan configuration* (say, in a partial linear space) is a configuration of four lines meeting in six points. O'Nan [597] proved that the full automorphism group of the Hermitian unital is $\mathsf{P\Gamma U}_3(q)$. Also, that this design does not contain O'Nan configurations. An immediate consequence is that $NU_3(q)$ (viewed as the block graph of the unital) has precisely two types of maximal cliques: cliques of size q^2 (meeting the Hoffman bound) consisting of all blocks on a fixed point, and cliques of size $q + 2$ consisting of the $q + 1$ blocks on a point p meeting a block B not on p, together with this block B.

Piper [619] conjectured that the Hermitian unital is characterized among the $S(2, q + 1, q^3 + 1)$ designs by the absence of O'Nan configurations. This conjecture remains open. Wilbrink [730] has partial results. See also [367] for another intrinsic characterization of the Hermitian unitals.

History

The above graphs were constructed in Chakravarti [189] for $n = 3, 4$. The chromatic number of $NU_3(q)$ was given by Soicher.

3.2 Graphs on half of the maximal singular subspaces

3.2.1 General observations

Let (X, Ω) be a finite embedded polar space of rank n and order $(q, 1)$. Recall from §2.2.12 that the graph $\Delta = \Delta(X, \Omega)$ has diameter n, and is bipartite, and hence that the halved graphs are distance-regular of diameter $\lfloor n/2 \rfloor$. In particular, they are strongly regular for $n = 4, 5$. We take a look at these cases separately, but we first show that the halved graphs are mutually isomorphic.

Lemma 3.2.1 *The two connected components $\Delta_{1/2}$ and $\Delta'_{1/2}$ of the distance-2 graph of Δ are isomorphic.*

Proof. Let X be given by its standard equation in $2n$-dimensional space V over the field $\mathbb{F}_q$:

$$X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-n}X_n = 0,$$

and let φ be the linear mapping interchanging the X_{-1}- and the X_1-coordinate of every vector (and leaving the rest as it is). Clearly φ preserves X and Ω. Then the maximal singular subspace W with equations $X_1 = X_2 = \cdots = X_n = 0$ is mapped onto the subspace W' with equations $X_{-1} = X_2 = X_3 = \cdots = X_n = 0$, which intersects W in an $(n-1)$-space. Hence W and W' correspond to adjacent vertices in Δ and hence to different connected components of the distance-2 graph of Δ. □

So from now on, we denote by $\Delta_{1/2}$ one of the two connected components of the distance-2 graph of Δ. When we want to emphasize the corresponding polar space $\mathsf{O}^+_{2n}(q)$ we write $\Delta_{1/2}(\mathsf{O}^+_{2n}(q))$.

We have the following isomorphism result.

Proposition 3.2.2 *The graph $\Delta_{1/2}(\mathsf{O}^+_{2n+2}(q))$ is isomorphic to the distance-$\{1,2\}$ graph of $\Delta(\mathsf{O}_{2n+1}(q))$. If q is even, then it is also isomorphic to the distance-$\{1,2\}$ graph of $\Delta(\mathsf{Sp}_{2n}(q))$.*

Proof. Let Q^+ be a hyperbolic quadric in a $(2n+2)$-dimensional space V over $\mathbb{F}_q$, $n \geq 2$. Let $Q = Q^+ \cap H$ be a hyperplane section of Q^+ with a hyperplane H such that Q is a nondegenerate parabolic quadric in H. Let Ω be the set of maximal singular subspaces of Q, and let Ω_1 and Ω_2 be the two natural classes of maximal singular subspaces of Q^+ (so any member of Ω_1 intersects any member of Ω_2 in a subspace of odd codimension in both, and $\Omega_1 \cup \Omega_2$ is the complete set of maximal singular subspaces—which are $(n+1)$-spaces). Let $M \in \Omega$. Then M is an n-dimensional singular subspace of Q^+ and hence contained in exactly two maximal singular subspaces (since Q^+ is hyperbolic). Clearly, exactly one of them belongs to Ω_1 (and the other to Ω_2). Conversely, for every member $M^+ \in \Omega_1$, the intersection $H \cap M^+$ belongs to Ω (since M^+ is not contained in H). This defines a natural bijection $\beta : \Omega \leftrightarrow \Omega_1$. Suppose $M^+, N^+ \in \Omega_1$ intersect in an $(n-1)$-space. Then $\beta(M^+)$ and $\beta(N^+)$ intersect in $(M^+ \cap H) \cap (N^+ \cap H) = (M^+ \cap N^+) \cap H$, which has dimension $n-1$ or $n-2$, i.e., $M^+ \cap H$ and $N^+ \cap H$ are at distance 1 or 2 in the graph, with self-explaining notation, $\Delta(Q) \cong \Delta(\mathsf{O}_{2n+1}(q))$. Conversely, let $M, N \in \Omega$ be at distance at most 2 in $\Delta(Q)$. Then $M \cap N$ has dimension at least $n-2$, and so $\beta(M) \cap \beta(N)$ has dimension at least $n-2$. Since the parity of the dimension of $\beta(M) \cap \beta(N)$ is that of $n+1$, the dimension of $\beta(M) \cap \beta(N)$ cannot be $n-2$ and hence is at least $n-1$. This means that $\beta(M)$ and $\beta(N)$ are adjacent in $\Delta(Q^+)$.

The last assertion follows from §2.6. □

3.2.2 The rank 4 case: the triality quadric

Suppose that the rank of (X, Ω) is 4. The next proposition says that the graph $\Delta_{1/2}$ is isomorphic to $\Gamma(X, \Omega)$. Usually, this is proved using a trilinear form (cf. [710], §2.4.6). We proceed with an explicit isomorphism.

Proposition 3.2.3 *If (X, Ω) is an embedded polar space of rank 4 and order $(q, 1)$, then the collinearity graph $\Gamma(X, \Omega)$ is isomorphic to each of the halved graphs of $\Delta(X, \Omega)$.*

Proof. Let X be given by the equation $X_{-1}X_1 + X_{-2}X_2 + X_{-3}X_3 + X_{-4}X_4 = 0$, and order the coordinates $(x_{-4}, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, x_4)$. Consider the following mapping τ from the point set X into the set Ω of maximal singular subspaces (every such subspace is given by a 4×8 matrix whose rows represent spanning vectors):

$$(1, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, -x_{-1}x_1 - x_{-2}x_2 - x_{-3}x_3) \mapsto \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & x_{-1} & x_{-2} & x_{-3} & 0 \\ 0 & 1 & 0 & 0 & -x_2 & x_1 & 0 & -x_{-3} \\ 0 & 0 & 1 & 0 & x_3 & 0 & -x_1 & -x_{-2} \\ 0 & 0 & 0 & 1 & 0 & -x_3 & x_2 & -x_{-1} \end{array}\right)$$

$$(0, 1, x_{-2}, x_{-1}; x_1, x_2, -x_{-1}x_1 - x_{-2}x_2, x_4) \mapsto \left(\begin{array}{cccc|cccc} x_1 & -x_{-2} & 1 & 0 & -x_4 & 0 & 0 & 0 \\ -x_2 & -x_{-1} & 0 & 1 & 0 & x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{-1} & x_{-2} & 1 & 0 \\ 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 & 1 \end{array}\right)$$

$$(0, 0, 1, x_{-1}; x_1, -x_{-1}x_1, x_3, x_4) \mapsto \left(\begin{array}{cccc|cccc} -x_1 & 1 & 0 & 0 & x_4 & 0 & 0 & 0 \\ x_3 & 0 & -x_{-1} & 1 & 0 & 0 & -x_4 & 0 \\ 0 & 0 & 0 & 0 & x_{-1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 & x_1 & 1 \end{array}\right)$$

$$(0, 0, 0, 1; 0, x_2, x_3, x_4) \mapsto \left(\begin{array}{cccc|cccc} x_2 & 1 & 0 & 0 & 0 & -x_4 & 0 & 0 \\ -x_3 & 0 & 1 & 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 & -x_2 & 1 \end{array}\right)$$

$$(0, 0, 0, 0; 1, x_2, x_3, x_4) \mapsto \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -x_4 & 0 & 0 & 0 \\ 0 & x_3 & x_2 & 1 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & -x_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 & 1 & 0 \end{array}\right)$$

$$(0,0,0,0;0,1,x_3,x_4) \;\mapsto\; \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -x_4 & 0 & 0 \\ 0 & x_3 & 1 & 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_3 & 1 & 0 \end{array}\right)$$

$$(0,0,0,0;0,0,1,x_4) \;\mapsto\; \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -x_4 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$(0,0,0,0;0,0,0,1) \;\mapsto\; \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

From the fact that the matrices on the right are diagonalized (up to permuting columns), we see that τ is injective in Ω. Now let Ω_1 be the collection of maximal singular subspaces intersecting the subspace W_1 with equations $X_{-1} = X_{-2} = X_{-3} = X_{-4} = 0$ in a subspace of even codimension (disjoint or intersecting in a line). We claim that τ is surjective on Ω_1. Indeed, let $W \in \Omega_1$ be arbitrary and let $p_{ijk\ell}$ be the Grassmann coordinate of W corresponding to the positions $i, j, k, \ell \in \{-4, -3, -2, -1, 1, 2, 3, 4\}$, where we write $\bar{i}$ instead of $-i$ for brevity. If W and W_1 are disjoint, then $p_{\bar{4}\bar{3}\bar{2}\bar{1}} \neq 0$, and so W is the image under τ of a point $(1, \ldots)$. If W intersects W_1 in a line, then we can pick a generating set of points of W such that the 4×4 matrix consisting of the first four coordinates of these four points is in diagonalized form, and has rank 2. Hence W is the image of a point of one of the following six shapes: $(0, 1, \ldots)$, $(0, 0, 1, \ldots)$, $(0, 0, 0, 1; \ldots)$, $(0, \ldots, 0; 1, \ldots)$, $(0, \ldots, 0; 0, 1, \ldots)$ or $(0, \ldots, 0; 0, 0, 1, \ldots)$. Finally, if $W = W_1$, then the inverse image of W is $(0, \ldots, 0, 1)$.

Now we claim that two points of X are collinear if and only if their images under τ intersect nontrivially. This follows from inspecting the 36 different cases for the shapes of the pair of points, according to the definition of τ. Let us do the most involved case, where the points, say u and v, have respective coordinates

$$\begin{array}{rl} & (1, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, -x_{-1}x_1 - x_{-2}x_2 - x_{-3}x_3) \\ \text{and} & (1, y_{-3}, y_{-2}, y_{-1}; y_1, y_2, y_3, -y_{-1}y_1 - y_{-2}y_2 - y_{-3}y_3). \end{array}$$

It is easy to calculate that u and v are collinear if and only if

$$(x_{-1} - y_{-1})(x_1 - y_1) + (x_{-2} - y_{-2})(x_2 - y_2) + (x_{-3} - y_{-3})(x_3 - y_3) = 0,$$

whereas the determinant of the matrix

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & x_{-1} & x_{-2} & x_{-3} & 0 \\ 0 & 1 & 0 & 0 & -x_2 & x_1 & 0 & -x_{-3} \\ 0 & 0 & 1 & 0 & x_3 & 0 & -x_1 & -x_{-2} \\ 0 & 0 & 0 & 1 & 0 & -x_3 & x_2 & -x_{-1} \\ 1 & 0 & 0 & 0 & y_{-1} & y_{-2} & y_{-3} & 0 \\ 0 & 1 & 0 & 0 & -y_2 & y_1 & 0 & -y_{-3} \\ 0 & 0 & 1 & 0 & y_3 & 0 & -y_1 & -y_{-2} \\ 0 & 0 & 0 & 1 & 0 & -y_3 & y_2 & -y_{-1} \end{array}\right)$$

is equal to

$$[(x_{-1} - y_{-1})(x_1 - y_1) + (x_{-2} - y_{-2})(x_2 - y_2) + (x_{-3} - y_{-3})(x_3 - y_3)]^2.$$

This shows the claim for u and v. Similar, but simpler, calculations hold for the other cases. □

It is rather cumbersome to calculate the inverse images of a given generic member of Ω_1. Except in characteristic 2. Indeed, in general, the image of the set of members of Ω_1 through a fixed line L on the Grassmannian of 4-spaces of V is a conic. When q is a power of 2, we can project the Grassmannian from the subspace generated by the nuclei of all such conics; this gives us precisely a point set projectively equivalent to X.

Proposition 3.2.4 *Let q be even and let $W \in \Omega_1$ be arbitrary. Let $p_{ijk\ell}$, $i,j,k,\ell \in \{\bar{4},\ldots,\bar{1},1,\ldots,4\}$, be as above. Then $\tau^{-1}(W)$ is the point with coordinates*

$$(p_{\bar{4}\bar{3}\bar{2}\bar{1}}^{1/2}, p_{\bar{2}\bar{1}34}^{1/2}, p_{\bar{3}\bar{1}24}^{1/2}, p_{\bar{3}\bar{2}14}^{1/2}; p_{\bar{4}\bar{1}23}^{1/2}, p_{\bar{4}\bar{2}13}^{1/2}, p_{\bar{4}\bar{3}12}^{1/2}, p_{1234}^{1/2}).$$

Proof. This follows immediately by calculating the relevant Grassmann coordinates of the images of a point, given in the definition of τ above. □

Remark One easily checks that the image under τ of the point set of a member W_1 of Ω_1 is a set of 4-spaces sharing 3-spaces with a fixed member W_2 of $\Omega_2 := \Omega \setminus \Omega_1$. We set $W_2 = \tau(W_1)$. The image under τ of the point set of W_2 is the set of members of Ω_1 containing a fixed point P and we can define $P = \tau(W_2)$. If we call points of (X,Ω) type 0 objects, members of Ω_1 type 1 objects and members of Ω_2 type 2 objects, then τ is an adjacency preserving and type rotating $(0 \to 1 \to 2 \to 0)$ map in Δ, with adjacency inherited by $\Gamma(X,\Omega)$ and the two halved graphs $\Delta_{1/2}$ and $\Delta'_{1/2}$. Such a map is called a *triality.* Trialities of order 3 play a special role since they give rise to generalized hexagons (i.e., the fixed lines with all their points form a possibly degenerate generalized hexagon).

3.2.3 Rank 5 hyperbolic polar spaces

Let (X,Ω) be a finite embedded polar space of rank 5 of order $(q,1)$ and set $\Gamma = \Gamma(X,\Omega)$ and $\Delta = \Delta(X,\Omega)$. This latter graph is bipartite. Let Ω_1 and Ω_2 be the two vertex classes, and let $\Delta_{1/2}$ be the halved graph of Δ with vertex set Ω_1. The graph $\Delta_{1/2}$ is strongly regular, with parameters given in Theorem 2.2.20.

Maximal cliques

The Hoffman bound yields $|C| \leq q^5+q^4+q^3+q^2+q+1$ for a maximal clique C of $\Delta_{1/2}$. But no maximal clique meets this bound; in fact, maximal cliques are much smaller as the following proposition shows.

Proposition 3.2.5 *Every maximal clique is either the set of members of Ω_1 containing a fixed line (and then has size q^3+q^2+q+1), or the set of members of Ω_1 intersecting a fixed member of Ω_2 in codimension 1 (and then has size $q^4+q^3+q^2+q+1$).*

Proof. Let $\mathscr{C}$ be a maximal clique of $\Delta_{1/2}$. Any two members of $\mathscr{C}$ meet in a plane. Let M_1, M_2, M_3 be three distinct elements of $\mathscr{C}$, not all on the same plane. If P is a point of $M_2 \cap M_3 \setminus M_1$, then $P^\perp \cap M_1$ has codimension 1 in M_1, so $(M_1 \cap M_2) \cup (M_1 \cap M_3)$ does not span M_1, so the two planes $M_1 \cap M_2$ and $M_1 \cap M_3$ span a hyperplane of M_1, i.e., a point of the dual projective space M_1^*.

Let M be a fixed element of $\mathscr{C}$, and let Π be the set of planes $M \cap M'$ for $M' \in \mathscr{C}$, $M' \neq M$. In the dual projective space M^*, the set Π is a set of lines pairwise intersecting in a point. Either all these lines have a common point, or are contained in a common plane. In the former case that point corresponds to a hyperplane H of M, which is contained in a unique $N \in \Omega_2$ (and all elements of $\mathscr{C}$ have a codimension 2, hence codimension 1 space in common with N); in the latter case the plane corresponds to a line L of M and all elements of Π, and hence of $\mathscr{C}$, contain L. □

We record a useful corollary of the above proposition.

Corollary 3.2.6 *Let u, v be two adjacent vertices of $\Delta_{1/2}$. Then the intersection of all maximal cliques containing u, v is the set of all members of Ω_1 containing a fixed projective plane of $\mathsf{O}_{10}^+(q)$.*

Maximal cocliques

The Hoffman bound for cocliques is $\frac{q^8-1}{q^3-1}$ which is not an integer. An obvious construction of (much smaller) maximal cocliques runs as follows.

Proposition 3.2.7 *Let p be a point of $\mathsf{O}_{10}^+(q)$. Then $\mathrm{Res}\,(p)$ is an embedded polar space in the quotient space $\mathsf{P}V/p$ isomorphic to $\mathsf{O}_8^+(q)$. Let $\mathscr{S}$ be a spread of $\mathsf{O}_8^+(q)$ contained in Ω_1. Then $\mathscr{S}$ corresponds to a maximal coclique (of size q^3+1) of $\Delta_{1/2}$.*

Larger cocliques exist, but the known examples are messy.

Automorphism group

The automorphism group of (X, Ω) is $\mathsf{P\Gamma O}_{10}^+(q)$. The subgroup that preserves the parts Ω_1 and Ω_2 has index 2 in $\mathsf{P\Gamma O}_{10}^+(q)$, and is a group of automorphisms of $\Delta_{1/2}$. We show that it is the full group. To this end, we introduce the notion of *clique-convex subgraph* of a graph: This is a subgraph closed under taking shortest paths between its vertices and such that, for every pair u, v of adjacent vertices of the subgraph, the intersection of all maximal cliques containing u and v is contained in the subgraph. The *clique-convex closure* of a subset of vertices is the intersection of all clique-convex subgraphs containing that subset; clearly this is a clique-convex subgraph.

The above claim about the automorphism group of $\Delta_{1/2}$ follows from the next proposition.

Proposition 3.2.8 *The family of clique-convex subgraphs of $\Delta_{1/2}$ which are the clique-convex closure of two vertices at distance 2 from each other, is in natural bijective correspondence with the set of vertices of Γ; moreover two such subgraphs are disjoint if and only if the corresponding vertices of Γ are not adjacent.*

Proof. Let $v \in V(\Gamma)$. Let $W_v = \{M \in \Omega_1 \mid v \in M\}$. We claim that the induced subgraph $\Delta_{1/2}(W_v)$ is clique-convex. Indeed, let M, N be vertices of $\Delta_{1/2}(W_v)$. Then $M \cap N$ is either a point or a plane. If it is a plane, then $M \sim N$ and Corollary 3.2.6 implies that each member of the intersection of all maximal cliques containing M and N contains v. If $M \cap N = \{v\}$, then their distance is 2 in $\Delta_{1/2}$. Let, in the latter case, $M \sim R \sim N$ for some vertex R of $\Delta_{1/2}$. Since R cannot contain two disjoint planes, we must have $M \cap N \cap R \neq 0$, so $v \in R$, that is, $R \in \Delta_{1/2}(W_v)$, showing the claim.

Now let again M, N be vertices of $\Delta_{1/2}$ with $M \cap N = \{v\}$. Let $\Delta_{1/2}(M, N)$ be the clique-convex closure of $\{M, N\}$. We claim $\Delta_{1/2}(M, N) = \Delta_{1/2}(W_v)$. The previous paragraph already implies $\Delta_{1/2}(M, N) \subseteq \Delta_{1/2}(W_v)$. Left to show is $\Delta_{1/2}(M, N) \supseteq \Delta_{1/2}(W_v)$. Considering the residue at v, and noting Proposition 3.2.3, we see that the claim is proved if we show that (embedded) polar space graphs have no proper clique-convex subgraphs containing two noncollinear points. Let us show this.

Let x, y be two noncollinear points of an embedded polar space E and let F be a clique-convex subgraph of the collinearity graph containing x and y. Then the definition of clique-convexity readily implies that $x^\perp \cup y^\perp$ belongs to F. Let $v \in E$ be an arbitrary point and suppose v does not belong to F. Consider two lines L, L' through x which are not contained in a singular plane. Then $v^\perp \cap (L \cup L')$ is a pair of noncollinear points belonging to F; hence by clique-convexity also v belongs to F and so $E = F$.

Since $W_u \cap W_v \neq 0$ if and only if u, v are collinear in Γ, the last assertion follows. □

3.2.4 Disjoint t.i. planes in $\mathsf{O}_7(q)$ and $\mathsf{Sp}_6(q)$

The dual polar graphs on the t.i. planes in the $\mathsf{O}_7(q)$ or $\mathsf{Sp}_6(q)$ geometry, adjacent when they meet in codimension 1, are distance-regular of diameter 3, with parameters $b_i = q^{i+1}(q^{3-i} - 1)/(q - 1)$ and $c_i = (q^i - 1)/(q - 1)$ and eigenvalues $\theta_i = (q^{4-i} - q^i)/(q - 1) - 1$ $(0 \le i \le 3)$, cf. §2.2.9. In particular, $\theta_2 = -1$. It follows from Proposition 1.3.12 that the distance-3 graph of each is strongly regular. Their parameters are

$$\begin{aligned} v &= (q^3 + 1)(q^2 + 1)(q + 1), & r &= q^2, \\ k &= q^6, & s &= -q^3, \\ \lambda &= q^2(q^3 - 1)(q - 1), & f &= q^2(q^4 + q^2 + 1), \\ \mu &= (q - 1)q^5, & g &= q(q + 1)^2. \end{aligned}$$

For the $\mathsf{O}_7(q)$ geometry, this graph is just the (complement of the) $\mathsf{O}_8^+(q)$ polar graph. Indeed, by triality that polar graph is isomorphic to the graph on one kind of t.i. solids, adjacent when they meet in a line, and hitting with a hyperplane we find the above description. For $\mathsf{Sp}_6(q)$ (with odd q) however, this graph is not isomorphic to graphs discussed earlier. The group is rank 4.

The subgraph of the dual polar graph for $\mathsf{O}_7(q)$ induced of the set of q^6 t.i. planes disjoint from a given plane is the Brouwer-Pasechnik graph described in [140], Proposition 3.1. It is distance-regular of diameter 3 with intersection array $\{q^3 - 1, q^3 - q, q^3 - q^2 + 1;\ 1, q, q^2 - 1\}$ and has eigenvalue -1. We see that the graph Γ on the t.i. planes of $\mathsf{O}_7(q)$, adjacent when disjoint, has local graphs that are strongly regular with parameters $v = q^6$, $k = (q^3-1)(q^3-q^2+1)$, $\lambda = \mu - (q^3-2q^2+2)$, $\mu = q^2(q-1)(q^3-q^2+1)$.

The subgraph of the dual polar graph for $\mathsf{Sp}_6(q)$ induced of the set of q^6 t.i. planes disjoint from a given plane is the symmetric bilinear forms graph on $V = \mathbb{F}_q^3$ ([123], Theorem 9.5.10). For even q the spaces $\mathsf{Sp}_6(q)$ and $\mathsf{O}_7(q)$ are isomorphic. So, let q be odd. Then the symmetric bilinear forms graph is the same as the quadratic forms graph (§3.4.3). We see for odd q that the graph Γ on the t.i. planes of $\mathsf{Sp}_6(q)$, adjacent when disjoint, is locally the complement of the quadratic forms graph on V, so that both Γ and its local graph are strongly regular. For Γ the parameters were given above. Its local graph has parameters $v = q^6$, $k = q^2(q^3-1)(q-1)$, $\lambda = \mu - q^2(q-2)$, $\mu = q^2(q-1)(q^3-q^2-1)$. In particular, Γ satisfies the 4-vertex condition.

3.3 Affine polar graphs

So far our graphs were mostly defined on projective points. Here we construct strongly regular graphs the vertices of which are vectors, where the vector space has a polar space on its hyperplane at infinity. These graphs are associated with two-weight codes, cf. §7.1.1.

3.3.1 Isotropic directions

Let V be a vector space of dimension $2m$ over $\mathbb{F}_q$, $m \geq 1$, provided with a nondegenerate quadratic form Q of type ε $(= \pm 1)$. Take as vertices the vectors in V, where two different vectors u and v are joined when $Q(v-u) = 0$. This yields a strongly regular graph Γ with parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$, where

$$\begin{aligned} v &= q^{2m}, & \theta_1 &= \varepsilon(q-1)q^{m-1} - 1, \\ k &= (q^m-\varepsilon)(q^{m-1}+\varepsilon), & \theta_2 &= -\varepsilon q^{m-1} - 1, \\ \lambda &= q(q^{m-1}-\varepsilon)(q^{m-2}+\varepsilon) + q - 2, & m_1 &= (q^m-\varepsilon)(q^{m-1}+\varepsilon) = k, \\ \mu &= q^{m-1}(q^{m-1}+\varepsilon), & m_2 &= q^{m-1}(q-1)(q^m-\varepsilon). \end{aligned}$$

Let us call these graphs $VO^{\varepsilon}_{2m}(q)$.

If we take the Hamming scheme $H(n,4)$ and call two vertices adjacent when their distance is even, we obtain a strongly regular graph (as was observed in [474]). But this is just the graph $VO^{\varepsilon}_{2n}(2)$, where $\varepsilon = (-1)^n$. Indeed, the weight of a quaternary digit is given by the (elliptic) binary quadratic form $x_1^2 + x_1x_2 + x_2^2$.

If $m = 1$, then $VO^{+}_2(q)$ is the $q \times q$ grid graph.

Rank 3 group action

Consider the graph $\Gamma(V, X)$ obtained by taking a vector space V as vertex set, and joining two vectors when the line joining them has a direction in X, where X is a subset of $\mathsf{P}V$. This graph has a transitive group (namely the additive group V). It will have a rank 3 group when the stabilizer of X in the collineation group of $\mathsf{P}V$ has precisely two orbits (namely X and its complement).

The graphs $VO^{\varepsilon}_{2m}(q)$ are obtained when X is the set of points on a quadric (and $\dim V$ is even).

Rank 3 action of the unitary group

If we take an m-dimensional vector space over $F = \mathbb{F}_{q^2}$ provided with a nondegenerate Hermitian form $f(x, y)$, then $Q(x) = f(x, x)$ is a nondegenerate quadratic form over $\mathbb{F}_q$ of type $\varepsilon = (-1)^m$. One finds that $VO^{\varepsilon}_{2m}(q)$ (with $\varepsilon = (-1)^m$) admits a rank 3 action of the group $V.(F^* \circ \mathsf{SU}(m, q))$.

Rank 3 action of the 7-dimensional orthogonal group

Take the graph $VO^+_8(q)$. We claim that it admits a rank 3 action of the group $V.(\mathbb{F}^*_q \circ \mathsf{PSO}_7(q))$, with $\mathsf{PSO}_7(q) \leq \mathsf{PGO}^+_8(q)$ the image under triality of a subgroup $G \leq \mathsf{PGO}^+_8(q)$ stabilizing a nondegenerate hyperplane W of V (or, equivalently, fixing a nonisotropic point).

Let W be a nondegenerate hyperplane of V. Then Q defines in W a polar space (X', Ω') of type $\mathsf{O}_7(q)$, which is a subspace of the polar space (X, Ω) related to Q. Write $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 and Ω_2 the two orbits in Ω under the action of $\mathsf{O}^+_8(q)$. Consider the group $G = \mathsf{PSO}_7(q)$ as subgroup of $\mathsf{PSO}^+_8(q)$ acting naturally on W. Each member of Ω' is contained in precisely one member of Ω_1, and each member of Ω_1 contains precisely one member of Ω'. Hence the group G acts transitively on Ω_1, since it acts transitively on Ω'. Conjugating with an appropriate triality $\tau\colon X \to \Omega_2 \to \Omega_1 \to X$ in $\mathrm{Out}(\mathsf{O}^+_8(q))$, we see that G^τ acts transitively on the singular points of $\mathsf{P}V$.

We now show that G^τ acts transitively on the nonsingular points of $\mathsf{P}V$. Since G^τ acts transitively on the singular points, it suffices to prove that, for some singular point x, the stabilizer $(G^\tau)_x$ acts transitively on the nonsingular points in $\langle x^\perp \rangle$ (here, $\perp$ is with respect to the $\mathsf{O}^+_8(q)$ geometry). We achieve this in two steps: First we show that the stabilizer in $(G^\tau)_x$ of some nonsingular line L in $\langle x^\perp \rangle$ through x acts transitively on the nonsingular points of L; then we show that $(G^\tau)_x$ acts transitively on the nonsingular lines in $\langle x^\perp \rangle$ through x.

We start with determining the order and structure of the kernel $K \leq (G^\tau)_x$ of the action of $(G^\tau)_x$ on the t.s. (totally singular) lines through x.

Set $Z_1 = x^{\tau^{-1}} \in \Omega_1$ and set $Z = Z_1 \cap W$. Let $H := \{g \in G \mid \tau^{-1} g \tau \text{ fixes } x$ and all t.s. lines on $x\}$, so that $K = H^\tau$. Then $H = \{g \in G \mid g \text{ fixes } Z_1$ and all t.s. lines in $Z_1\} = \{g \in G \mid g \text{ fixes } Z_1 \text{ pointwise}\}$.

Taking for Q the standard quadratic form

$$X_{-1}X_1 + X_{-2}X_2 + X_{-3}X_3 + X_{-4}X_4,$$

for W the hyperplane with equation $X_{-4} + X_4 = 0$ and for Z_1 the solid (4-space) with equations $X_1 = X_2 = X_3 = X_4 = 0$, it is easily checked that H corresponds to the family of linear maps with generic matrix (action on the right)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 1 & 0 & 0 & 0 \\ 0 & c & 0 & -b & 0 & 1 & 0 & 0 \\ 0 & 0 & -c & -a & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the coordinates are ordered $(x_{-4}, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, x_4)$. Hence both H and K are elementary abelian groups of order q^3.

Let S_1 be the t.s. solid with equations $X_{-2} = X_{-1} = X_3 = X_4 = 0$. Then $S_1 \in \Omega_1$ since $N := S_1 \cap Z_1$ is a line. Computing the images of S_1 under the action of H, we see that H acts transitively on the q t.s. solids in Ω_1 on N distinct from Z_1, so K acts transitively on $M \setminus \{x\}$ for $M = N^\tau$.

The t.s. solid Z_2 with equations $X_1 = X_2 = X_3 = X_{-4} = 0$ contains Z, belongs to Ω_2, and H fixes Z_2 pointwise, hence fixes all solids in Ω_1 incident with Z_2. Applying τ we find a t.s. solid $S = {Z_2}^\tau$ through x pointwise fixed by K.

Considering a plane through M and a point of S not collinear to all points of M, we deduce that there also exists a nonsingular line L such that K acts transitively on $L \setminus \{x\}$. This shows Step 1.

Now the order of G is $q^9(q^6-1)(q^4-1)(q^2-1)$. Since there are (q^3+1) $(q^2+1)(q+1)$ singular planes in a polar space of type $\mathsf{O}_7(q)$, the stabilizer $(G^\tau)_x$ has order $q^9(q^3-1)(q^2-1)(q-1)$. Hence, since $|K| = q^3$, the quotient group $(G^\tau)_x/K$, which is canonically isomorphic to G_{Z_1}/H, has order $q^6(q^3-1)$ $(q^2-1)(q-1)$. But G_{Z_1}/H acts faithfully on Z_1, and is thus isomorphic to a subgroup of $\mathsf{PGL}(Z_1)_Z$. Since the latter has the same order, G_{Z_1}/H coincides with $\mathsf{PGL}(Z_1)_Z$.

The perp of a nonsingular line on x in $\langle x^\perp \rangle$ induces in $x^\perp$ a degenerate polar space with radical x of which the quotient space with respect to x is of type $\mathsf{O}_5(q)$. The triality τ^{-1} takes the point set of such a space (i.e., the t.s. lines on x of the corresponding degenerate polar space) to the set of lines in Z_1 t.i. with respect to a nondegenerate symplectic form. Hence Step 2 is equivalent to showing that the stabilizer in $\mathsf{PGL}(Z_1) \cong \mathsf{PGL}_4(q)$ of the plane Z acts transitively on the family of symplectic polar spaces of type $\mathsf{Sp}_4(q)$ in $Z_1 \cong \mathsf{PG}(3,q)$, which follows from the transitivity of $\mathsf{PGL}_4(q)$ on this family and the transitivity of the group $\mathsf{Sp}_4(q)$ on the points of $\mathsf{PG}(3,q)$.

Hence we have shown that G^τ acts transitively on the nonsingular points of $\mathsf{P}V$.

Note that, if we would have started with $\mathsf{O}_7(q)$ instead of $\mathsf{PSO}_7(q)$, then we would have ended up with $\mathsf{PSL}_4(q)$, which does not act transitively on the symplectic polar spaces in $\mathsf{PG}(3,q)$ if q is odd.

The Suzuki-Tits ovoid at infinity

There is another rank 3 graph with the same parameters and similar construction as $VO_4^-(q)$. Let O be the Suzuki-Tits ovoid (see §2.5) embedded in $\mathsf{P}V$, where V is a 4-dimensional vector space over the field $\mathbb{F}_q$ with $q = 2^{2e-1}$, and let $VSz(q)$ be the graph $\Gamma(V, O)$ defined as above.

Since O is an ovoid of a symplectic quadrangle, the totally isotropic lines with respect to the corresponding alternating form (hence those of the symplectic quadrangle) intersect O in exactly one point. The first paragraph of the proof of Proposition 2.5.1 shows that nonisotropic lines intersect O in zero or two points. Hence O is an ovoid of $\mathsf{P}V$. There are as many planes that meet O in $q+1$ points forming an oval as there are nonisotropic points. Every such plane contains a unique nucleus of the corresponding oval, which is a point contained in all tangent lines to the oval. These tangent lines are, by the above discussion, totally isotropic, hence a point is the nucleus of exactly one oval. This implies that there is a bijective correspondence between the ovals on O and the points off

O. As the Suzuki group $\mathsf{Sz}(q)$ acts transitively on the ovals, it acts transitively on the points off O, and hence $V.(\mathbb{F}_q^* \circ \mathsf{Sz}(q))$ acts rank 3 on $VSz(q)$.

3.3.2 Square directions

Let V be a vector space of dimension $2m$ over $\mathbb{F}_q$, where q is odd, provided with a nondegenerate quadratic form Q of type ε $(= \pm 1)$. Take as vertices the vectors in V, where two vectors u and v are joined when $Q(v-u)$ is a nonzero square. This yields a strongly regular graph Γ with parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ s^g$, where

$$\begin{aligned} v &= q^{2m}, & \lambda &= \mu + \varepsilon q^{m-1}, \\ k &= \frac{1}{2}(q-1)(q^m - \varepsilon)q^{m-1}, & \mu &= \frac{1}{4}q^{m-1}(q-1)(q^m - q^{m-1} - 2\varepsilon), \\ r &= \frac{1}{2}q^{m-1}(q+\varepsilon), & f &= \frac{1}{2}(q^{2m} - q^m + q^{m-1} - 1) - \frac{1}{2}\varepsilon(q^{2m-1} - 1), \\ s &= -\frac{1}{2}q^{m-1}(q-\varepsilon), & g &= \frac{1}{2}(q^{2m} + q^m - q^{m-1} - 1) + \frac{1}{2}\varepsilon(q^{2m-1} - 1). \end{aligned}$$

Here $f = k$ if $\varepsilon = 1$, and $g = k$ if $\varepsilon = -1$.

Let us call these graphs $VNO^{\varepsilon}(2m, q)$. They have a rank 4 group.

3.3.3 Affine half spin graphs

The first subconstituent of the affine polar graph $VO^{\varepsilon}_{2m}(q)$ is a $(q-1)$-clique extension of the graph $\Gamma(\mathsf{O}^{\varepsilon}_{2m}(q))$. There is also an affine graph that is locally a $(q-1)$-clique extension of the graph $\Delta_{1/2} = \Delta_{1/2}(\mathsf{O}^{+}_{10}(q))$. In order to define and construct this graph, which we shall denote by $VD_{5,5}(q)$, we need to represent the vertex set of $\Delta_{1/2}$ as 1-spaces in a vector space (and not as a set of higher dimensional subspaces, as we did above).

Let $V = V_1 \oplus V_2$ be a 16-dimensional vector space, written as the direct sum of two 8-dimensional subspaces V_1, V_2, over the finite field $\mathbb{F}_q$ (but everything that follows, except for the counts, holds over an arbitrary field). Let $\iota \colon V_1 \to V_2$ be an isomorphism, identify V_1 with $\mathbb{F}_q^8$, labeling the coordinates X_i, with $i \in \{-4, -3, -2, -1, 1, 2, 3, 4\}$ and consider the quadratic form

$$Q \colon V_1 \to \mathbb{F}_q : (x_{-4}, x_{-3}, \dots, x_4) \mapsto X_{-1}X_1 + X_{-2}X_2 + X_{-3}X_3 + X_{-4}X_4.$$

Let $\Phi = \{u \in V_1 \mid Q(u) = 0\}$ be the corresponding hyperbolic quadric in V_1. Recall the map τ defined in the proof of Proposition 3.2.3 sending the 1-spaces in Φ to 4-spaces of V_1 contained in Φ, and define $\rho(u) = \tau(\langle u \rangle)$ for $u \in \Phi \setminus \{0\}$. Let S be the union over all $u \in \Phi \setminus \{0\}$ of the 5-dimensional subspaces $\langle u, \iota\rho(u) \rangle$.

The vertex set of $VD_{5,5}(q)$ is V, and two vectors u_1 and u_2 are adjacent when $u_1 - u_2 \in S$.

Proposition 3.3.1 *The graph $VD_{5,5}(q)$ is a rank 3 strongly regular graph with parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ s^g$, where*

$$\begin{aligned} v &= q^{16}, & r &= q^8 - q^3 - 1, \\ k &= (q^8 - 1)(q^3 + 1), & s &= -(q^3 + 1), \\ \lambda &= q^8 + q^6 - q^3 - 2, & f &= (q^8 - 1)(q^3 + 1) = k, \\ \mu &= q^3(q^3 + 1), & g &= q^3(q^8 - 1)(q^5 - 1). \end{aligned}$$

The proof of this proposition will occupy the rest of this subsection. It will reveal some interesting structure of $VD_{5,5}(q)$ and its underlying geometry.

Lemma 3.3.2 *With the above notation, choose coordinates in V_2 so that ι maps a vector in V_1 to a vector with the same coordinates in V_2. Let the coordinates of a generic vector in V be labeled as*

$$(x_{-4}, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, x_4 \,|\, y_{-4}, y_{-3}, y_{-2}, y_{-1}; y_1, y_2, y_3, y_4).$$

Then S is given by the intersection of the null sets of the following quadratic forms:

$$X_{-4}X_4 + X_{-3}X_3 + X_{-2}X_2 + X_{-1}X_1, \tag{3.1}$$
$$Y_{-4}Y_4 + Y_{-3}Y_3 + Y_{-2}Y_2 + Y_{-1}Y_1, \tag{3.2}$$
$$X_{-4}Y_4 + X_{-3}Y_{-3} + X_{-2}Y_{-2} + X_{-1}Y_{-1}, \tag{3.3}$$
$$X_{-4}Y_3 - X_{-3}Y_{-4} - X_2Y_{-1} + X_1Y_{-2}, \tag{3.4}$$
$$X_{-4}Y_2 + X_3Y_{-1} - X_{-2}Y_{-4} - X_1Y_{-3}, \tag{3.5}$$
$$X_{-4}Y_1 - X_3Y_{-2} + X_2Y_{-3} - X_{-1}Y_{-4}, \tag{3.6}$$
$$X_4Y_{-1} - X_{-3}Y_2 + X_{-2}Y_3 - X_1Y_4, \tag{3.7}$$
$$X_4Y_{-2} + X_{-3}Y_1 - X_2Y_4 - X_{-1}Y_3, \tag{3.8}$$
$$X_4Y_{-3} - X_3Y_4 - X_{-2}Y_1 + X_{-1}Y_2, \tag{3.9}$$
$$X_4Y_{-4} + X_3Y_3 + X_2Y_2 + X_1Y_1. \tag{3.10}$$

Proof. Let T denote the intersection of the null sets of the quadratic forms in the statement of the lemma. We show $S \subseteq T$ and $T \subseteq S$.

Part 1: $S \subseteq T$

We present an algebraic argument. This consists of going through the possible coordinate shapes of a vector u of Φ, $u \neq 0$, and then show that $\langle u, \iota\rho(u)\rangle$ is contained in T. Let us do this for the most involved case, i.e., when u has coordinates

$$(1, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, -x_{-1}x_1 - x_{-2}x_2 - x_{-3}x_3).$$

Then a generic vector of $\langle u, \iota\rho(u)\rangle$ has, according to the proof of Proposition 3.2.3, the following coordinates:

$$\begin{aligned}&(1, x_{-3}, x_{-2}, x_{-1};\; x_1,\; x_2,\; x_3,\; -x_{-1}x_1 - x_{-2}x_2 - x_{-3}x_3 \,|\\ &y_{-4}, y_{-3}, y_{-2}, y_{-1};\; y_{-4}x_{-1} - y_{-3}x_2 + y_{-2}x_3,\;\; y_{-4}x_{-2} + y_{-3}x_1 - y_{-1}x_3,\\ &\qquad y_{-4}x_{-3} - y_{-2}x_1 + y_{-1}x_2,\;\; -y_{-3}x_{-3} - y_{-2}x_{-2} - y_{-1}x_{-1}).\end{aligned}$$

An elementary calculation shows that this vector vanishes under all of the given quadratic forms.

Part 2: $T \subseteq S$

Let there now be given a vector w with coordinates

$$(x_{-4}, x_{-3}, x_{-2}; x_{-1}; x_1, x_2, x_3, x_4 \,|\, y_{-4}, y_{-3}, y_{-2}, y_{-1}; y_1, y_2, y_3, y_4)$$

vanishing under all of the given quadratic forms. Since both sets S and T are projective, we may assume that the first nonzero coordinate is equal to 1. We again treat the most involved case. Suppose $x_{-4} \neq 0$, then we assume $x_{-4} = 1$. Expressing that w is in the null set of the quadratic forms 3.1, 3.3, 3.4, 3.5 and 3.6 implies

$$\begin{aligned} x_4 &= -x_{-1}x_1 - x_{-2}x_2 - x_{-3}x_3, \\ y_4 &= -y_{-3}x_{-3} - y_{-2}x_{-2} - y_{-1}x_{-1}, \\ y_3 &= y_{-4}x_{-3} - y_{-2}x_1 + y_{-1}x_2, \\ y_2 &= y_{-4}x_{-2} + y_{-3}x_1 - y_{-1}x_3, \\ y_1 &= y_{-4}x_{-1} - y_{-3}x_2 + y_{-2}x_3, \end{aligned}$$

and yields the coordinates of a generic vector of $\langle u, \iota\rho(u)\rangle$, with u as in Part 1 of this proof. □

Let us denote by G the automorphism group of $VD_{5,5}(q)$ induced by $\mathsf{AGL}(V)$, and by G_0 the stabilizer in G of the zero vector of V; so $G_0 = G \cap \mathsf{GL}(V)$.

Lemma 3.3.3 *The group G_0 acts transitively on the set of 1-spaces in S.*

Proof. Each of the quadratic forms 3.1–3.10 defines a hyperbolic quadric of type $\mathsf{O}_8^+(q)$ in an 8-dimensional subspace of V (generated by the basis vectors corresponding to the variables appearing in the quadratic form). We first show that G_0 acts transitively on this set of ten quadrics.

We define a graph Υ on the set of basis vectors of V by declaring two basis vectors e en f adjacent if $e + f \in S$ (hence Υ is the graph on the basis vectors induced by Γ). It is easy to see that two basis vectors are adjacent in Υ if and only if the corresponding coordinate variables do not appear together in a common term of one of the forms 3.1–3.10. One now checks that the correspondence

$X_{-4} \mapsto 00000$	$X_1 \mapsto 10010$	$Y_{-4} \mapsto 00011$	$Y_1 \mapsto 01111$
$X_{-3} \mapsto 11000$	$X_2 \mapsto 01010$	$Y_{-3} \mapsto 00101$	$Y_2 \mapsto 10111$
$X_{-2} \mapsto 10100$	$X_3 \mapsto 00110$	$Y_{-2} \mapsto 01001$	$Y_3 \mapsto 11011$
$X_{-1} \mapsto 01100$	$X_4 \mapsto 11110$	$Y_{-1} \mapsto 10001$	$Y_4 \mapsto 11101$

yields an isomorphism of Υ (where we indicated every basis vector with its corresponding coordinate variable) to the Clebsch graph, which is the graph on the set of even weight binary vectors of length 5, adjacent when the Hamming distance is 2, see §10.7. We now claim that the full automorphism group of Υ acts on (extends to) Γ.

We define $g_1, g_2, g_3 \in \mathsf{GL}(V)$ by their action on a generic vector

$$u = (x_{-4}, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, x_4 \,|\, y_{-4}, y_{-3}, y_{-2}, y_{-1}; y_1, y_2, y_3, y_4)$$

as follows:

$$\begin{aligned} g_1 \;:\; u &\mapsto (-x_{-4}, x_{-3}, x_{-2}, x_{-1}; y_{-1}, y_{-2}, y_{-3}, -y_4 \\ &\qquad |\, -y_{-4}, x_3, x_2, x_1; y_1, y_2, y_3, -x_4), \\ g_2 \;:\; u &\mapsto (-x_{-4}, x_{-1}, -x_2, x_3; -x_{-3}, x_{-2}, -x_1, x_4 \\ &\qquad |\, y_{-1}, y_{-4}, y_{-3}, y_{-2}; y_2, y_3, y_4, y_1), \\ g_3 \;:\; u &\mapsto (x_{-1}, -x_{-4}, -x_{-3}, x_{-2}; x_2, -x_3, -x_4, x_1 \\ &\qquad |\, y_1, -y_4, -y_{-3}, y_{-2}; y_2, -y_3, -y_{-4}, y_{-1}). \end{aligned}$$

Then one checks that g_i, $i = 1, 2, 3$, stabilizes the set of quadratic forms 3.1–3.10. Moreover, g_1, g_2 fix the coordinate X_{-4}, hence they permute the five binary coordinate positions in the representation of Υ given above. The action of g_2 is a 4-cycle on the first four coordinate positions, whereas g_1 induces the transposition related to the last two positions. Hence $\langle g_1, g_2\rangle$ induces the full stabilizer S_5 of 00000 in Aut(Υ). Since g_3 moves the basis vector corresponding to the coordinate X_{-4}, we conclude that $\langle g_1, g_2, g_3\rangle$ induces the full automorphism group of Υ and our claim, to which we will refer as Observation 1, is proved.

Now we observe that, since ρ is a triality, every automorphism $\varphi \in \mathsf{GL}(V_1)$ of Φ preserving each of the natural systems of maximal singular subspaces, induces an automorphism $\iota(\varphi) \in \mathsf{GL}(V_2)$ of $\iota(\Phi)$, unique up to a scalar, such that $(\varphi, \iota(\varphi))$, acting on $V_1 \oplus V_2$, preserves S. We refer to this as Observation 2. We denote the group of automorphisms of Φ preserving the systems of maximal singular subspaces by Aut°(Φ).

We note that Witt's theorem implies that the stabilizer in Aut°(Φ) of a maximal singular subspace W of Φ (as an embedded polar space) acts transitively on the 1-spaces of W. This will be referred to as Observation 3.

Now let $u \in S$ be given by coordinates as above and different from the zero vector. We establish an automorphism of S mapping u to a vector in V_1. The transitivity of Aut(S) on the 1-spaces of S then follows from Observation 2.

First note that, if $(x_{-4}, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, x_4) = (0, 0, 0, 0; 0, 0, 0, 0)$, then $u \in \iota(\Phi)$ and, using Observation 1, we may use an automorphism of S interchanging V_1 and V_2; this automorphism does the job. Henceforth we assume $(x_{-4}, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0; 0, 0, 0, 0)$.

Then note that, by Observation 2, we may assume that

$$(x_{-4}, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, x_4) = (1, 0, 0, 0; 0, 0, 0, 0).$$

Let $e \in V$ be the first basis vector (all coordinates 0 except for the first, which is 1). Then Observation 2, combined with Observation 3 (applied to the maximal singular subspace $\langle \iota\rho(e)\rangle$), implies that, if $u \notin V_1$, we may assume that

$$(y_{-4}, y_{-3}, y_{-2}, y_{-1}; y_1, y_2, y_3, y_4) = (1, 0, 0, 0; 0, 0, 0, 0).$$

Hence u is contained in the quadric corresponding to one of the quadratic forms 3.4, 3.5 or 3.6. But then Observation 1 yields an automorphism of S mapping u in Φ. □

Now we can start looking at the parameters of $VD_{5,5}(q)$. Clearly, $v = |V| = q^{16}$. Also, $k = |S| = |\Phi \setminus \{0\}| \cdot (q^4 + 1) = (q^8 - 1)(q^3 + 1)$.

Proposition 3.3.4 $\lambda = q^8 + q^6 - q^3 - 2$.

Proof. Obviously, λ is equal to $q - 2$ plus $q^2 - q$ times the number N_u of 2-spaces entirely contained in S and containing a given vector u of S. By the previous lemma we may assume $u \in \Phi$. Let us briefly call two vectors or 1-spaces of S spanning a 2-space entirely contained in S *collinear*. Then N_u can be written as $N_1 + N_2 + N_3$, where N_1 is the number of 2-spaces in Φ through u, N_2 is the number of 1-spaces in $\iota\rho(u)$, and $N_3 = N_u - N_1 - N_2$. Clearly

$$\begin{aligned} N_1 &= (q^2 + 1)(q^2 + q + 1), \\ N_2 &= q^3 + q^2 + q + 1. \end{aligned}$$

Taking into account that every 1-space of $V \setminus (V_1 \cup V_2)$ lies on a unique 2-space intersecting both V_1 and V_2 nontrivially, we deduce that a vector w of $V \setminus (V_1 \cup \langle u, \iota\rho(u)\rangle)$ is collinear to u if and only if it is contained in a 3-space intersecting V_2 in a 1-space T on $\iota\rho(u)$ and V_1 in a 2-space U containing u and contained in Φ. There are q^3+q^2+q+1 possibilities for T, and fixing T, there are q^2+q+1 possibilities for U. This yields

$$N_3 = (q-1)(q^3+q^2+q+1)(q^2+q+1) = (q^4-1)(q^2+q+1).$$

An easy calculation now completes the proof of the proposition. □

Lemma 3.3.5 *Every* 1*-space of* S *is contained in exactly* q^3+q^2+q+1 *quadrics of type* $\mathsf{O}_8^+(q)$ *entirely contained in* S *and contained in the orbit of* Φ *under* G_0*.*

Proof. Let $u \in S$, $u \neq 0$, be arbitrary. By Lemma 3.3.3 we may assume $u \in \Phi$ is the first basis vector of the standard basis. Let e be the eighth basis vector of the standard basis. Then $W = \iota\rho(e)$ is a 4-space in $\iota(\Phi)$ disjoint from $\iota\rho(u)$. The 5 standard basis vectors of $\langle e, \iota\rho(e)\rangle$ are each contained in a quadric of type $\mathsf{O}_8^+(q)$ entirely contained in S and contained in the orbit of Φ under G_0, by Observation 1 of the proof of Lemma 3.3.3. Since the stabilizer in $\mathrm{Aut}^\circ(\iota(\Phi))$ of the 4-space $\iota\rho(u)$ acts transitively on the 1-spaces of $\iota(\rho(e))$, we deduce that every 1-space of $\iota\rho(e)$ is together with u contained in a quadric of type $\mathsf{O}_8^+(q)$ entirely contained in S and contained in the orbit of Φ under G_0. Again using Observation 1 of the proof of Lemma 3.3.3, we see that the stabilizer of u in G_0 acts transitively on the set of 1-spaces of $\langle e, \iota\rho(e)\rangle$, and hence deduce that each 1-space of the latter is together with u contained in a quadric of type $\mathsf{O}_8^+(q)$ entirely contained in S and contained in the orbit of Φ under G_0. This yields $q^4+q^3+q^2+q+1$ such quadrics. Denote by $\mathscr{Q}$ this set of quadrics.

Now we claim that two such quadrics intersect in a singular subspace. Indeed, by transitivity we may assume that one of them is Φ. Now two noncollinear 1-spaces of Φ are only collinear with common 1-spaces of Φ, as follows from the construction of S. This yields the claim.

We conclude that there are precisely $q^6(q^4+q^3+q^2+q+1)$ 1-spaces of S not collinear to u contained in some member of $\mathscr{Q}$. But that is exactly equal to the number of 1-spaces on S (namely, $(q^4+1)(q^3+1)(q^2+1)(q+1)$), minus the number of 1-spaces of S spanned by or collinear to u (and that is equal to $1+q(q^4+q^3+q^2+q+1)(q^2+1)$), which concludes the proof of the lemma. □

We can now finish the proof of Proposition 3.3.1. It remains to determine the value of μ.

Proposition 3.3.6 *The group* G_0 *acts transitively on* $V \setminus S$*. Also, for a given vector* $w \in V \setminus S$*, precisely* $\frac{1}{2}(q^6+q^3)$ 2*-spaces containing* w *intersect* S *in exactly two* 1*-spaces (and we call such a* 2*-space a secant), and no* 2*-space through* w *intersects* S *in at least three* 1*-spaces. This implies* $\mu = q^6+q^3$*.*

Proof. It is easy to see directly from the construction of S that no 2-space of V intersecting V_1 in a 1-space outside Φ has more than one 1-space in common with S. This first implies, after a moment's thought, that no vector outside S is contained in at least two 8-spaces spanned by members of $\mathscr{Q}$. Second, this implies that all secants through $w \in V_1 \setminus \Phi$ are contained in V_1. This easily yields $\frac{1}{2}(q^6+q^3)$ secants through such w. Now we count the number of 1-spaces contained in at least (and then in precisely) one 8-space spanned by a member of $\mathcal{Q}$.

An elementary double count reveals that

$$|\mathcal{Q}| = (q^4+q^3+q^2+q+1)(q^4+1).$$

This gives rise to $(q^4+q^3+q^2+q+1)(q^{11}-q^3)$ 1-spaces all nonzero vectors w of which satisfy the proposition. But this is exactly equal to the number of 1-spaces in $V \setminus S$, as one easily calculates.

The transitivity of G_0 on $V \setminus S$ now follows from the transitivity of G_0 on $\mathcal{Q}$ together with the transitivity of $\mathsf{GO}_8^+(q)$ on the nonisotropic vectors.

Since the common neighbors in $VD_{5,5}(q)$ of 0 and w are given by the nonzero vectors $u \in S$ such that $w-u \in S$, each secant through w defines two such common neighbors. Hence $\mu = q^6+q^3$. □

Automorphism group

The additive group of V, which is isomorphic to the elementary abelian group q^{16}, acts simply transitively on the vertex set of $VD_{5,5}(q)$. The full isomorphism group of $VD_{5,5}(q)$ is the group of index 2 in $q^{16}:\mathrm{Aut}(\mathsf{GO}_{10}^+(q))$ preserving the systems of maximal singular subspaces.

Cliques and cocliques

The maximal cliques correspond to the maximal subspaces of V contained in S, and these have dimensions 5 and 4, each forming a single orbit. Examples of the former are $\langle u, \iota\rho(u)\rangle$, with $u \in \Phi$; examples of the latter are the maximal singular subspaces of $\iota(\Phi)$ not in the natural system containing $\iota\rho(u)$, for some $u \in \Phi$.

There are cocliques of size q^4 obtained by the span of two 2-spaces; one in V_1 intersecting Φ trivially, and one in V_2 intersecting $\iota(\Phi)$ trivally. We conjecture that these are maximal (but there are several orbits).

Note that the sizes of the cliques and cocliques mentioned above are much smaller than the Hoffman bound q^8.

3.4 Forms graphs

3.4.1 Bilinear forms graphs

The *bilinear forms graph* $H_q(d,e)$ is the graph of which the vertices are the $d \times e$ matrices over the field $\mathbb{F}_q$, adjacent when the difference has rank 1. This graph has q^{de} vertices, and is distance-transitive of diameter $\min(d,e)$, cf. [123], Theorem 9.5.2. The neighbors of the zero matrix are the rank 1 matrices $xy^\top$, where $x \in \mathbb{F}_q^d$ and $y \in \mathbb{F}_q^e$. If we fix y and vary x, or fix x and vary y, we find cliques of sizes q^d and q^e.

The bilinear forms graph $H_q(d,e)$ is isomorphic to the graph on the d-subspaces of a $(d+e)$-space that are disjoint from a fixed e-space E, adjacent when they meet in codimension 1. There are two types of maximal cliques:

those of size q^d (all vertices contained in a fixed $(d+1)$-space that meets E in a single point), and those of size q^e (all vertices containing a fixed $(d-1)$-space disjoint from E).

In particular, for $d = 2$ and $e \geq 2$ we get a strongly regular graph with parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ s^g$, where

$$\begin{aligned} v &= q^{2e}, & r &= q^e - q - 1, \\ k &= (q+1)(q^e - 1), & s &= -q - 1, \\ \lambda &= q^e + (q-2)(q+1), & f &= k, \\ \mu &= q(q+1), & g &= v - k - 1 = q(q^e - 1)(q^{e-1} - 1). \end{aligned}$$

The large cliques reach the Hoffman bound, and we have a partial geometry $pg(K, R, T)$ with $K = q^e$, $R = q + 1$, $T = q$ (cf. §8.6). This is a net, a dual transversal design. The small cliques are the lines of a semipartial geometry (cf. §8.7.2, (vii)).

This graph is its own Delsarte dual (cf. §7.1.3).

For $d = e = 2$, the condition $\operatorname{rk} M \leq 1$ for a $d \times e$ matrix M is equivalent to $m_{11}m_{22} - m_{12}m_{21} = 0$, and the bilinear forms graph is the strongly regular graph $VO_4^+(q)$ with vertex set $\mathbb{F}_q^4$ where two vertices are adjacent when the line joining them hits the hyperplane at infinity in a point of a fixed hyperbolic quadric.

Let $q = p^r$ with p prime and r an integer. The full automorphism group of the graphs $H_q(d, e)$ is $G = p^{rde} : (q-1) : (\mathsf{PGL}_d(q) \times \mathsf{PGL}_e(q)) : r$ when $d \neq e$, and is $G.2$ when $d = e$. This group acts distance-transitively ([123], Theorem 9.5.1). In particular, for $d = 2$ and $e \geq 2$ the group G is rank 3. For $q = 2$, it is easy to see that the group $2^{2e} : (\mathsf{S}_3 \times H)$ still acts rank 3 for any subgroup H of $\mathsf{PGL}_e(2)$ acting transitively on the set of points and on the set of lines of $\mathsf{PG}(e-1, 2)$. For example, for $e = 2, 3, 5$ one can take $H = 3,\ 7,\ 31 : 5$. For $e = 4$ the smallest rank 3 group is $2^8 : (3 \times \mathsf{A}_7)$.

3.4.2 Alternating forms graphs

The *alternating forms graph* on $\mathbb{F}_q^n$ is the graph of which the vertices are the skew-symmetric matrices with zero diagonal of order n over the field $\mathbb{F}_q$, adjacent when the difference has rank 2. This graph has $q^{n(n-1)/2}$ vertices, and is distance-transitive of diameter $\lfloor n/2 \rfloor$, cf. [123], Theorem 9.5.6. In particular, we get a strongly regular graph for $n = 4$ and $n = 5$. The parameters for $n = 4$ are

$$\begin{aligned} v &= q^6, & r &= q^3 - q^2 - 1, \\ k &= (q^2+1)(q^3 - 1), & s &= -q^2 - 1, \\ \lambda &= q^4 + q^3 - q^2 - 2, & f &= k, \\ \mu &= q^2(q^2+1), & g &= v - k - 1 = q^2(q^3 - 1)(q - 1). \end{aligned}$$

The parameters for $n = 5$ are

$$\begin{aligned} v &= q^{10}, & r &= q^5 - q^2 - 1, \\ k &= (q^2+1)(q^5 - 1), & s &= -q^2 - 1, \\ \lambda &= q^5 + q^4 - q^2 - 2, & f &= k, \\ \mu &= q^2(q^2+1), & g &= v - k - 1 = q^2(q^5 - 1)(q^3 - 1). \end{aligned}$$

This graph is its own Delsarte dual (cf. §7.1.3).

For $n = 4$, the condition $\mathrm{rk}\, A \leq 2$ for an alternating matrix A is equivalent to $a_{12}a_{34} + a_{14}a_{23} + a_{13}a_{42} = 0$, and the alternating forms graph is the strongly regular graph $VO_6^+(q)$ with vertex set $\mathbb{F}_q^6$ where two vertices are adjacent when the line joining them hits the hyperplane at infinity in a point of a fixed hyperbolic quadric. In the special case $n = 4$, $q = 2$ we find the complement of the folded halved 8-cube.

3.4.3 Quadratic forms graphs

The *quadratic forms graph* on $V = \mathbb{F}_q^n$ is the graph of which the vertices are the quadratic forms on V, adjacent when the rank of the difference is 1 or 2. It has $v = q^{n(n+1)/2}$ vertices, and is distance-regular of diameter $\lfloor (n+1)/2 \rfloor$ (Egawa [305]; cf. [123], Theorem 9.6.3). In particular, we get a strongly regular graph for $n = 3$ and $n = 4$.

The quadratic forms graph on $\mathbb{F}_q^n$ is distance-regular with the same parameters as the alternating forms graph on $\mathbb{F}_q^{n+1}$, but these graphs are nonisomorphic for $n \geq 3$, $(n, q) \neq (3, 2)$, as the former is not distance-transitive.

3.4.4 Hermitian forms graphs

Let $q = u^2$, where u is a prime power. Let $\overline{x} = x^u$. A *Hermitian matrix* is a matrix A satisfying $\overline{A} = A^\top$. The *Hermitian forms graph* on $\mathbb{F}_q^d$ is the graph of which the vertices are the Hermitian matrices of order d over the field $\mathbb{F}_q$, adjacent when the difference has rank 1. This graph has u^{d^2} vertices, and is distance-transitive of diameter d, with parameters $b_i = (u^{2d} - u^{2i})/(u + 1)$, $c_i = u^{i-1}(u^i - (-1)^i)/(u + 1)$ $(0 \leq i \leq d)$ ([123], Theorem 9.5.7). In particular, we get a strongly regular graph for $d = 2$. The parameters are

$$\begin{aligned} v &= u^4, & r &= u - 1, \\ k &= (u^2 + 1)(u - 1), & s &= -u^2 + u - 1, \\ \lambda &= u - 2, & f &= u(u - 1)(u^2 + 1), \\ \mu &= u(u - 1), & g &= k. \end{aligned}$$

Let Δ be the collinearity graph of the dual polar space $\mathsf{U}(2d, q)$. Then Δ is distance-regular of diameter d, and the Hermitian forms graph is the graph induced on the vertices at distance d from a fixed vertex of Δ ([123], Theorem 9.5.10).

When lines (q-cliques) are given, one can use this to characterize the Hermitian forms graph:

Theorem 3.4.1 (Ivanov & Shpectorov [458]) *Let Γ be a distance-regular graph with the parameters of the Hermitian forms graph, and assume that each edge in Γ is contained in a clique of size q. If $d \geq 3$, then u is a prime power, and Γ is the Hermitian forms graph on $\mathbb{F}_q^d$. If $d = 2$, then Γ is the subgraph induced on the vertices at distance 2 from a fixed vertex in a generalized quadrangle $\mathsf{GQ}(q, q^2)$.*

3.4.5 Baer subspaces

Let V be a vector space of dimension m over $\mathbb{F}_{q^2}$, so that $|V| = q^{2m}$, and let X be a Baer subspace of the hyperplane $\mathsf{P}V$ at infinity, so that $|X| = \frac{q^m-1}{q-1}$. For hyperplanes H, the intersection size $|X \cap H|$ takes the two values $m_1 = \frac{q^{m-1}-1}{q-1}$ and $m_2 = \frac{q^{m-2}-1}{q-1}$. It follows (cf. §7.1.1) that the graph with vertex set V, where two vectors $x, y \in V$ are joined when $\langle y - x\rangle \in X$, is strongly regular with parameters

$$\begin{aligned} v &= q^{2m}, & r &= q^m - q - 1, \\ k &= (q+1)(q^m - 1), & s &= -q - 1, \\ \lambda &= q^m + q^2 - q - 2, & f &= (q+1)(q^m - 1), \\ \mu &= q(q+1), & g &= q(q^{m-1} - 1)(q^m - 1). \end{aligned}$$

This graph is isomorphic to the bilinear forms graph $H_q(2, m)$.

More generally, let V be a vector space of dimension e over $\mathbb{F}_{q^d}$, so that $|V| = q^{de}$, and let X be an $\mathbb{F}_q$-subspace of dimension e of the hyperplane $\mathsf{P}V$ at infinity, so that $|X| = \frac{q^e-1}{q-1}$. Then the graph with vertex set V, where two vertices $x, y \in V$ are joined when $\langle y - x\rangle \in X$, is isomorphic to $H_q(d, e)$. The q^d-cliques are the lines of V in the direction of X. The q^e-cliques are the $\mathbb{F}_q$-subspaces of dimension e with X as hyperplane at infinity.

The special case $d = 3$, $e = 2$ occurs in the classification of rank 3 groups because $\mathsf{PGL}_2(r)$ has two orbits (of sizes $r + 1$ and $r^3 - r$) on $\mathsf{PG}(1, r^3)$.

In fact $\mathsf{P}V$ has a partition into $\frac{q^m+1}{q+1}$ Baer subspaces. Each hyperplane H hits one in $\frac{q^{m-1}-1}{q-1}$ points, and $\frac{q^m-q}{q+1}$ in $\frac{q^{m-2}-1}{q-1}$ points. Let D be the union of t of these Baer subspaces, where $0 < t < \frac{q^m+1}{q+1}$. Then $|D \cap H|$ takes the two values $t\frac{q^{m-2}-1}{q-1}$ and $q^{m-2} + t\frac{q^{m-2}-1}{q-1}$. Let Γ be the graph on V where $x, y \in V$ are joined when $\langle y - x\rangle \in D$. Then Γ is strongly regular with parameters

$$\begin{aligned} v &= q^{2m}, & r &= q^m - t(q+1), \\ k &= t(q+1)(q^m - 1), & s &= -t(q+1), \\ \lambda &= q^m + t(q+1)(tq + t - 3), & f &= t(q+1)(q^m - 1), \\ \mu &= t(q+1)(tq + t - 1), & g &= (q^m - 1)(q^m + 1 - tq - t). \end{aligned}$$

3.4.6 A hyperoval at infinity

Let V be a 3-dimensional vector space over $\mathbb{F}_q$, where q is even, and let X be a fixed hyperoval of the hyperplane $\mathsf{P}V$ at infinity, so that $|X| = q + 2$. Now $|X \cap H|$ takes the two values 0 and 2 for lines H. It follows (cf. §7.1.1) that the graph with vertex set V, where two vectors $x, y \in V$ are joined when $\langle y-x\rangle \in X$, is strongly regular with parameters

$$\begin{aligned} v &= q^3, & r &= q - 2, \\ k &= (q-1)(q+2), & s &= -q - 2, \\ \lambda &= q - 2, & f &= \frac{1}{2}(q^2 - 1)(q+2), \\ \mu &= q + 2, & g &= \frac{1}{2}q(q-1)^2. \end{aligned}$$

These graphs are the collinearity graphs of generalized quadrangles with parameters $(q-1, q+1)$. See [6].

3.5 Grassmann graphs

The graph on the d-subspaces of an n-space, adjacent when they meet in a $(d-1)$-space, is distance-regular of diameter d (for $n \geq 2d$). The case $d=2$ yields strongly regular graphs.

3.5.1 Lines in a projective space

Let Γ be the graph on the lines in $\mathsf{PG}(n-1,q)$, where $n \geq 4$, adjacent when they meet. (This is the Grassmann graph $J_q(n,2)$, cf. §1.2.4.) Then Γ is strongly regular, with parameters $v = \begin{bmatrix} n \\ 2 \end{bmatrix}$, $k = (q+1)(\begin{bmatrix} n-1 \\ 1 \end{bmatrix} - 1)$, $\lambda = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} + q^2 - 2$, $\mu = (q+1)^2$, and eigenvalues k, $r = q^2 \begin{bmatrix} n-3 \\ 1 \end{bmatrix} - 1$, $s = -q-1$ with multiplicities, 1, $f = \begin{bmatrix} n \\ 1 \end{bmatrix} - 1$, $g = \begin{bmatrix} n \\ 2 \end{bmatrix} - \begin{bmatrix} n \\ 1 \end{bmatrix}$.

For $n=4$, the lines can be seen as points on the Klein quadric, and Γ is isomorphic to the $\mathsf{O}_6^+(q)$ graph.

Group

The full automorphism group $\mathrm{Aut}\,\Gamma$ of Γ is $\mathsf{PGL}_n(q)$ if $n > 4$ and $\mathsf{PGL}_n(q).2$ if $n = 4$.

Cliques

Maximal cliques are maximal sets of pairwise intersecting lines, and come in two types: (i) all lines on a given point, and (ii) all lines in a given plane. Sets of type (i) have size $\begin{bmatrix} n-1 \\ 1 \end{bmatrix}$ (and reach the Hoffman bound), those of type (ii) have size $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Both types are in the same $\mathrm{Aut}\,\Gamma$-orbit for $n=4$.

Cocliques

Maximal cocliques are maximal sets of pairwise disjoint lines. If n is even, the largest of these are line spreads, of size $(q^n-1)/(q^2-1)$. If n is odd, the largest are partial spreads of size $(q^n - q^3)/(q^2-1) + 1$ (Beutelspacher [66]).

Chromatic number

If the set of all lines can be partitioned into spreads, then n is even and Γ has chromatic number $\chi(\Gamma) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}$. Such a partition is known as a *line packing* or *parallelism*. The existence of a parallelism is known for $n=4$ (Denniston [282]), for $n = 2^e$, $e \geq 2$ (Beutelspacher [65]), for $q=2$, n even (Baker [34]; see also [728]), and for $(q,n) = (3,6)$ (Etzion & Vardy [308]).

For odd $n \geq 5$, and $q=2$, Meszka [561] showed that $\chi(\Gamma) = 2^{n-1} + 2$.

3.6 The case $q = 2$

3.6.1 Local structure

We have precise information about the local structure of the polar graphs $O^\varepsilon_m(2)$.

Proposition 3.6.1 (Brouwer & Shult [142])

$$\begin{aligned} TO^\varepsilon_m(2) &= 1 + O^\varepsilon_m(2) + O^\varepsilon_m(2) + 1 \\ VO^\varepsilon_{2n}(2) &= 1 + O^\varepsilon_{2n}(2) + NO^\varepsilon_{2n}(2) \quad \textit{and} \quad VO_{2n+1}(2) = TO_{2n+1}(2) \\ NO^\varepsilon_{2n}(2) &= 1 + O_{2n-1}(2) + TO^{-\varepsilon}_{2n-2}(2) \\ O^\varepsilon_m(2) &= 1 + O^\varepsilon_{m-2}(2).2 + VO^\varepsilon_{m-2}(2), \end{aligned}$$

Here we indicate the subgraphs found at a given distance from a fixed point, writing $\Gamma = \Gamma_0(x) + \Gamma_1(x) + \Gamma_2(x) + \cdots$. The graphs occurring here are $O^\varepsilon_m(2)$, the graph on the singular points, adjacent when orthogonal, $NO^\varepsilon_{2n}(2)$, the graph on the nonsingular points, adjacent when orthogonal, $VO^\varepsilon_m(2)$, the graph on $\mathbb{F}_2^m$ where distinct vectors x, y are adjacent when $Q(y - x) = 0$, and $TO^\varepsilon_m(2)$, the Taylor extension of $O^\varepsilon_m(2)$. The notation $\Gamma.2$ denotes the 2-clique extension of Γ.

Small cases are $O^-_2(2) = K_0$, $O^-_4(2) = \overline{K_5}$, $O^+_2(2) = \overline{K_2}$, $O^+_4(2) = 3 \times 3$, $NO^-_2(2) = \overline{K_3}$, $NO^-_4(2) = \overline{T(5)}$, $NO^+_2(2) = K_1$, $NO^+_4(2) = K_{3,3}$, $NO^+_6(2) = \overline{T(8)}$, $VO^-_2(2) = \overline{K_4}$, $VO^+_2(2) = 2 \times 2$, $TO^-_2(2) = \overline{K_2}$, $TO^+_2(2) = C_6$.

As a consequence, the size of the largest cocliques in $O^\varepsilon_m(2)$ depends on m (mod 8). See also [746].

Proposition 3.6.2 *For even $m \geq 2$, the largest cocliques in the graphs $TO^\varepsilon_m(2)$, $NO^\varepsilon_m(2)$, $VO^\varepsilon_m(2)$, and $O^\varepsilon_m(2)$ have sizes given in the following table*

$m \pmod 8$	0	2	4	6
$TO^-_m(2)$	m	m	$m+2$	$m+1$
$TO^+_m(2)$	$m+2$	$m+1$	m	m
$NO^-_m(2)$	$m-1$	$m+1$	m	$m-1$
$NO^+_m(2)$	m	$m-1$	$m-1$	$m+1$
$VO^-_m(2)$	m	$m+2$	$m+1$	m
$VO^+_m(2)$	$m+1$	m	m	$m+2$
$O^-_m(2)$	$m-1$	$m-1$	$m+1$	m
$O^+_m(2)$	$m+1$	m	$m-1$	$m-1$

except for the empty graph $O^-_2(2)$, where the largest coclique has size 0.

If we call this maximum $c_{\max}$, then the smaller maximal cocliques have all possible sizes $c_0 \leq c < c_{\max}$ with $c \equiv c_0 \pmod 4$, where $c_0 = 2, 3, 4, 5$ for the cases $TO^\varepsilon_m(2)$, $NO^\varepsilon_m(2)$, $VO^\varepsilon_m(2)$, $O^\varepsilon_m(2)$, respectively. When size c occurs, there is a single orbit of maximal cocliques of size c. □

For example, $O^-_{14}(2)$ has single orbits of maximal cocliques of sizes 5, 9, 13, 14. In particular, the bound obtained from Theorem 2.6.3 holds with equality for $O^-_{8t+4}(2)$ and $O^+_{8t}(2)$.

Proposition 3.6.3 *The maximal cocliques in $VO_{2n+1}(2)$, $n \geq 0$, have all even sizes c with $2 \leq c \leq 2n+2$. The maximal cocliques in $O_{2n+1}(2)$, $n \geq 1$ have all odd sizes c with $3 \leq c \leq 2n+1$. When size c occurs, there is a single orbit of maximal cocliques of size c. The graph $O_1(2)$ has no vertices. The graph $Sp_{2n}(2)$ is isomorphic with $O_{2n+1}(2)$.* □

The maximal cliques of $O^-_{2n}(2)$, $O^+_{2n}(2)$, $O_{2n+1}(2)$ have size $2^{n-1}-1$, 2^n-1, 2^n-1, respectively. The maximal cliques of $VO^-_{2n}(2)$, $VO^+_{2n}(2)$, $VO_{2n+1}(2)$ have size 2^{n-1}, 2^n, 2^n, respectively. The maximal cliques of $NO^\varepsilon_{2n}(2)$ have size 2^{n-1}. In all cases they form a single orbit.

Above we gave the partition of a binary orthogonal graph around a point. There are further such partitions, induced by the perp of a nonsingular point. These give rise to regular sets

$$O^\varepsilon_{2n}(2) = TO^\varepsilon_{2n-2}(2) + O_{2n-1}(2).$$

3.6.2 Symmetric groups

Let V be the m-dimensional vector space over $\mathbb{F}_2$, provided with the quadratic form $Q(x) = \sum_{i<j} x_i x_j$. Then the symmetric group S_m acts on V by coordinate permutation.

We determine the type of the corresponding polar space. Let $\mathrm{wt}(x)$ be weight of the vector x, i.e., its number of nonzero coordinates. Then $Q(x) = \binom{\mathrm{wt}(x)}{2}$ and $B(x,y) = \mathrm{wt}(x)\mathrm{wt}(y) + \sum_i x_i y_i$. In particular, $B(x,\mathbf{1}) = (m+1)\mathrm{wt}(x)$. It follows that $V^\perp = \{\mathbf{1}\}$ if m is odd, while the bilinear form B is nondegenerate if m is even. A vector x is singular when $\mathrm{wt}(x) \equiv 0$ or $1 \pmod 4$, and nonsingular when $\mathrm{wt}(x) \equiv 2$ or $3 \pmod 4$. In particular, the space V is degenerate only when $m \equiv 1 \pmod 4$.

Proposition 3.6.4 *(i) If $m = 4t$, the polar space $\mathsf{P}V$ and the quotient $\mathbf{1}^\perp/\langle\mathbf{1}\rangle$ are both nondegenerate of Witt type $(-1)^t$.*

(ii) If $m = 4t+1$, the quotient $V/\langle\mathbf{1}\rangle$ is nondegenerate of Witt type $(-1)^t$.

(iii) If $m = 4t+2$, the polar space $\mathsf{P}V$ is nondegenerate of Witt type $(-1)^t$.

(iv) If $m = 4t+3$, the polar space $\mathsf{P}V$ is nondegenerate parabolic. □

Chapter 4

Buildings

Generalizing the situation of projective spaces and polar spaces, Tits associated a *building* to arbitrary Chevalley groups and classified the resulting groups and geometries in [694]. Finite buildings of type E_6 have strongly regular collinearity graphs that are most easily and naturally described in this buildings setup.

4.1 Geometries

A *Buekenhout-Tits geometry* (or just *geometry*) Γ is a set X of *objects* together with a *type function* $t\colon X \to I$, where I is the set of *types*, and a symmetric and reflexive *incidence* relation $*$ such that if $x * y$ and $x \neq y$, then $t(x) \neq t(y)$. The corresponding intuition is that one has objects of several types, maybe points and lines and planes and circles, and that objects of different types may be incident; conventionally each object is incident with itself.

The *rank* of a geometry is the cardinality $|I|$ of its set of types.

A *flag* F in a geometry is a set of pairwise incident objects. If $t(F) = I$ (that is, if F contains one object of each type), then F is called a *chamber*. The *residue* $\mathrm{Res}_\Gamma F$ (or just $\mathrm{Res}\, F$) of a flag F in a geometry $\Gamma = (X, I, t, *)$ is the geometry $\Delta = (X', I', t', *')$, where X' is the set of objects not in F incident with each element of F, and $I' = I \setminus t(F)$, and t', $*'$ are the restrictions of $t, *$ to X' and $X' \times X'$, respectively. We say that I' is the *type* of $\mathrm{Res}\, F$.

A geometry is called *connected* when its *incidence graph* (with the objects as vertices, different objects joined when they are incident) is connected. A geometry is called *residually connected* when all of its residues of rank at least 2 are connected, and all of its residues of rank at least 1 are nonempty (i.e., have a nonempty set of objects). A residually connected geometry is called *thick* (resp. *thin*) when all of its residues of rank 1 have at least three (resp. precisely two) objects.

A *subgeometry* of a geometry $\Gamma = (X, I, t, *)$ is a geometry $(Y, J, t', *')$ with $Y \subseteq X$, $J \subseteq I$, and $t', *'$ the restrictions of $t, *$ to Y and $Y \times Y$, respectively.

4.1.1 Generalized polygons

A *generalized polygon* (*generalized d-gon*) with $d \geq 3$ is a partial linear space with an incidence graph of diameter d and girth $2d$. For example, a generalized 3-gon is a projective plane.

A generalized polygon *of order* (s, t) is one where each line has $s + 1$ points, and each point is on $t + 1$ lines. If $s = t$ one says *of order* s.

The *dual* of a generalized polygon (P, L) is the generalized polygon (L, P) obtained by interchanging the roles of points and lines.

The standard reference for generalized polygons is Van Maldeghem [710].

Example: the Fano plane

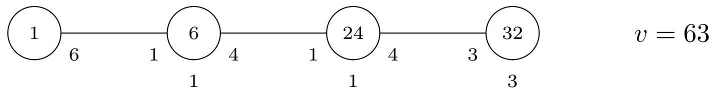
The *Fano plane* is the (unique) projective plane of order 2. It has 7 points and 7 lines. One can take as points the integers mod 7, and as lines the sets $\{0, 1, 3\} + i$ (mod 7). In the notation of §6.2, it is an $S(2, 3, 7)$.

Example: the generalized quadrangle of order 2

There is a unique generalized quadrangle of order 2. It has 15 points and 15 lines. One can take as points the $\binom{6}{2} = 15$ pairs from a set Ω of 6 symbols, and as lines the partitions of Ω into three pairs, where a point is incident with a line when the pair is one of the parts of the partition.

Example: the generalized hexagons of order 2

There are precisely two nonisomorphic generalized hexagons of order 2, one the dual of the other, so that the incidence graph is uniquely determined (Cohen & Tits [205]). They have 63 points and 63 lines. Diagram of the collinearity graph:



Combinatorially the two can be distinguished by looking at the subgraph of the collinearity graph induced by the vertices at distance 3 from a given point. In what one calls the classical $\mathsf{G}_2(2)$ generalized hexagon, this subgraph is connected. In its dual this subgraph has two connected components of size 16. See also [114].

The classical generalized hexagon of order 2 is found by taking the $7 + 7 + 21 + 28 = 63$ points, lines, flags, and antiflags of the Fano plane as points, and whenever (p, L) is a flag of the Fano plane and $L = \{p, q, r\}$ and p is on L, M, N, taking the sets $\{p, L, (p, L)\}$ and $\{(p, L), (q, M), (r, N)\}$ as lines ([711]).[1]

According to [691], the dual of the classical generalized hexagon of order 2 is found by taking as points the nonisotropic points of $\mathsf{PG}(2, 9)$ provided with a nondegenerate Hermitian form, and as lines the orthogonal bases. See also [710], (1.3.12) and [123], p. 384.

Terminology: what we have called here the 'classical' generalized hexagon of order 2 (to distinguish it from its dual) is known in the literature as the 'short root' or 'split Cayley' or 'symplectic' generalized hexagon (the latter in characteristic 2), whereas its dual is called the 'long root' or 'dual split Cayley' generalized hexagon. For a construction of the split Cayley hexagon over an arbitrary field, see §4.8.

[1] This construction shows that the classical generalized hexagon of order 2 contains the generalized hexagon of order $(1, 2)$. Its dual does not—this is another way to distinguish them.

4.1.2 Diagrams

A *diagram* for a geometry is a labeled directed graph on the set of types. It is interpreted as an axiom system for the geometry, as follows: the label on the pair (i, j) is a class of rank 2 geometries Γ_{ij} with set of types $\{1, 2\}$ (thought of as {points, lines}) such that each residue of rank 2 with set of types $\{i, j\}$ is isomorphic to a member of Γ_{ij} under an isomorphism that maps i to 1 and j to 2.

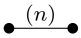
Below a dictionary of traditional labels.

 Every point is incident to every line.

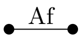
 Points and lines of a projective plane.

 Points and lines of a generalized quadrangle.

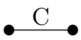
 Points and lines of a generalized hexagon.



Points and lines of a generalized n-gon, $n \geq 2$



Points and lines of an affine plane.

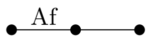


Points and edges of a complete graph.

Examples

The geometry of points, lines and planes in a 3-dimensional projective space satisfies the axioms given by the diagram .

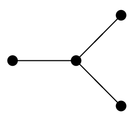
(That is: the lines and planes on a point form the points and lines of a projective plane; every point on a line is on every plane containing that line; the points and lines on a plane are the points and lines of a projective plane.)

The geometry of points, lines and planes in a 3-dimensional affine space satisfies the axioms given by the diagram 

.

The geometry of 8 vertices, 12 edges and 6 faces of a cube satisfies the axioms given by the diagram . This is a thin geometry.

The geometry of totally singular points, lines, planes and solids in a geometry of type $\mathsf{O}_8^+(F)$ satisfies the axioms given by the diagram .

The geometry of totally singular points, lines, solids of the first kind, and solids of the second kind in a geometry of type $\mathsf{O}_8^+(F)$ satisfies the axioms given by the diagram  . The solids are 4-spaces (as vector spaces) and two solids of the same type have an intersection of even (vector space) dimension. Two solids of different types are incident when they meet in a plane (that is, in a 3-space).

4.1.3 Simple properties

In principle, the diagram is a labeled complete graph. However, we omit the edges labeled with the label of invisibility which denotes a generalized digon (every point incident to every line). Now it makes sense to talk about connected components of the diagram.

Proposition 4.1.1 (BUEKENHOUT [155]) *Let $\Gamma = (X, I, t, *)$ be a residually connected Buekenhout-Tits geometry of finite rank. Let $X_i = t^{-1}(i)$ be the set of objects of type i.*

(i) For any two distinct types $i, j \in I$, the subgraph of the incidence graph induced on $X_i \cup X_j$ is connected.

(ii) If the types i, j belong to different connected components of the diagram, then each i-object is incident with each j-object.

Proof. (i) Induction on the rank. The case of rank at most 2 holds by definition. Since Γ is connected, we can join two objects in $X_i \cup X_j$ by a chain $x_0 * x_1 * \cdots * x_l$. Next, for each x_h in this chain with a type different from i and j, we can replace x_h by a chain in $X_i \cup X_j$ in $\text{Res}(x_h)$ (by the induction hypothesis and residual connectedness).

(ii) Induction on the rank. The case of rank at most 2 holds by definition. Using part (i) we can join two objects $x \in X_i$ and $y \in X_j$ by a chain $x = x_0 * x_1 * \cdots * x_l = y$ contained in $X_i \cup X_j$ (so that the types alternate between i and j). Let the length l be chosen minimal, and suppose that $l > 1$. Let k be a third type different from i and j. We may suppose that j and k belong to different connected components of the Buekenhout-Tits diagram. In $\text{Res}(x_1)$ we can replace $x_0 * x_1 * x_2$ by a path $x_0 = x'_0 * x'_1 * \cdots * x'_m = x_2$ using only types i and k. Now x_3 and its two predecessors in the chain have types k-i-j, and by the induction hypothesis we can omit the middle object (of type i). Then x_3 and its two predecessors have types i-k-j, and again we can omit the middle object. It follows after m steps that $x_0 * x_3$, so that l was not minimal. □

After this preparation, it is an easy exercise to prove the Veblen-Young axiom from the A_n diagram, so that a (thick) residually connected geometry satisfying the A_n diagram is a projective space.

Buildings (§4.5) provide the prototypes of diagram geometries.

4.1.4 Shadow geometries

Consider a geometry $(X, I, t, *)$ and fix an element $i \in I$, calling the objects of that type *points*. Let the *shadow* of any flag F be the set of points p incident with all elements of F. Let *lines* be the shadows of the flags of cotype $\{i\}$ (i.e., of type $I \setminus \{i\}$). In this way, a Buekenhout-Tits geometry yields a point-line geometry (where lines are sets of points) if we specify the point type.

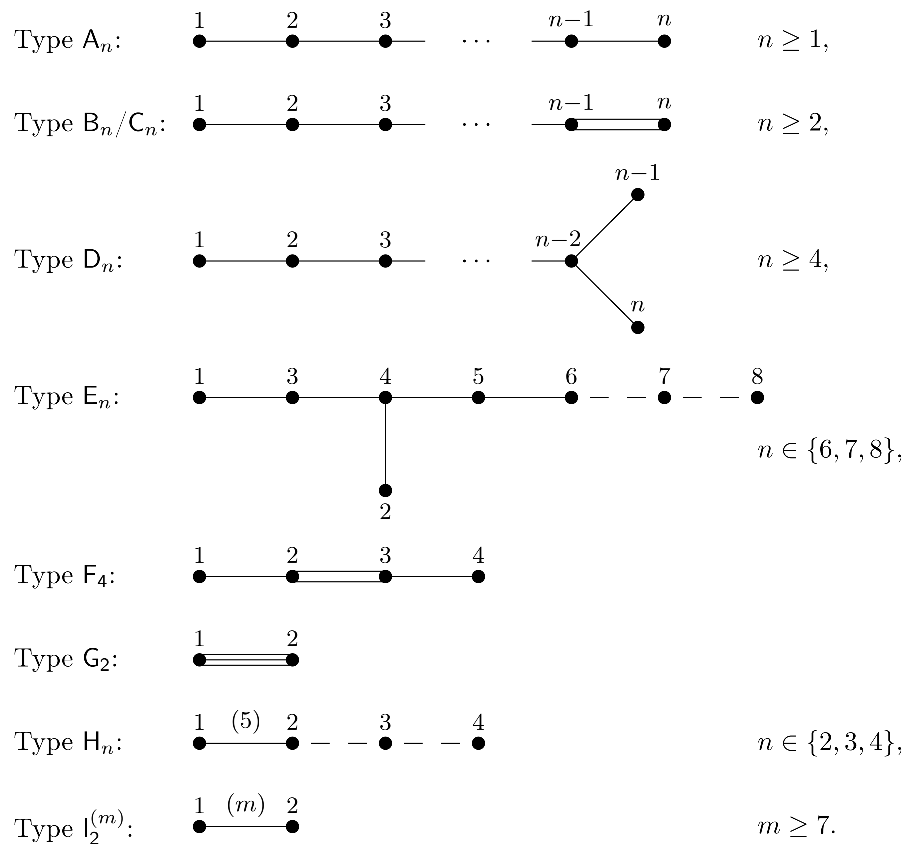
4.2 Coxeter systems

Let W be a group generated by a finite nonempty set $S = \{s_1, \ldots, s_n\}$ of involutions and let, for each pair $(s_i, s_j) \in S \times S$, the number m_{ij} be the order of the product $s_i s_j$ (setting $m_{ij} = \infty$ if $s_i s_j$ generates an infinite group). Then (W, S) is a *Coxeter system*, and W is a *Coxeter group*, if W has the presentation by generators and relations $W = \langle S : (s_i s_j)^{m_{ij}} = 1, \forall i, j \in \{1, 2, \ldots, n\} \rangle$. The natural number n is called the *rank* of the system. Two Coxeter systems (W, S) and (W', S') are *isomorphic* if there is a bijection $S \to S'$ extending to an isomorphism $W \to W'$.

The symmetric matrix $(m_{ij})_{1 \le i,j \le n}$ is called the *Coxeter matrix* belonging to (W, S). The *Coxeter diagram* is the edge labeled graph $\Gamma(W, S)$ with vertex set S and no edge between s_i and s_j if $m_{ij} = 2$; otherwise an edge with label (m_{ij}) between s_i and s_j, for all $i, j \in \{1, 2, \ldots, n\}$. The labels of edges with label (3) are usually omitted, those with label (4) are usually drawn as a double edge, and those with label (6) are sometimes drawn as a triple edge. Note that the Coxeter diagram completely determines the Coxeter group and system. However, it is not true that any Coxeter group determines a unique isomorphism class of Coxeter systems, as distinct sets of generators may lead to different Coxeter diagrams. For example, the Coxeter group $\mathsf{D}_{12} = \langle s_1, s_2 : s_1^2 = s_2^2 = (s_1s_2)^6 = 1\rangle$ is also generated by the involutions $r_1 := s_1$, $r_2 := s_2s_1s_2$, and $r_3 := s_2s_1s_2s_1s_2s_1$, and the group can be presented as $\langle r_1, r_2, r_3 : r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^3 = (r_1r_3)^2 = (r_2r_3)^2 = 1\rangle$.

Let (W, S) be a Coxeter system. If $S = S_1 \cup S_2$, with $W = \langle S_1\rangle \times \langle S_2\rangle$ (then automatically $S_1 \cap S_2 = \emptyset$), then we say that (W, S) is *reduced.* If (W, S) is not reduced, then it is called *irreducible.* For instance, the above Coxeter group D_{12} is the direct product $\langle r_1, r_2\rangle \times \langle r_3\rangle$.

We will only be concerned with finite Coxeter groups. These were classified by Coxeter [237], and the Coxeter diagrams of the irreducible ones are the following.



Standard references for Coxeter groups are Bourbaki [102] and Humphreys [448].

Remarks

- Most finite irreducible Coxeter systems (W, S) are related to an irreducible *crystallographic root system*, i.e., a finite set R of vectors spanning the real Euclidean n-space $\mathbb{R}^n$, $n = |S|$, not contained in the union of two nontrivial orthogonal subspaces and satisfying the following three conditions: (1) if $v \in R$ and $rv \in R$, for some $r \in \mathbb{R}$, then $r \in \{1, -1\}$; (2) if $v, w \in R$, then $w - 2\frac{\langle v,w\rangle}{\langle v,v\rangle} v \in R$; and (3) if $v, w \in R$, then $2\frac{\langle v,w\rangle}{\langle v,v\rangle} \in \mathbb{Z}$. Given an irreducible crystallographic root system R, there exists a basis $B \subseteq R$ of $\mathbb{R}^n$, called a *fundamental basis*, such that every element of R can be expressed as a linear combination of members of B only using either nonnegative integer coefficients, or nonpositive integer coefficients. The set S of reflections about the hyperplanes perpendicular to the members of B generates the automorphism group W of R, and (W, S) is a Coxeter system. The Coxeter systems arising as such are the ones of types A to G above.

- The reason why the second diagram has two names (B_n and C_n) is because this particular Coxeter system is related to two nonisomorphic root systems, one of type B_n and one of type C_n. A root system of type B_n (resp. C_n) can be obtained from one of type C_n (resp. B_n) by multiplying the shortest vectors by 2.

- The *Dynkin diagram* of a crystallographic root system is an edge labeled graph with vertices the elements of a fundamental basis, and an edge with label (k) joining two vertices if the angle between the corresponding basis vectors is equal to $\frac{k-1}{k}\pi$. Edges with label (2) are usually omitted. It is easy to see that basis vectors corresponding to vertices joined by an edge with label (3) have the same length. If the label is (4) or (6), then the length of one vector is $\sqrt{2}$ or $\sqrt{3}$, respectively, times that of the other. No other labels are possible. An edge with label (4) or (6) is further furnished with an arrow pointing from the longer to the shorter vector. By removing the arrows of the Dynkin diagram one obtains the Coxeter diagram of the corresponding Coxeter system.

- Coxeter groups of type A_n are isomorphic to the full symmetric group $\mathsf{Sym}(n+1)$; those of type B_n are the full automorphism group of the n-cube; the one of type F_4 is the automorphism group of the 24-cell in $\mathbb{R}^4$; those of type H_n, $n = 2, 3, 4$ are the automorphism group of a regular pentagon in $\mathbb{R}^2$, a dodecahedron or icosahedron in $\mathbb{R}^3$, and a 120-cell or 600-cell in $\mathbb{R}^4$, respectively.

- The Coxeter groups of types $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ are isomorphic to the groups $\mathsf{GO}_6^-(2)$, $2 \times \mathsf{GO}_7(2)$, $2.\mathsf{GO}_8^+(2)$, respectively.

- Coxeter systems of type $\mathsf{I}_2^{(m)}$ are dihedral groups D_{2m} with generators two reflections about axes forming an angle of π/m. Occasionally one denotes the types $\mathsf{A}_2, \mathsf{B}_2, \mathsf{G}_2, \mathsf{H}_2$ by $\mathsf{I}_2^{(3)}$, $\mathsf{I}_2^{(4)}$, $\mathsf{I}_2^{(6)}$, $\mathsf{I}_2^{(5)}$, respectively.

We mention some fundamental properties of Coxeter systems, the first one of which is called the *deletion condition*.

Proposition 4.2.1 *Let (W, S) be a Coxeter system and let $w \in W$ be arbitrary. Let $\ell(w)$ be the minimum length of an expression in the generators (members of S) producing w. Suppose $w = s_1 s_2 \cdots s_m$, with $m > \ell(w)$. Then there exist $i, j \in \{1, 2, \ldots, m\}$, with $i < j$, such that $w = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_m$.*

Proposition 4.2.2 *Every symmetric matrix $M = (m_{ij})_{1 \le i,j \le n}$, with $m_{ij} \in \mathbb{Z}_{>1} \cup \{\infty\}$, for all $i \neq j$, and $m_{ii} = 1$, for all i, is the Coxeter matrix belonging to a Coxeter system. In other words, if $S = \{s_1, s_2, \ldots, s_n\}$ and $W = \langle S : (s_i s_j)^{m_{ij}} = 1, \forall i, j \in \{1, 2, \ldots, n\}\rangle$, then (W, S) is a Coxeter system with Coxeter matrix M; in particular, the order of the product $s_i s_j$ is exactly equal to m_{ij}. Also, for any subset $S' \subseteq S$, the system $(\langle S' \rangle, S')$ is a Coxeter system with Coxeter matrix the restriction of M to S', with self-explaining terminology.*

A consequence of these properties is the following.

Corollary 4.2.3 *Let (W, S) be a Coxeter system, and let $w \in W$ be arbitrary. Then all expressions of w in the elements of S of length $\ell(w)$ contain exactly the same elements of S.*

Proof. Induction on $\ell(w)$. Set $\ell = \ell(w)$ and let $s_1 s_2 \cdots s_\ell$ and $r_1 r_2 \cdots r_\ell$ be two expressions of w in the elements of S. We have $s_1 \cdots s_{\ell-1} = r_1 r_2 \cdots r_\ell s_\ell$, and the right-hand side is not reduced, while the expression $r_1 r_2 \cdots r_\ell$ is, so s_ℓ can be canceled against some factor r_i, and $s_1 \cdots s_{\ell-1} = r_1 \cdots r_{i-1} r_{i+1} \cdots r_\ell$. Similarly, $s_2 \cdots s_\ell = r_1 \cdots r_{j-1} r_{j+1} \cdots r_\ell$ for some j, proving (by induction) that the s_i occur among the r_j. □

This also implies that, with the terminology of Proposition 4.2.2, $\langle S' \rangle \cap S = S'$. Another consequence is the following.

Corollary 4.2.4 *Let (W, S) be a Coxeter system. Let $R, T \subseteq S$. Then $\langle R \rangle \cap \langle T \rangle = \langle R \cap T \rangle$.*

Proof. Clearly $\langle R \cap T \rangle \le \langle R \rangle \cap \langle T \rangle$. Conversely, let $w \in \langle R \rangle \cap \langle T \rangle$. The set S_w of elements occurring in one (and then each) minimal expression of w is contained in R and in T, hence in $R \cap T$. It follows that $w \in \langle R \cap T \rangle$. □

4.3 Coxeter geometries

Let (W, S) be a Coxeter system. A *standard parabolic subgroup* is a subgroup of W generated by a proper subset of S. A *parabolic subgroup* is a conjugate of a standard parabolic subgroup. A *maximal* standard parabolic subgroup is one not properly contained in another one, i.e., generated by all but one elements of S. We shall use the notation $P_T = \langle S \setminus T \rangle$ for $T \subset S$, and $P_s = P_{\{s\}}$.

Let (W, S) be a Coxeter system. We define a *Coxeter geometry* $\Gamma(X, S, t, *)$ as follows. The set X is the set of all right cosets of any maximal standard parabolic subgroup. Two members of X are incident if they are, as subsets of W, not disjoint. The type function is defined by $t(P_s w) = s$ for $s \in S$ and $w \in W$. We have the following results.

Lemma 4.3.1 *Let (W, S) be a Coxeter system, and let $T \subseteq S$. If the cosets $P_t w_t$ (for $t \in T$) meet pairwise, then $\bigcap_{t \in T} P_t w_t$ is nonempty. If $T = S$, then this intersection is a singleton.*

Proof. Let $U = \{t \in T \mid w_t = 1\}$. Apply induction on $|T \setminus U|$.

If $|T \setminus U| = 0$, then $\bigcap_{t \in T} P_t w_t = P_T$, as desired.

If $t \in T \setminus U$, then let w_t be a shortest representative of $P_t w_t$. Since $P_t w_t$ meets P_u for all $u \in U$, w_t can be written without u for all $u \in U$, so that $w_t \in P_U$. Now multiply on the right by w_t^{-1} to reduce to the case $U' = U \cup \{t\}$. Finally, if $T = S$, then we reduce to $P_S = \{1\}$. □

Proposition 4.3.2 *Let (W, S) be a Coxeter system. Then the corresponding Coxeter geometry $\Gamma(X, S, t, *)$ is a residually connected thin Buekenhout-Tits geometry of rank $|S|$.*

Proof. It is clear that the type function t is well defined, as W is not generated by a proper subset of S. In the previous lemma we showed:

(*) *Any flag of type $T \subseteq S$ can be written as $\{P_t w : t \in T\}$ for some $w \in W$, that is, is the collection of objects of type t (with $t \in T$) containing $P_T w$.*

It follows that a chamber is just a coset of the trivial subgroup; hence the chambers are in one-to-one correspondence with the elements of W. Let $s \in S$ and let F be a flag of type $S \setminus \{s\}$. Then (*) implies that F is the set of cosets of maximal standard parabolics containing a fixed coset of $\{1, s\}$, say $\{1, s\}w$, $w \in W$. Then only $P_s w$ and $P_s sw$ complete F to chambers, corresponding to w and sw, respectively. Hence $\Gamma(X, S, t, *)$ is thin.

Now let $T \subseteq S$, $|T| < |S| - 1$, and let F be a flag of type T. Using an appropriate translate, we may assume that F is the set $\{P_t : t \in T\}$. Since $\bigcap_{t \in T} P_t = \langle S \setminus T \rangle$, the set of elements incident with every member of F can be identified with the set of maximal standard parabolics of $\langle S \setminus T \rangle$. Hence $\operatorname{Res} F$ is the Coxeter geometry corresponding to the Coxeter system $(\langle S \setminus T \rangle, S \setminus T)$. Hence, to show local connectivity, it suffices to show that every Coxeter geometry of rank at least 2 is connected.

Consider the graph on X where adjacency is incidence. For each w, all objects $P_s w$ ($s \in S$) are mutually adjacent, and hence are in the same connected component. If $w \neq 1$, say $w = rv$ with $\ell(w) = \ell(v) + 1$, then the factor r can be absorbed in P_s whenever $s \neq r$. So induction on $\ell(w)$ shows that this graph is connected. □

Since Coxeter geometries are residually connected and thin, all rank 2 residues of a Coxeter geometry are (as incidence graphs) even length cycles or (bi)infinite paths. In the finite case only cycles appear, and the cycle has 2ℓ vertices if and only if the corresponding Coxeter group (of the residue) is the dihedral group $\mathsf{D}_{2\ell}$. A cycle with 2ℓ vertices is the incidence graph of a geometry called an *ordinary polygon*.

Hence the Coxeter diagram can be interpreted geometrically as follows: the label of the diagram of a Coxeter system (W, S) between nodes i and j is (k) if and only if each residue in the corresponding Coxeter geometry $(X, S, t, *)$ of type $\{i, j\}$ is an ordinary k-gon.

4.4 Coxeter geometries of types A_n, D_n and E_6

We describe the Coxeter geometries of types A_n, D_n and E_6 and find that they belong to the complete graph K_{n+1}, the complete n-partite graph $K_{n\times 2}$, and the Schläfli graph (§10.10).

An object of type i will be called an i-object. Given a diagram X_n and a point type i we denote the corresponding shadow geometry by $\mathsf{X}_{n,i}$.

A_n

Let $\Omega = \{1, \dots, n+1\}$. The Coxeter group (W, S) of type A_n can be taken to be the symmetric group $\mathsf{Sym}(\Omega)$ of order $(n+1)!$, with set of generators $S = \{s_1, \dots, s_n\}$, where s_i is the transposition $(i, i+1)$ interchanging i and $i+1$.

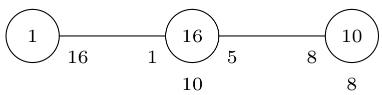
The standard i-object can be identified with the i-subset $\{1, \dots, i\}$ of Ω fixed by the standard maximal parabolic $\mathsf{Sym}\{1, \dots, i\} \times \mathsf{Sym}\{i+1, \dots, n+1\}$. Then the i-objects are the i-subsets of Ω, collinear when they meet in an $(i-1)$-set (and have an $(i+1)$-set as union). It follows that the collinearity graph of the shadow geometry of type $\mathsf{A}_{n,i}$ is the Johnson graph $J(\Omega, i)$.

D_n

Let Ω be the set $\{1, \dots, n\} \times \{\pm 1\}$. The Coxeter group (W, S) of type D_n can be taken to be the group W of shape $2^{n-1}{:}\mathsf{Sym}(n)$ acting on Ω by permutation of $\{1, \dots, n\}$, and changing an even number of signs, together with the generators $S = \{s_1, \dots, s_n\}$, where s_i interchanges $(i, \pm 1)$ and $(i+1, \pm 1)$ (preserving signs) for $1 \le i \le n-1$, and s_n interchanges $(n-1, \pm 1)$ and $(n, \mp 1)$.

Let Γ be the complete n-partite graph $K_{n\times 2}$ on Ω (with $(i, 1)$ and $(i, -1)$ nonadjacent $(1 \le i \le n)$). The i-objects can be identified with the i-cliques in Γ $(1 \le i \le n-2)$. The $(n-1)$-objects and n-objects can be identified with the n-cliques in Γ containing an even (odd) number of vertices with second coordinate -1, adjacent when they differ by a single sign change. We see that Γ is the collinearity graph of the shadow geometry of type $\mathsf{D}_{n,1}$, and find for $\mathsf{D}_{n,n-1}$ and $\mathsf{D}_{n,n}$ the halved graphs of the Hamming graph $H(n, 2)$.

E_6



$v = 27$

The Weyl group $W(\mathsf{E}_6)$ is isomorphic to $\mathsf{GO}_6^-(2)$, and the collinearity graph Γ of the shadow geometry of type $\mathsf{E}_{6,1}$ is the Schläfli graph, the noncollinearity graph of an elliptic quadric in $\mathsf{PG}(5, 2)$ (§10.10).

The i-objects of the Coxeter geometry of type E_6 $(1 \le i \le 6)$ can be identified with the vertices, 6-cliques, edges, triangles, maximal 5-cliques and subgraphs $K_{5\times 2}$ in Γ, respectively.

4.5 Buildings

Buildings provide a geometrical setting e.g. for groups of Lie type. They were introduced in Tits [694]. See also [4], [147], [628], [727].

4.5.1 Generalities

Let (W, S) be a Coxeter system with corresponding Coxeter geometry $(X, S, t, *)$, which will be called the *standard apartment*. A *building of type* (W, S) is a geometry $(B, S, t, *)$ endowed with a family $\mathscr{A}$ of subgeometries, called *apartments*, over the type set S, all isomorphic (preserving types) to $(X, S, t, *)$, such that

(B1) Every pair of flags of $(B, S, t, *)$ is contained in a member of $\mathscr{A}$;

(B2) If two flags F, F' are both contained in two apartments Σ, Σ', then there exists an isomorphism (preserving types) $\Sigma \to \Sigma'$ fixing $F \cup F'$ vertexwise.

Note that it does no harm to have the same notation for the type map and the incidence relation in the building and in the standard apartment.

The Coxeter group W is sometimes also called the *Weyl group* of the building. If (W, S) is of type X_n, with $\mathsf{X} \in \{\mathsf{A}, \ldots, \mathsf{G}\}$ and n appropriate, then the building is also said to be of type X_n itself.

The family of apartments is not necessarily unique, but in finite buildings it always is.

To gain more insight into the structure of a building, we now determine its diagram, using the list of traditional labels. So we ought to look at the residues.

First note that any set B can be seen as a building of rank 1 by considering every pair of elements of B as an apartment; the Weyl group is the group of order 2.

Proposition 4.5.1 *Any nonempty residue of a building is a building.*

Proof. Let F be a flag of the building $\Delta = (B, S, t, *)$ of type (W, S) and set of apartments $\mathscr{A}$, and assume that F is not a chamber. Endow $\operatorname{Res} F$ with the family of apartments $\mathscr{A}_F = \{\operatorname{Res}_\Sigma F : F \subseteq \Sigma \in \mathscr{A}\}$. Pick two flags G, G' in $\operatorname{Res}_\Delta F$. Then $F \cup G$ and $F \cup G'$ are contained in a common apartment Σ, and so G and G' are contained in the common apartment $\operatorname{Res}_\Sigma F$ of $\operatorname{Res} F$.

Now assume that G and G' are both contained in two apartments $\operatorname{Res}_\Sigma F$ and $\operatorname{Res}_{\Sigma'} F$, with $\Sigma, \Sigma' \in \mathscr{A}$. Then any type preserving isomorphism $\Sigma \to \Sigma'$ fixing $F \cup G \cup G'$ vertexwise induces a type preserving isomorphism $\operatorname{Res}_\Sigma F \to \operatorname{Res}_{\Sigma'} F$ fixing $G \cup G'$ vertexwise. □

A straightforward example of a building is a Coxeter geometry. The easiest thick examples are those of rank 2 related to finite dihedral groups.

Proposition 4.5.2 *Let (W, S) be a Coxeter system of rank 2 with W finite. Then every building of type (W, S) is a generalized polygon, more exactly a generalized $\frac{|W|}{2}$-gon. Conversely, every generalized polygon is a building of rank 2 with finite dihedral Weyl group.*

Proof. Let $(B, S, t, *)$ be a rank 2 building with finite Weyl group W, say $|W| = 2n$. The graph $\Gamma = (B, *)$ is bipartite and the bipartition classes correspond to the types S. According to the definition in §4.1.1, we only need to show that Γ has diameter n and girth $2n$. In fact, it suffices to show that the diameter is at most n and the girth is exactly $2n$.

Note that apartments of $(B, S, t, *)$ are $2n$-cycles in Γ. Since every pair of vertices is contained in an apartment by (B1), we see that the diameter of Γ is at most n. Also, the girth is even, say $2g$, and at most $2n$.

Let $\gamma = (v_1, v_2, \ldots, v_{2g})$ be any $2g$-cycle. Obviously, by the definition of girth, the distance between two vertices of γ in Γ equals the distance between these vertices in γ (as a subgeometry). Let $j < g$ be maximal with the property that every apartment through two vertices of γ at distance j from each other contains all vertices on the shortest path between them in γ. Note that j is well defined since obviously $j \geq 1$. Suppose for a contradiction that $j < g - 1$. Without loss we may assume that there is an apartment Σ containing v_1 and v_{j+2} not containing any of $v_2, \ldots, v_{j+1}$. Any apartment Σ' through $\{v_1, v_2\}$ and v_{j+2} contains the path $\mu = (v_1, v_2, \ldots, v_{j+2})$. Now μ together with its image under any isomorphism $\Sigma' \to \Sigma$ fixing v_1 and v_{j+2} forms a cycle of length $2j + 2 < 2g$, a contradiction. Hence $j = g - 1$.

Now consider an apartment Σ'' through the chambers $\{v_1, v_2\}$ and $\{v_{g+1}, v_{g+2}\}$. By the previous paragraph, Σ'' contains γ, so $g = n$ as desired.

The converse is easy (the apartments being the $2n$-cycles of the incidence graph). □

We can now recover the diagram of any building.

Corollary 4.5.3 *The diagram of any building as a Buekenhout-Tits geometry coincides with the diagram of its Weyl group as the corresponding Coxeter system.*

Proof. Follows directly from Propositions 4.5.1 and 4.5.2. □

4.5.2 Spherical buildings

A building is called *spherical* if its Weyl group is finite. Non-spherical buildings are necessarily infinite, hence we now take a closer look at the spherical ones, in particular the finite ones.

By Corollary 4.5.3, spherical buildings are geometries of type A_n, $n \geq 1$, B_n, $n \geq 2$, D_n, $n \geq 4$, E_n, $n = 6, 7, 8$, F_4, H_n, $n = 3, 4$, or $\mathsf{I}_2^{(m)}$, $m \geq 5$. In each case the Coxeter geometry of the corresponding type is a finite thin example. Thick buildings of type H_3 and H_4 do not exist (see [695]). Also, thick finite buildings of type $\mathsf{I}_2^{(m)}$ with $m \neq 2, 3, 4, 6, 8$, do not exist by [316] (by an eigenvalue argument).

We give the identification of finite thick buildings with classical geometries.

- Thick buildings of type A_n, $n \geq 2$, are the projective spaces. With the numbering of the nodes of the diagram as before, elements of type i correspond to subspaces of projective dimension $i - 1$ and incidence is symmetrized containment.

- Thick buildings of type B_n, $n \geq 2$, are the thick polar spaces, i.e., polar spaces of order (s, t) with $s, t \geq 2$. With the numbering of the nodes of the diagram as before, elements of type i correspond to singular subspaces of projective dimension $i - 1$ and incidence is symmetrized containment.

- Thick buildings of type D_n, $n \geq 4$, are the oriflamme geometries of the non-thick polar spaces, i.e., of the polar spaces of order $(s, 1)$, $s \geq 2$. The *oriflamme geometry* of a non-thick polar space of rank n is the geometry of rank n where the elements of type i, $1 \leq i \leq n - 2$, are the singular subspaces of projective dimension $i-1$, and where the elements of type $n-1$ and n correspond to the partition of maximal singular subspaces into the two classes given by the bipartite graph of Theorem 2.2.17. Incidence is given by containment when at least one element has type $i \leq n-3$; two elements of types $n-1$ and n are incident if they intersect in a singular subspace of projective dimension $n-2$. The moral here is that we throw away the $(n-2)$-spaces as elements of the geometry, but they sneak in again via the incidence (in graph theoretical language: they cease to be vertices and become edges).

- Thick buildings of type E_n, $n \in \{6,7,8\}$, and F_4 are called of *exceptional type*. They do not correspond to classical objects. Only type E_6 will be of interest to us, and we provide an explicit construction below (§4.9.3).

In the not necessarily finite case, thick buildings of type A_n also include vector spaces over skew fields, and for $n = 2$ also non-Desarguesian projective planes. Similarly, the projective spaces that occur as residues in arbitrary thick buildings of types B_n and F_4 need not be defined over fields and for B_3 and F_4 can be non-Desarguesian. On the other hand, thick buildings of types D_n $(n \geq 4)$ and E_n are always defined over a field and uniquely determined by that field. In the infinite case, the s and t in the order of polar spaces can be infinite cardinal numbers.

In the finite case there is a unique building of type A_n, D_n $(n \geq 3)$ and $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ such that the rank 2 residues are projective planes that have order q. We denote such buildings by $\mathsf{X}_n(q)$, $\mathsf{X} \in \{\mathsf{A}, \mathsf{D}, \mathsf{E}\}$. The corresponding shadow geometry with respect to type i (in the labeling given in the list of diagrams in Section 4.2) is denoted by $\mathsf{X}_{n,i}(q)$. If we do not want to specify the field, then we write $\mathsf{X}_{n,i}$.

4.5.3 Characterizations

Tits [696] characterizes various buildings as residually connected geometries with given diagram and point type such that the shadows (cf. §4.1.4) satisfy certain axioms. Brouwer & Cohen [124] show that in the case of E_6 these axioms are automatically satisfied. Hence

Proposition 4.5.4 *Every residually connected geometry of type* A_n, $n \geq 2$, D_n, $n \geq 4$, *or* E_6 *is a building.* □

For other spherical diagrams quotients exist that are not buildings. However, an eigenvalue argument shows that in the finite case quotients do not occur.

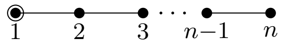
Today only one example is known of a finite residually connected thick geometry of rank at least 3 with a spherical Coxeter diagram and not the quotient of any building. It is the famous *Neumaier geometry*, with 7 points, 35 lines and 15 planes constituting a geometry of type B_3 with full automorphism group A_7 ([590], [16]; cf. §6.2.2).

4.5.4 Chain calculus

The chain calculus due to Tits [690] allows one to obtain results on the diameter of a geometry from its diagram.

We shall talk about chains $x_0 * x_1 * \cdots * x_l$ (in some residually connected Buekenhout-Tits geometry satisfying a given diagram) by just giving the sequence of types t_0-t_1-$\cdots$-t_l, where the object x_i is of type t_i.

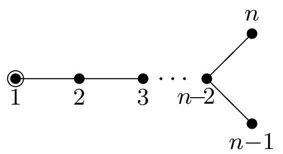
A sequence of types given as a statement, denotes the claim that arbitrary objects x_0 and x_l of the types occurring first and last can be joined by a chain of objects of the indicated types, each incident with the preceding and following. In the proofs we shall modify chains, but always keep the ends fixed. A main ingredient is Proposition 4.1.1, which we shall not explicitly quote.



A_n.

Proposition A_n: *For* $2 \le i \le n$ *we have* 1-i-$(i-1)$. *In particular, for* $n \ge 2$, *we have* 1-2-1.

Proof. If $i < n$, then by induction we find that if 1-2-1-i-$(i-1)$, then 1-2-$(i+1)$-i-$(i-1)$, hence 1-$(i+1)$-$(i-1)$, hence 1-i-$(i-1)$, so chains 1-2-1-i-$(i-1)$ can be shortened to 1-i-$(i-1)$, and by residual connectedness we are done. By definition of A_2 we have 1-2-1 in A_2. There remains the case $i = n \ge 3$. But in that case 1-2-1-$(n-1)$, so 1-2-n-$(n-1)$, so 1-n-$(n-1)$, by induction and since 1-2-1 holds. □



D_n.

Proposition D_n: *Let* $n \ge 2$. *Then the following hold.*

(a) 1-$(n-1)$-n.

(b) 1-i-$(i-1)$-i *for* $2 \le i \le n-2$. *In particular:* 1-2-1-2.

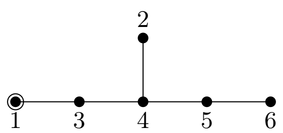
(c) If n *is even, then* $(n-1)$-1-n. *If* n *is odd, then* n-1-n.

Proof. In D_2 we have 1-2, implying all our claims. For $n = 3$ everything follows from Proposition A_3. Now use induction on n. For part (a) we find by induction and Proposition A_n: if 1-2-1-$(n-1)$-n then 1-2-n-$(n-1)$-n so 1-n-$(n-2)$-n so 1-$(n-1)$-$(n-2)$-n so 1-$(n-1)$-n, proving part (a).

For part (b): by part (a) 1-$(n-1)$-n-i, so 1-$(n-1)$-$(i-1)$-i, so 1-i-$(i-1)$-i.

For part (c): if n is even, then (by part (a) and induction): $(n-1)$-n-1-n, so $(n-1)$-n-2-n, so $(n-1)$-1-2-n, so $(n-1)$-1-n, and if n is odd, then n-1-$(n-1)$-n, so n-2-$(n-1)$-n, so n-2-1-n, so n-1-n. □

For E_6, E_7, E_8 we shall omit the '-' in type sequences.



E_6.

Proposition E_6: *(a)* 161,

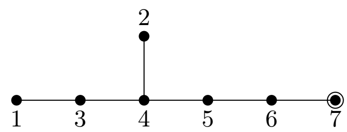
(b) 13126,

(c) if 1316 *then* 126.

Proof. (c) 1316 yields 1326 and then 126.

(a) 13161 yields 1261, 1251, 1651, 161.

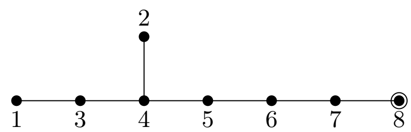
(b) 1616 yields 15216, 15236, 131236, 131436, 131426, 13126. □



E_7.

Proposition E_7: *(a)* 7671*,*
(b) 7176*,*
(c) if 76767 *then* 717*.*

Proof. (c) 76767 yields 76167 and then 717.
(a) 767671 yields 7171, 7161, 76761, 7671.
(b) 76767676 yields 717676, 717616, 71716, 71616, 767616, 76716, 767676, 7176. □



E_8.

Proposition E_8: *(a)* 8181*,*
(b) if 81878 *then* 87878*,*
(c) 878787*.*

Proof. (a) 181878 yields 1817678, 181768, 1878768, 187868, 187878, 1767878, 176878, 1767178, 1787178, 1818.
(b) 81878 yields 817678, 81768, 878768, 87868, 87878.
(c) 81817 (by (a)), 818787, 878787 (by (b)). □

For the collinearity graph Γ of the shadow geometry for the circled node (vertices: objects of the circled type, say i; adjacency: both in the residue of some flag of cotype i — in our cases this is equivalent to both incident to some object of type j, where j is the unique neighbor of i in the diagram) the above means the following:

$\mathsf{A}_{n,1}$: Γ is a clique (has diameter 1).

$\mathsf{D}_{n,1}$: Γ has diameter 2; any line carries a point at distance at most one from a given point.

$\mathsf{E}_{6,1}$: Γ has diameter 2 — indeed, any two vertices are in a $\mathsf{D}_{5,1}$ subgraph.

$\mathsf{E}_{7,7}$: Γ has diameter 3; any two vertices at distance 2 are in a $\mathsf{D}_{6,1}$ subgraph; any line carries a point at distance at most two from a given point.

$\mathsf{E}_{8,8}$: Γ has diameter 3; if x and y are two points at distance 2 in a $\mathsf{D}_{7,1}$ subgraph, then y has no neighbors at distance 3 from x; any line carries a point at distance at most two from a given point.

For the relation between points x and symplecta S (objects of type $6, 1, 1$ in E_6, E_7, E_8, respectively), the above implies:

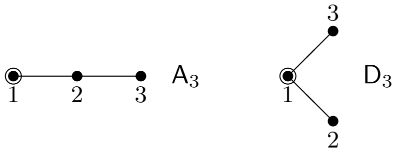
$\mathsf{E}_{6,1}$: $x^\perp \cap S$ is either empty or a projective 4-space.

$\mathsf{E}_{7,7}$: $x^\perp \cap S$ is either a single point or a projective 5-space.

$\mathsf{E}_{8,8}$: $x^\perp \cap S$ is either empty or a line or a projective 6-space.

We established that the collinearity graph of a geometry of type $\mathsf{E}_{6,1}$ has diameter 2. In the finite case, it will turn out to be strongly regular.

4.6 The Klein quadric and Klein correspondence



The A_3 and D_3 diagrams are the same, and hence they describe the same buildings. The circled node differs: different objects are called 'points'. The A_3 diagram (for a finite geometry) is that of the points, lines, and planes of projective 3-space. The D_3 diagram (for a finite geometry) is that of the points and the totally singular planes (of two kinds) of a hyperbolic quadric in projective 5-space.

In coordinates the correspondence goes as follows. Let V be a 4-dimensional vector space over $\mathbb{F}_q$ with basis $e_1, \ldots, e_4$. Let $W = V \wedge V$ be the 6-dimensional vector space over $\mathbb{F}_q$ with basis $f_{ij} = e_i \wedge e_j$ $(1 \leq i < j \leq 4)$. A vector $w = \sum a_{ij} f_{ij}$ is of the form $u \wedge v$ when $Q(w) = 0$, where $Q(w) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$ is a nondegenerate quadratic form on W. If $\langle u \rangle$, $\langle v \rangle$ are distinct points in $\mathsf{P}V$, then $\langle u \wedge v \rangle$ is a point in $\mathsf{P}W$ corresponding to the line $\langle u, v \rangle$ of $\mathsf{P}V$. Thus, projective lines in $\mathsf{PG}(3, q)$ correspond to singular points on this hyperbolic quadric. The quadric is called the *Klein quadric*, and this correspondence the *Klein correspondence*.

Ovoids and spreads

Let B be the symmetric bilinear form derived from Q, so that $B(w, w') = Q(w + w') - Q(w) - Q(w')$. Put $f = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Then $w \wedge w' = B(w, w')f$. Two singular points w, w' are orthogonal if and only if they correspond to intersecting lines. An *ovoid* in $\mathsf{P}W$, that is, a set of $q^2 + 1$ pairwise nonorthogonal singular points, corresponds to a *spread* in $\mathsf{P}V$, that is, a set of $q^2 + 1$ pairwise disjoint lines (a partition of the space).

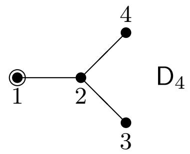
Symplectic forms

Each $w \in W$ defines a symplectic form f_w on V via $f_w(u, v) = B(u \wedge v, w)$, and conversely all symplectic forms occur in this way. The nonsingular points correspond to the nondegenerate symplectic forms. The isotropic lines for f_w correspond to the singular points in $w^\perp$. Thus the points and lines of the $\mathsf{Sp}(4, q)$ generalized quadrangle correspond to the lines and points of the $\mathsf{O}_5(q)$ generalized quadrangle.

Groups

The linear group $\mathsf{PGL}_4(q)$ corresponds to the subgroup of the orthogonal group $\mathsf{PGO}_6^+(q)$ that preserves both types of maximal singular planes. The simple groups are isomorphic: $\mathsf{L}_4(q) \simeq \mathsf{O}_6^+(q)$.

4.7 Triality



By the classification of buildings of type D_4 (Veldkamp [715], Tits [694]) there is for each field F up to isomorphism a unique building $\mathsf{D}_4(F)$. It is the geometry $\mathsf{O}_8^+(F)$ of points, lines, and totally singular solids (of two kinds) of a hyperbolic quadric in projective 7-space.

By the symmetry of the diagram, also the objects of types 3 and 4 can be viewed as the singular points on a quadric in projective 7-space, and the building admits *trialities*, non-type-preserving automorphisms that permute the types $1 \to 3 \to 4 \to 1$ and $2 \to 2$.

In order to give a compact algebraic description, we now first introduce split octonion algebras. These will also be used later to construct buildings of type E_6 and the split Cayley generalized hexagons.

4.7.1 Split octonion algebras

Composition algebras

An *algebra* is a vector space provided with a bilinear multiplication. A *composition algebra* C is an algebra with two-sided identity element e and a nondegenerate quadratic form N such that $N(xy) = N(x)N(y)$ for all x, y. Define a symmetric bilinear *inner product* by $f(x,y) = N(x+y) - N(x) - N(y)$, and define $\overline{x} = f(x,e)e - x$. Then $x^2 - f(x,e)x + N(x)e = 0$ for all x, and $\overline{\overline{x}} = x$, and $x\overline{x} = \overline{x}x = N(x)e$, and $\overline{xy} = \overline{y}\,\overline{x}$.† If f is degenerate, one can show that its radical $R = C^\perp$ is a field, and then that $C = R$. Assume that f is nondegenerate. One can show that $\dim C \in \{1,2,4,8\}$ (and the real numbers, complex numbers, quaternions and octaves are examples over $\mathbb{R}$ where $N(x)$ is positive definite). The composition algebra C is called *split* when there is a nonzero x with $N(x) = 0$. For each $\dim C \in \{2,4,8\}$ there is a unique split example, given the underlying field.

Split octonion algebras

We introduce the *split octonion algebra* or *split Cayley algebra* over the field F. Let $M = \mathscr{M}^{2\times 2}(F)$ be the algebra of 2×2 matrices over the field F. Then the split Cayley algebra $O(F)$ over F consists of pairs $(A,B) \in M \times M$ with componentwise addition, and multiplication given by

$$(A,B)\cdot(C,D) = (AC + DB^{\mathsf{Ad}}, A^{\mathsf{Ad}}D + CB)$$

†In $N(xy) = N(x)N(y)$ replace y by $y+z$ and expand to get $f(xy,xz) = N(x)f(y,z)$. Replace x by $x+w$ and expand to get $f(xy,wz) + f(wy,xz) = f(x,w)f(y,z)$. With $w = e$ this becomes $f(xy,z) = f(y,(f(x,e)e - x)z) = f(y,\overline{x}z)$. Similarly, $f(yx,z) = f(y,z\overline{x})$. Since $N(e) = 1$ and $N(\overline{x}) = N(x)$ and $\overline{\overline{x}} = x$, one finds $f(x,\overline{y}) = f(\overline{x},y)$. Now $f(\overline{xy},z) = f(xy,\overline{z}) = f(x,\overline{z}\,\overline{y}) = f(zx,\overline{y}) = f(z,\overline{y}\,\overline{x})$ for all z, so that $\overline{xy} - \overline{y}\,\overline{x}$ belongs to the radical of f. Using $2N(w) = f(w,w)$ for $w = \overline{xy}$ we see that $N(\overline{xy} - \overline{y}\,\overline{x}) = 0$, and hence, since N is nondegenerate, $\overline{xy} = \overline{y}\,\overline{x}$. From $N(x)f(y,z) = f(xy,xz) = f(\overline{x}(xy),z)$ and symmetry one gets $\overline{x}(xy) = N(x)y = (yx)\overline{x}$. With $y = e$ this proves all claims.

for $A, B, C, D \in M$, where Ad denotes the adjoint operator, i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathsf{Ad}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

so that $AA^{\mathsf{Ad}} = A^{\mathsf{Ad}}A = (\det A)I$ and $(AB)^{\mathsf{Ad}} = B^{\mathsf{Ad}}A^{\mathsf{Ad}}$.

We call this multiplication the *Cayley-Dickson multiplication*, as it is the result of the so-called Cayley-Dickson process in composition algebras. In the literature, a traditional direct definition of this multiplication is the following. Denote by $\mathbf{v}.\mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$ the ordinary dot product and vector product,[3] respectively, of vectors $\mathbf{v}, \mathbf{w} \in F^3$, and by $a\mathbf{v}$ the scalar multiplication, $a \in F$, $\mathbf{v} \in F^3$. Define the following multiplication in the set of mixed matrices

$$\left\{ \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix} \mid a, b \in F,\ \mathbf{v}, \mathbf{w} \in F^3 \right\} :$$

Let $\mathbf{v}, \mathbf{v}', \mathbf{w}, \mathbf{w}' \in F^3$ and let a, a', b, b' be scalars (elements of F). Then

$$\begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix} \odot \begin{pmatrix} a' & \mathbf{v}' \\ \mathbf{w}' & b' \end{pmatrix} = \begin{pmatrix} aa' + \mathbf{v}.\mathbf{w}' & a\mathbf{v}' + b'\mathbf{v} + \mathbf{w} \times \mathbf{w}' \\ a'\mathbf{w} + b\mathbf{w}' + \mathbf{v}' \times \mathbf{v} & bb' + \mathbf{w}.\mathbf{v}' \end{pmatrix}.$$

In fact, the Cayley-Dickson multiplication and the traditional multiplication $\odot$ are opposite multiplications[4] under the identification (denoting the components of the vector $\mathbf{v} \in F^3$ by (v_1, v_2, v_3) and similar for $\mathbf{w}$)

$$\begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix} \longleftrightarrow \left(\begin{pmatrix} a & w_1 \\ v_1 & b \end{pmatrix}, \begin{pmatrix} v_2 & v_3 \\ -w_3 & w_2 \end{pmatrix} \right).$$

For $x = (A, B) \in O(F)$, we define $\overline{x} = (A^{\mathsf{Ad}}, -B)$ (in terms of mixed matrices, this amounts to the adjoint defined in the obvious way). Now $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ for all $x, y \in O(F)$. Let I and O be the identity matrix and zero matrix, respectively, in $\mathscr{M}^{2\times 2}(F)$. We can identify F with $F' = \{(aI, O) \mid a \in F\} \subseteq O(F)$. Then the addition and multiplication of F coincides with the addition and multiplication of $O(F)$ restricted to F'. One easily calculates that, using this identification, $x + \overline{x} \in F$ and $x \cdot \overline{x} = \overline{x} \cdot x \in F$ for all $x \in O(F)$. The mapping $x \mapsto \overline{x}$ is called the *standard involution* in $O(F)$. The multiplication in $O(F)$ is not associative, but it is *alternative*, i.e., for all $x, y \in O(F)$ it is true that

$$\begin{cases} x \cdot (x \cdot y) &= (x \cdot x) \cdot y, \\ x \cdot (y \cdot x) &= (x \cdot y) \cdot x, \\ y \cdot (x \cdot x) &= (y \cdot x) \cdot x. \end{cases}$$

Note that $O(F)$ is in the natural way an 8-dimensional vector space over F. The scalar multiplication $(c, x) \mapsto cx$ is, for $x = (A, B)$, given by $cx = (cA, cB) = (cI, O) \cdot (A, B)$.

With $N(x) = x\overline{x}$, the algebra $O(F)$ is an 8-dimensional composition algebra over F. For $x = (A, B)$, we have $N(x) = \det A - \det B$ and $T(x) := x + \overline{x} = \operatorname{tr} A$.

[3]That is, $\mathbf{v}.\mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$, and $(\mathbf{v} \times \mathbf{w})_i = v_jw_k - v_kw_j$ for $i, j, k \in \{1, 2, 3\}$ where $(i, j, k) = (i, i+1, i+2) \pmod 3$.

[4]The *opposite multiplication* of $(a, b) \mapsto ab$ is $(a, b) \mapsto ba$.

4.7.2 Triality

The two previous paragraphs imply that the norm map in $O := O(\mathbb{F}_q)$ is a quadratic form that defines the $\mathsf{O}_8^+(q)$ geometry, where the t.i. vectors are given by the elements of O with norm 0. The perp of a vector x is given by the elements $y \in O$ such that $f(x,y) = x\overline{y} + y\overline{x} = 0$, or equivalently, $f(x,y) = \overline{x}y + \overline{y}x = 0$. For $a \in O$, define the linear maps $\varphi_a : O \to O : x \mapsto xa$ and ${}_a\varphi : O \to O : x \mapsto ax$.

Noting that two t.i. vectors $x, y \in O$ are collinear if and only if $x + y$ is t.i., one deduces that $\operatorname{Ker}\varphi_a$ and $\operatorname{Im}\varphi_a = Oa$ are singular subspaces, and hence their dimension is at most 4. Since clearly $\operatorname{Im}\varphi_a \subseteq \operatorname{Ker}\varphi_{\bar{a}}$ we conclude that $\operatorname{Im}\varphi_a = \operatorname{Ker}\varphi_{\bar{a}}$ is a maximal singular subspace. In fact, it turns out that the following facts hold (see §2 of [70]):

(i) *Every maximal singular subspace is of the form Oa or aO, for a unique $a \in O$ with $N(a) = 0$.*

(ii) *Two distinct maximal singular subspaces have a plane in common if and only if they are of the form aO and Ob, with $ab = 0$.*

(iii) *Two distinct maximal singular subspaces intersect in a line if and only if they are of the form either Oa and Ob, or aO and bO, with $N(a) = N(b) = N(a+b) = 0$.*

So we can view the maximal singular subspaces of the form Ox, $N(x) = 0$, as the elements of type 3 of the corresponding building of type D_4. It follows that $x \mapsto Ox$ induces an isomorphism from the shadow geometry $\mathsf{D}_{4,1}(q)$ to the shadow geometry $\mathsf{D}_{4,3}(q)$. It is precisely the map given in the proof of Proposition 3.2.3 when interchanging columns i and $-i$, $i \in \{1,2,3\}$, and negating the last column. One checks that this isomorphism maps Ox back to x and interchanges xO and $\overline{x}O$.

Hence the mapping $x \mapsto O\overline{x} \mapsto \overline{x}O \mapsto x$ induces a triality of order 3. We say a little more about this triality in Section 4.8.

4.8 A construction of $\mathsf{G}_2(q)$

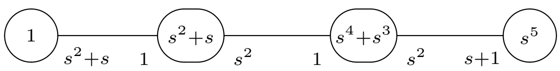
The only known finite generalized hexagons of order s are the split Cayley hexagon $\mathsf{G}_2(s)$ and its dual (and then s is any prime power). It is self-dual if and only if s is a power of 3 (see Section 3.5 of [710]). It arises as the absolute geometry of a suitable triality of order 3, like the one in Subsection 4.7.2 defined on the t.i. points of $O(\mathbb{F}(s))$ under the bilinear form defined by the norm and given by $\tau : x \mapsto O\overline{x} \mapsto \overline{x}O \mapsto x$, $N(x) = 0$. A point $\langle x \rangle$, $x \in O$, $N(x) = 0$, is absolute for τ if and only if $x \in O\overline{x}$, or equivalently $x \in \operatorname{Ker}\varphi_x$, which is clearly equivalent to $x^2 = 0$, and then to $f(x,e) = 0$, or $x + \overline{x} = 0$. Consequently, the points of $\mathsf{G}_2(s)$ are the t.i. points in the hyperplane $e^\perp$, hence the points of a parabolic polar space $\mathsf{O}_7(s)$. The lines fixed under τ are the 2-spaces spanned by two vectors $x, y \in O$ with $x^2 = y^2 = xy = yx = 0$. This can be calculated explicitly, and then one obtains the following description, first given in [691].

As we already deduced, the points of $\mathsf{G}_2(s)$ are the points of a parabolic polar space $\mathsf{O}_7(s)$. In order to describe the lines it is convenient to fix the corresponding quadratic form $\beta : V \to \mathbb{F}_s$ of the 7-dimensional vector space V over $\mathbb{F}_s$ as

$$(x_0, x_1, \ldots, x_6) \mapsto x_0x_4 + x_1x_5 + x_2x_6 - x_3^2.$$

Then the lines are given by the singular 2-spaces of V whose Plücker coordinates satisfy $p_{12} = p_{34}$, $p_{20} = p_{35}$, $p_{01} = p_{36}$, $p_{03} = p_{56}$, $p_{13} = p_{64}$ and $p_{23} = p_{45}$, where $p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$, for independent vectors $(x_0, x_1, \ldots, x_6)$ and $(y_0, y_1, \ldots, y_6)$ of the 2-space in question. This representation of $\mathsf{G}_2(s)$, or any isomorphic one, will be called the *standard representation of* $\mathsf{G}_2(s)$.

The diagram of the collinearity graph of both $\mathsf{G}_2(s)$ and its dual is



$v = s^5{+}s^4{+}s^3{+}s^2{+}s{+}1$

and we see that $k = s^2 + s = b_2 + c_3 - 1$. Proposition 1.3.12 implies that the graph with vertices the points of $\mathsf{G}_2(s)$ or its dual, adjacent when they are at distance 3 from one another in the collinearity graph, is strongly regular. For $\mathsf{G}_2(s)$, this graph is the complement of the $\mathsf{O}_7(s)$ graph and is rank 3. For the dual of $\mathsf{G}_2(s)$, if s is not a power of 3, this graph has the same parameters of the complement of the $\mathsf{O}_7(s)$ graph but is not isomorphic to it. By Theorem 4 of Govaert & Van Maldeghem [359], the full group of this graph equals $\mathrm{Aut}\,\mathsf{G}_2(s)$. It is rank 4.

Another rank 4 permutation group is obtained by considering the action of $\mathsf{SO}_7(q)$ on the set of standard representations of the split Cayley hexagons on $O_7(q)$. There are $q^3(q^4-1)$ such representations. The group $\mathsf{O}_7(q)$ has $\gcd(2, q-1)$ orbits on this set. The suborbits can be seen geometrically as follows. Let ω be a fixed split Cayley hexagon on $\mathsf{O}_7(q)$ and let Ω be the orbit of ω under the action of $\mathsf{O}_7(q)$.

- ω contains $\frac{1}{2}q^3(q^3-1)$ Hermitian spreads, and each Hermitian spread is the intersection of the line set of ω with the line set of every member of a set of $\frac{q+1}{\gcd(2,q-1)} - 1$, that is, q (if q is even) or $\frac{q-1}{2}$ (if q is odd) split Cayley hexagons from Ω.

- ω contains $\frac{1}{2}q^3(q^3+1)$ non-thick subhexagons of order $(1, q)$, and the line set of each such subhexagon is the intersection of the line set of ω with the line set of every member of a set of $\frac{q-1}{\gcd(2,q-1)} - 1$, that is, $q-2$ (if q is even) or $\frac{q-3}{2}$ (if q is odd) split Cayley hexagons from Ω.

- For each point x of ω, the set of lines at distance 1 from x (that is, the lines not containing x but containing a point collinear to x) is the intersection of the line set of ω with the line set of every member of a set of $q-1$ split Cayley hexagons from Ω.

An elementary count reveals that the union of the subsets of Ω described above (also considering ω) is Ω.

The group $\mathsf{G}_2(q)$, seen as an automorphism group of ω, acts transitively on each of the three above subsets of Ω, hence we obtain a rank 4 permutation group of $\mathsf{O}_7(q)$ on the cosets of its subgroup $\mathsf{G}_2(q)$. However, the number $\frac{q-1}{\gcd(2,q-1)} - 1$ equals 0 if and only if $q \in \{2, 3\}$, in which case we obtain a rank 3 group. The corresponding strongly regular graphs are $\overline{NO_8^+(2)}$ and $NO_8^+(3)$ and they have larger full automorphsm group, to be precise $\mathsf{O}_8^+(2):2$ and $\mathsf{PGO}_8^+(3)$.

4.9 The $\mathsf{E}_{6,1}(q)$ graph

We study the collinearity graph of the shadow geometry $\mathsf{E}_{6,1}(q)$.

The literature contains several constructions of $\mathsf{E}_{6,1}(q)$ (or, more generally, $\mathsf{E}_{6,1}(F)$). The standard construction is as coset geometry in an algebraic group of type E_6. Alternatively, one can use the blueprint construction of Ronan & Tits [629]. The geometry $\mathsf{E}_{6,1}(q)$ admits an embedding in $\mathsf{PG}(26,q)$, of which there exists a construction using a trilinear form, see Aschbacher [15]. One can also construct it as an intersection of quadrics, see Cohen [204] and the remarks in §4.9.3 below. Here, we provide a construction of $\mathsf{E}_{6,1}(F)$ over an arbitrary field F using a split octonion algebra.

4.9.1 Parameters

The parameters can be read off from the diagram.

Proposition 4.9.1 *The collinearity graph of $\mathsf{E}_{6,1}(q)$ is strongly regular with parameters (v,k,λ,μ) and spectrum $k^1\ r^f\ s^g$, where*

$$v=\frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)},\qquad r=q^8+q^7+q^6+q^5+q^4-1,$$
$$k=q(q^3+1)\begin{bmatrix}8\\1\end{bmatrix},\qquad s=-q^3-1,$$
$$\lambda=q^2(q^2+1)\begin{bmatrix}5\\1\end{bmatrix}+q-1,\quad f=q^{11}+q^8+q^7+q^5+q^4+q,$$
$$\mu=(q^3+1)\begin{bmatrix}4\\1\end{bmatrix},\qquad g=q^2(q^6+1)(q^4+1)\begin{bmatrix}5\\1\end{bmatrix}.$$

In the thin case ($q=1$) we find the Schläfli graph (§10.10).

Proof. It suffices to find k, λ, and μ.

The local structure is clear by inspection of the diagram: the residue of $\mathsf{E}_{6,1}$ at a point (that is, of $\mathsf{E}_6(q)$ at a type 1 vertex, taking as points of the residue the vertices corresponding to the lines of $\mathsf{E}_{6,1}(q)$) is a geometry of type $\mathsf{D}_{5,5}$. By Theorem 2.2.20 we have $v(\mathsf{D}_{5,5}(q))=(q^3+1)\begin{bmatrix}8\\1\end{bmatrix}$ and $k(\mathsf{D}_{5,5}(q))=q\begin{bmatrix}5\\2\end{bmatrix}$. The points of $\mathsf{D}_{5,5}(q)$ are lines in $\mathsf{E}_{6,1}(q)$, and each contributes q neighbors, so $k=q\cdot v(\mathsf{D}_{5,5}(q))$ and $\lambda=q\cdot k(\mathsf{D}_{5,5}(q))+q-1$.

That $\mu(\mathsf{E}_{6,1}(q))=\mu(\mathsf{D}_{5,1}(q))$ follows from the fact that the symplecton (object of type 6) on two noncollinear points is unique. (Indeed, if s_1,s_2 are symplecta on the points p_1,p_2, then $\{p_1,s_1\}$ and $\{p_2,s_2\}$ are flags contained in an apartment. The apartment consists of the 27 vertices, 72 6-cliques, 216 edges, 720 triangles, 216 maximal 5-cliques and 27 subgraphs of the form $\Gamma_2(x)$ for a vertex x in the Schläfli graph Γ, the noncollinearity graph of the generalized quadrangle $\mathsf{GQ}(2,4)$, cf. §10.10. Now $\{p_1,p_2,s_1\}$ and $\{p_1,p_2,s_2\}$ correspond to 3-cocliques in Γ, hence to lines in the $\mathsf{GQ}(2,4)$, hence $s_1=s_2$.) □

We saw that the local graph of the collinearity graph of $\mathsf{E}_{6,1}(q)$ is the q-clique extension of the collinearity graph of $\mathsf{D}_{5,5}(q)$.

4.9.2 Cliques, cocliques and regular sets

Cliques

The maximal cliques correspond to the maximal singular subspaces of the shadow geometry $\mathsf{E}_{6,1}(q)$, which, on their turn, correspond to the objects of types 2 and 5 in the corresponding building and hence contain $q^5+q^4+q^3+q^2+q+1$ points (singular subspaces of projective dimension 5) and $q^4+q^3+q^2+q+1$ points (singular subspaces of projective dimension 4), respectively.

Cocliques

COOPERSTEIN [225] shows that the existence of an ovoid in the $\mathsf{O}^+_{10}(q)$ hyperbolic quadric implies the existence of a coclique of largest possible size q^8+q^4+1 in the collinearity graph of $\mathsf{E}_{6,1}(q)$. However, no such ovoid is known (for $q>1$), and for many q nonexistence has been established, see for instance Proposition 2.6.17.

A regular set of type F_4

The geometry $\mathsf{E}_{6,1}(q)$ has exactly three types (orbits) of geometric hyperplanes. Two types have the property that they contain all points collinear to some fixed point, and hence these cannot give rise to a regular bipartition. The third type does give rise to a regular bipartition. In fact, such a geometric hyperplane H has the following property. Let $\mathscr{L}_H$ be the set of lines contained in H and lying in at least two maximal singular subspaces of projective dimension 5 which are also entirely contained in H. Then $(H,\mathscr{L}_H)$ is isomorphic to the point-line geometry $\mathsf{F}_{4,4}(q)$. Also, the stabilizer in $\mathrm{Aut}\,\mathsf{E}_{6,1}(q)$ of H acts transitively on the complement of H (this is true in general for any field, see [285]).

Hence H is a regular set of size $(q+1)(q^2+1)(q^4+1)(q^8+q^4+1)$ with degree $q(q^3+1)(q^6+q^5+q^4+q^3+q^2+q+1)$ and nexus $(q+1)(q^2+1)(q^3+1)(q^4+1)$.

A regular set of type ${}^3\mathsf{D}_4$

Let $O:=O(\mathbb{F}_{q^3})$ be the split Cayley algebra over $\mathbb{F}_{q^3}$. Then the absolute points and fixed lines of the triality map $\tau: x\mapsto O\overline{x}^q\mapsto \overline{x}O\mapsto x^{q^2}$, $N(x)=0$, constitute a $\mathsf{GH}(q^3,q)$. Since the line grassmannian of the $\mathsf{O}^+_8(q^3)$ quadric embeds in the shadow geometry $\mathsf{F}_{4,1}(q^3)$, we obtain an embedding of the dual $\mathsf{GH}(q,q^3)$ in $\mathsf{F}_{4,1}(q^3)$. It turns out that this embedding is contained in the subgeometry isomorphic to $\mathsf{F}_{4,1}(q)$ obtained by field restriction.

Now the regular set of type F_4 described in the previous paragraph gives rise to an embedding of the $\mathsf{F}_{4,1}(q)$ shadow geometry into the $\mathsf{E}_{6,2}(q)$ shadow geometry. It follows that there is a representation of the $\mathsf{GH}(q,q^3)$ in $\mathsf{E}_{6,1}(q)$ where points p are maximal (projective) 5-spaces U_p and the lines L are planes π_L, with natural incidence. The lines of $\mathsf{GH}(q,q^3)$ incident with a given point p form a symplectic spread in U_p. This symplectic spread is pointwise fixed by a (Singer) group of order q^2+q+1, and all elements of that group extend to elements of $\mathrm{Aut}\,\mathsf{E}_{6,1}(q)$ pointwise stabilizing $\mathsf{GH}(q,q^3)$ (i.e., stabilizing the plane π_L, for each line L of $\mathsf{GH}(q,q^3)$). Let W be the union of all such planes. Together with the cyclic group generated by the field automorphism $x\mapsto x^q$ acting on $\mathsf{GH}(q,q^3)$, which becomes a linear automorphism in $\mathsf{E}_{6,1}(q)$, we obtain a group $((q^2+q+1)\times {}^3\mathsf{D}_4(q)):3$ stabilizing W. (This is a maximal subgroup of $\mathsf{E}_6(q)$.)

Let x be a point in W. Then x is contained in a unique plane π_L, with L a line of $\mathsf{GH}(q,q^3)$. If M is a line of $\mathsf{GH}(q,q^3)$ opposite L, then π_L and π_M are opposite in $\mathsf{E}_{6,1}(q)$ and no point of π_M is collinear to x. If M is concurrent to L, then π_M is contained in a projective 5-space together with x and so all points of π_M are collinear to x. Finally, if M is at distance 1 from L (meaning that the minimal distance in the collinearity graph of $\mathsf{GH}(q,q^3)$ between points of M and points of L is 1; so there exist unique collinear points $u\in L$ and $v\in M$), then x is collinear to the points of a solid S of U_v. Since $\pi_{uv}\subseteq S$, we see that $S\cap\pi_M$ is a point. (Note that we used the fact that a point outside a given projective 5-space U is either collinear with a unique point of U, or with the points of a unique solid in U, see Fact 4.2.10 in [285].) This yields the degree of the graph induced on W, namely

$$(q^2+q)+(q+1)q^3(q^2+q+1)+(q+1)q^7=q(q^3+1)(q^4+q^3+q^2+q+1).$$

Now let x be a point off W, and let p be a point of $\mathsf{GH}(q,q^3)$. Then, as mentioned earlier, x is collinear to either a unique point of U_p, or all points of a solid in U_p. In the former case, we see that x is collinear to either one or zero points of the planes π_L contained in U_p; in the latter case x is collinear to either 1 or $q+1$ points of the planes π_L contained in U_p and there are exactly $q+1$ such planes in U_p containing $q+1$ points collinear to x (if $x^\perp$ would contain a plane π_L, then it would follow that $x\in W$). It can now be argued that the projective 5-spaces U_p with $x^\perp\cap W$ a solid and the planes π_L with $x^\perp\cap\pi_L$ a line, form a subhexagon $\mathscr{H}$ of order q of $\mathsf{GH}(q,q^3)$. Moreover, $x^\perp\cap\pi_L$ is a point if and only if L is a line not contained in $\mathscr{H}$ but incident with a point of $\mathscr{H}$. We see that

$$|x^\perp\cap W|=(q+1)^2(q^4+q^2+1)+(q^3-q)(q+1)(q^4+q^2+1)=(q^3+1)^2(q^2+q+1).$$

Hence W is a regular set of size $(q^2+q+1)(q^3+1)(q^8+q^4+1)$ with degree $q(q^3+1)(q^4+q^3+q^2+q+1)$ and nexus $(q^3+1)^2(q^2+q+1)$.

4.9.3 Construction of $\mathsf{E}_{6,1}(q)$

Let $F^2\times O(F)^3$ be a model for the 26-dimensional affine space $\mathsf{AG}(26,F)$ over F, with projective completion $\mathsf{PG}(26,F)$. We use 27-tuples over F to describe the points of $\mathsf{PG}(26,F)$ and order them so that a point with coordinates $(1,\ldots)$ belongs to $\mathsf{AG}(26,F)$, and the coordinates following the 1 belong to $F^2\times O(F)^3$. It is convenient to write a semicolon between the third and fourth position, separating the coordinates in F from those in $O(F)$. Also, we denote the zero element of $O(F)$ simply by 0.

For every pair $(x,y)\in O(F)\times O(F)$, we define the point $p(x,y)$ of $\mathsf{AG}(26,F)$ by $p(x,y)=(1,x\overline{x},y\overline{y};x\overline{y},x,y)$. We set $S_1=\{p(x,y)\mid x,y\in O(F)\}$.

For every pair $((x_1,y_1),(x_2,y_2))\in(O(F)\times O(F))^2$ with

$$(*)\begin{cases}(x_1-x_2)(\overline{x}_1-\overline{x}_2) &= 0,\\ (y_1-y_2)(\overline{y}_1-\overline{y}_2) &= 0,\\ (x_1-x_2)(\overline{y}_1-\overline{y}_2) &= 0,\end{cases}$$

we define the point $p(x_1,y_1,x_2,y_2)=p(x_1,y_1)-p(x_2,y_2)$. The set of all points $p(x_1,y_1,x_2,y_2)$ with $((x_1,y_1)(x_2,y_2))$ satisfying $(*)$, is denoted by S_2.

Finally, let S_3 be the set of points with coordinates $(0, a, b; x, 0, 0)$ satisfying $ab = x\overline{x}$ (S_3 is a nonsingular hyperbolic quadric Q in a 9-dimensional projective subspace, an element of type 6 in the corresponding building of type E_6). Then $S := S_1 \cup S_2 \cup S_3$, endowed with all projective lines contained in it, is a model for $\mathsf{E}_{6,1}(F)$.

Remarks

- If $|F| > 2$, then S_2 is just the set of points lying on the projective extension of an affine line of $\mathsf{AG}(26, F)$ entirely contained in S_1. Likewise, S_3 is the set of points lying on a line of which all points but one are contained in S_2. Now all lines of $\mathsf{PG}(26, F)$ all but possibly one of whose points belong to $S_1 \cup S_2 \cup S_3$ are entirely contained in $S_1 \cup S_2 \cup S_3$. This procedure can be seen as the *Zariski closure* of the set S_1, viewed as a variety of low degree.

- The set $S_2 \cup S_3$ is a geometric hyperplane of S with S_3 as its set of deep points. A *deep point* of a geometric hyperplane is a point p with the property that all points collinear to p also belong to the hyperplane. The geometric hyperplane $S_2 \cup S_3$ arises as the intersection of the hyperplane $H_1 := \mathsf{PG}(26, F) \setminus \mathsf{AG}(26, F)$.

- The orbit of H_1 under the group $\mathsf{E}_6(F)$ forms a set of points in the dual of $\mathsf{PG}(26, F)$ which is isomorphic to S. This exhibits the duality of the building of type E_6 apparent in its diagram.

- There are two other orbits of hyperplanes; the first is the orbit of the hyperplane H_2 spanned by all points of S_1 collinear in $\mathsf{E}_{6,1}(F)$ to $(1, 0, 0, 0, 0, 0)$ (these all have coordinates of the form $(1, 0, 0, 0, x, y)$, with $x\overline{x} = y\overline{y} = x\overline{y} = 0$) and the points of a nonsingular parabolic subquadric of Q in an 8-dimensional projective subspace. The point $(1, 0, 0, 0, 0, 0)$ is the unique deep point of the corresponding geometric hyperplane; hence every geometric hyperplane in this orbit has a unique deep point. But unlike the situation with H_1, where the set of deep points determines the geometric hyperplane, here there are many geometric hyperplanes (of the same orbit) having the same deep point as the one corresponding to H_2. The second orbit is an orbit of a hyperplane H_3 such that $H_3 \cap S$ has no deep points, and does not contain any element of type 6. The stabilizer is a group of type F_4. If we restrict the set of lines to the set of lines contained in at least two 5-spaces entirely contained in $S \cap H_3$, then we obtain a shadow geometry of type $\mathsf{F}_{4,4}$.

- Let $\mathsf{GQ}(2, 4) = (\mathscr{P}, \mathscr{L})$ be the unique generalized quadrangle of order $(2, 4)$. The complement of its collinearity graph is the Schläfli graph (§10.10). Recall that a *spread* is a set of lines that partitions the point set. There are two isomorphism classes of spreads in $\mathsf{GQ}(2, 4)$ ([144]). One isomorphism class contains spreads $\mathscr{S}$, called *regular* or *Hermitian spreads*, with the property that, given any pair of lines $L_1, L_2 \in \mathscr{S}$, the unique line L_3 composed of the three points outside $L_1 \cup L_2$ that are collinear with collinear points of $L_1 \cup L_2$, also belongs to $\mathscr{S}$. We consider such a spread $\mathscr{S}$. Let a basis of the projective space $\mathsf{PG}(26, F)$ be indexed by the 27 points of $\mathsf{GQ}(2, 4)$. Hence an arbitrary point of $\mathsf{PG}(26, F)$ has coordinates of the form $(x_i)_{i\in\mathscr{P}}$, $x_i \in F$, for all $i \in \mathscr{P}$. Given a point $i \in \mathscr{P}$, we define the quadric Q_i with equation

$$x_{j_1}x_{j_2} + x_{j_3}x_{j_4} + x_{j_5}x_{j_6} + x_{j_7}x_{j_8} = x_{j_9}x_{j_0},$$

where $\{i, j_1, j_2\}, \{i, j_3, j_4\}, \{i, j_5, j_6\}, \{i, j_7, j_8\}$ are the four lines of $\mathsf{GQ}(2,4)$ on i not belonging to $\mathscr{S}$, and $\{i, j_9, j_0\} \in \mathscr{S}$. Then the set S constructed above is projectively equivalent to the intersection of the 27 quadrics Q_i, with i ranging over $\mathscr{P}$. Up to the numbering, it is the same set of quadrics as given by COHEN [204].

- A brief algebraic way to note down the set of 27 quadrics of the previous remark is to label a generic point of $\mathsf{PG}(26, F)$ with the coordinates $(x_1, x_2, x_3, X_1, X_2, X_3) \in F^3 \times O(F)^3$, up to an F-multiple. Then $\mathsf{E}_{6,1}(F)$ is given by the set of points whose coordinates satisfy $X_i\overline{X}_i = x_{i+1}x_{i+2}$ and $x_i\overline{X}_i = X_{i+1}X_{i+2}$, for all $i \in \{1, 2, 3\}$ mod 3.

Chapter 5

Fischer spaces

Fischer classified the groups generated by a conjugacy class D of 3-transpositions (involutions such that the product of any two has order at most 3) and discovered three new sporadic groups that bear his name. These groups are rank 3 groups: D carries in a natural way the structure of a geometry with lines of length 3 and the structure of a rank 3 graph.

5.1 Definition

Let $(X, \mathscr{L})$ be a partial linear space. A subset Y of X, together with the lines contained in it, is called a *subspace* when Y contains each line that meets it in at least two points. A *Fischer space* is a partial linear space such that (i) each line has size 3, and (ii) any two intersecting lines span a subspace, called a *plane*, that is isomorphic either to the dual affine plane of order 2 (with 6 points and 4 lines), or to the affine plane of order 3 (with 9 points and 12 lines).

Consider a partial linear space $(X, \mathscr{L})$ with three points on each line. Each point x defines a permutation s_x of X defined by $s_x(y) = z$ when $\{x, y, z\} \in \mathscr{L}$, and $s_x(y) = y$ otherwise. Now $s_x^2 = 1$ and s_x is an involution (unless there are no lines on x, and $s_x = 1$). If $(X, \mathscr{L})$ is a Fischer space, then each s_x induces an automorphism of each plane on x, and hence an automorphism of $(X, \mathscr{L})$. If x, y are not collinear then $s_x s_y = s_y s_x$ and $(s_x s_y)^2 = 1$. If $\{x, y, z\}$ is a line, then $s_z = s_x s_y s_x = s_y s_x s_y$ and $(s_x s_y)^3 = s_z^2 = 1$, so that $\langle s_x, s_y \rangle$ acts on $\{x, y, z\}$ as the symmetric group S_3.

We see that if the Fischer space is connected, then all s_x are conjugate, and each product $s_x s_y$ has order at most 3. Conversely, suppose that G is a group generated by a conjugacy class of involutions D, such that the product of any two elements of D has order at most 3. (Then D is called a *class of 3-transpositions.*) Make a partial linear space with point set D and lines of size 3 given by $\{s, t, sts\}$ when s, t are distinct involutions in D that do not commute. Now the group G acts by conjugation, and the partial linear space is a connected Fischer space. (See also Example (vi) below.)

The *Fischer graph* of a Fischer space is its noncollinearity graph, that is, is the commuting involutions graph of D. The *Fischer group* of a Fischer space $(X, \mathscr{L})$ is the group $G = \langle s_x \mid x \in X \rangle$.

Examples

We list examples of groups with a class D of 3-transpositions. Detailed parameter information is given below.

(i) Let D be the class of transpositions (ij) in the symmetric group S_n, $n \geq 2$. The corresponding Fischer graph is $\overline{T(n)}$, the complement of the triangular graph $T(n)$.

(ii) Let V be a $2n$-dimensional vector space over $\mathbb{F}_2$ provided with a nondegenerate symplectic form. Let D be the class of transvections $t_v\colon x \mapsto x + (x,v)v$, where $v \neq 0$, in the symplectic group $\mathsf{Sp}_{2n}(2)$ acting on V. If $(v,w) = 0$, then t_v and t_w commute. Otherwise, $(v,w) = 1$, and $t_v t_w t_v = t_{v+w}$. The Fischer graph is the collinearity graph of the symplectic space provided with its totally isotropic lines, and the lines of the Fischer space are the hyperbolic lines of the geometry.

(iii) Let V be a $2n$-dimensional vector space over $\mathbb{F}_2$ provided with a nondegenerate quadratic form Q of type $\varepsilon = \pm 1$. Let D be the class of transvections $t_v\colon x \mapsto x + (x,v)v$, where $Q(v) = 1$, in the orthogonal group $\mathsf{O}^{\varepsilon}_{2n}(2)$ acting on V. If $(v,w) = 0$, then t_v and t_w commute. Otherwise, $(v,w) = 1$, and $t_v t_w t_v = t_{v+w}$. The Fischer graph and Fischer spaces here are the induced ones from the symplectic example.

(iv) Let V be an n-dimensional vector space over $\mathbb{F}_4$ provided with a nondegenerate Hermitian form, linear in the first coordinate. Let D be the class of transvections $t_v\colon x \mapsto x + (x,v)v$, where $(v,v) = 0$, $v \neq 0$, in the unitary group $\mathsf{SU}_n(2)$. Here t_v and t_w commute when $(v,w) = 0$. Otherwise, $t_v t_w t_v = t_u$, where $u = v + (v,w)w$. The Fischer graph is the collinearity graph of the unitary space provided with its totally isotropic lines, and the lines of the Fischer space are the triples of isotropic points on nondegenerate lines.

(v) Let V be an n-dimensional vector space over $\mathbb{F}_3$ provided with a nondegenerate quadratic form Q of type $\varepsilon = \pm 1$. Let $\eta = \pm 1$. Let D_η be the class of reflections $t_v\colon x \mapsto x + \frac{(x,v)}{(v,v)}v$, where $Q(v) = \eta$ (that is, $(v,v) = -\eta$), in the orthogonal group $\mathsf{O}^{\varepsilon}_n(3)$ acting on V. Here t_v and t_w commute when $(v,w) = 0$. Otherwise, $t_v t_w t_v = t_u$, where $u = v - \eta(v,w)w$. The subgroup of $\mathsf{O}^{\varepsilon}_n(3)$ generated by D_η is called $\mathsf{O}^{\varepsilon,\eta}_n(3)$. The Fischer graph is the orthogonality graph on the set X of nonsingular points of one kind. The Fischer space has as lines the intersections with X of tangent lines.

(v)′ The group $\mathsf{S}_6 = \mathsf{Sp}_4(2) = \mathsf{O}^{-,+}_4(3)$ appears three times on the list above. It is most familiar as S_6, where it has an outer automorphism interchanging transpositions (ij) and synthemes $(ab)(cd)(ef)$. Both classes give a Fischer group. The geometries and graphs are the same.

The existence of the outer automorphism is best understood in terms of $\mathsf{O}^{-,+}_4(3)$. Each 3-transposition group $\mathsf{O}^{-,+}_{2m}(3)$ has two generating classes of 3-transpositions, the reflections of D_+ and negative reflections of $-D_-$. As Q and $-Q$ are isometric in even dimension, the two groups are canonically isomorphic, and these two classes are switched by an outer automorphism.

(vi) The noncommuting graph Δ of a set S of 3-transpositions is often called the *diagram* of S since the group $\langle S \rangle$ must be a quotient of the Coxeter group with simply laced diagram Δ. In particular, the generating reflection class of the finite Weyl groups $W(\mathsf{A}_m)$, $W(\mathsf{D}_m)$, and $W(\mathsf{E}_m)$ are all classes of 3-transpositions. The connected diagrams on three vertices are A_3 with Weyl group S_4—yielding as Fischer space the dual affine plane on 6 points—and

$\widetilde{A}_2$ with affine Weyl group $(\mathbb{Z} \times \mathbb{Z}):\mathsf{S}_3$, whose 3-transposition quotients are $(2 \times 2):\mathsf{S}_3 = \mathsf{S}_4$ and $(3 \times 3):\mathsf{S}_3 = \mathsf{SU}_3(2)'$—yielding the dual affine plane again and the 9 point affine plane. This justifies the earlier claim that every 3-transposition group yields a Fischer space.

History

Fischer introduced classes of 3-transpositions—aiming to characterize the transposition class of the symmetric group—and was led to his broad classification (Theorem 5.2.2 below). Soon after that Buekenhout [154] introduced the concept of Fischer space in order to provide a uniform geometric context for Fischer's examples.

Maximal cliques in a Fischer graph

Let $(X, \mathscr{L})$ be a finite Fischer space with Fischer graph Γ and Fischer group G. Then G acts transitively on the set of maximal cliques in Γ. More precisely, if M and M' are two maximal cliques, then there is a $g \in G$ mapping M to M' and fixing $M \cap M'$ pointwise. (Indeed, since $M \cup M'$ is not a clique, we can choose $x \in M \setminus M'$ and $y \in M' \setminus M$ joined by a 3-line $\{x, y, z\}$. Now $s_z = s_x s_y s_x$ fixes $M \cap M'$ pointwise and maps x to y, so that it sends M to a maximal clique with larger intersection with M'.)

Let G_M be the stabilizer in G of the maximal clique M. Then G_M contains the elementary abelian 2-group $\langle\, s_x \mid x \in M \,\rangle$ as a normal subgroup.

If $(X, \mathscr{L})$ is connected, then G is transitive on X and hence on pairs (x, M) with $x \in M$. It follows that G_M is transitive on M.

Subspaces

Every subspace of a Fischer space is again a Fischer space. If $(X, \mathscr{L})$ is a Fischer space, then the subset Y is a subspace if and only if Y is invariant under s_y for all $y \in Y$.

The connected components of Fischer spaces are subspaces. Conversely, given a collection of Fischer spaces $(X_i, \mathscr{L}_i)$, where the X_i are disjoint, the union $(\bigcup_i X_i, \bigcup_i \mathscr{L}_i)$ is a Fischer space. If the $(X_i, \mathscr{L}_i)$ are connected, they are the connected components of their union.

Also the connected components of Fischer graphs are subspaces. (If C is such a component, and $\{x, y, z\}$ is a line with $x, y \in C$, then s_x maps a path from x to y into a path from x to z, so that also $z \in C$.)

Let $(X, \mathscr{L})$ be a Fischer space with Fischer graph Γ. For each $x \in X$, the set $\Gamma(x)$ of neighbors of x in Γ, that is, the set of points noncollinear with x in $(X, \mathscr{L})$, is a subspace. (And so is $x^\perp = \{x\} \cup \Gamma(x)$.)

Diameter

The connected components of the collinearity and noncollinearity graphs of a Fischer space have diameter at most 2.

(Indeed, if $a \sim b \sim c \sim d$ is an induced path in the collinearity graph, then both a and d are collinear with the third point of the line bc.

If $a \sim b \sim c \sim d$ is an induced path in the Fischer graph, then let e be the third point of the line ac (then $b \sim e$), f the third point of de, and g the third point of bf. The plane acd shows that $a \sim f$, so $a \sim g$. The plane bde shows that $d \sim g$. So $a \sim g \sim d$ is a shorter path.)

Quotient spaces and imprimitivity

Let $F = (X, \mathscr{L})$ be a Fischer space with Fischer group G, and let Π be a G-invariant partition of X. Then each $Y \in \Pi$ is a subspace of F. Moreover, Π, together with the lines $\{Y, Y', Y''\}$ where Y, Y', Y'' are distinct elements of Π and there are points $y \in Y$, $y' \in Y'$, $y'' \in Y''$ with $\{y, y', y''\} \in \mathscr{L}$, is again a Fischer space, called the *quotient space* F/Π of F with respect to Π.

(Indeed, if $\{y, y', y''\} \in \mathscr{L}$, then s_y interchanges Y' and Y'', so for each $z \in Y'$ there is a line $\{y, z, s_y(z)\}$ with $s_y(z) \in Y''$. Similarly, y' and y'' are collinear with each point of Y, and $Y \cup Y' \cup Y''$ is a subspace of F.)

There are three sources of nontrivial invariant partitions, two of which were mentioned above:

(i) *The connected components of a disconnected Fischer space.* The quotient Fischer space is a collection of points with no lines.

(ii) *The connected components of the Fischer graph.* The quotient Fischer space has complete collinearity graph. (See the discussion of Hall triple systems below.)

(iii) *Degenerate forms on classical spaces.* In examples (ii)–(v) above, the form in question can be degenerate as long as the points of the Fischer space (transvection and reflection centers v) are chosen outside the radical. Nontrivial blocks of the invariant partition consist of the points in the same coset of the radical. In the characteristic 2 examples (ii)–(iv) the resulting group will have a noncentral normal 2-subgroup that respects the partition, while in case (v) the corresponding noncentral normal subgroup will be a 3-group.

Rank 3

Let $(X, \mathscr{L})$ be a finite Fischer space. Its Fischer group G is transitive (in its permutation action on X) when the space is connected. Suppose that moreover the action of G on X (with $|X| > 3$) is primitive. Then this action is rank 3 (cf. Fischer [327] (3.3.5)).

5.2 Fischer's classification

Since ${}^g(s_x) = s_{gx}$, it follows that s_x is central in G_x. From this, and Iwasawa's Lemma, we see that if G acts primitively on X, it is close to being simple.

Lemma 5.2.1 ('Iwasawa's Lemma', cf. [461], [678] (1.2)) *Let G be a group acting primitively on a set X. Let $x \in X$ and suppose that G_x has an abelian normal subgroup A such that $G = \langle {}^gA \mid g \in G \rangle$. If $N \trianglelefteq G$, then $N \leq G_{[X]}$ (the pointwise stabilizer of X) or $N \geq G'$ (the commutator subgroup of G). In particular, if $G = G'$, then $G/G_{[X]}$ is simple.*

In our case (i.e., G primitive), we can take $A = \langle s_x \rangle$ and $G_{[X]} = 1$ so that any nontrivial normal subgroup of G contains G'. Also, either $G = G'$ or G' has index 2 in G; in fact one easily proves that if $1 < N \trianglelefteq G$ then $G = N \cup N s_x$.

(As follows: G is primitive, so N is transitive, and $G = NG_x$. Now any element of G is a product of conjugates of s_x, i.e., of the form $(n_1 g_1 s_x g_1^{-1} n_1^{-1}).(n_2 g_2 s_x g_2^{-1} n_2^{-1})....$, where $n_i \in N$ and $g_i \in G_x$. And this reduces to $(n_1 s_x n_1^{-1}).(n_2 s_x n_2^{-1})....$ since s_x is central in G_x. If the number of factors is even, this is in N (since N is normal), otherwise in $N s_x$.)

Since G'' is normal in G, either $G'' = G'$ or $G'' = 1$. But in the latter case G' is abelian and transitive, hence regular and we find that $|X| = p^h$ and G' is elementary abelian; since X is connected, G' contains elements of order 3, so $p = 3$; now our linear space is obtained from $\mathsf{AG}(h, 3)$ by replacing all 3-lines in some parallel classes by 2-lines, and G' is the translation group. Clearly, s_x preserves parallelism, so that each parallel class of lines is a system of imprimitivity for G, a contradiction unless $|X| = 1$ or $|X| = 3$. This shows that in all cases, if G is primitive on X, then $G' = G''$.

Let $Z(G)$ be the center of G and $O_p(G)$ the largest normal p-subgroup of G. (Then we saw $Z(G) = O_p(G) = 1$ unless $|X| = 3$.) Now we can state the main theorem of Fischer.

Theorem 5.2.2 (Fischer [327]) *Let G be a finite group, generated by a conjugacy class D of 3-transpositions. If $O_2(G)$ and $O_3(G)$ are both contained in the center $Z(G)$ of G, then $G/Z(G)$ is one of the following:*

(i) the trivial group,

(ii) a symmetric group S_n with $n \geq 5$,

(iii) a symplectic group $\mathsf{Sp}_{2n}(2)$ with $n \geq 3$,

(iv) a unitary group $\mathsf{PSU}_n(2)$ with $n \geq 5$,

(v) an orthogonal group $\mathsf{O}^{\pm}_{2n}(2)$ with $n \geq 4$,

(vi) $\mathsf{PO}^{\pm,+}_n(3)$, the subgroup of index 2 in $\mathsf{PO}^{\pm}_n(3)$ generated by the reflections in norm 1 vectors, where $n \geq 5$,

(vii) $\Omega^+_8(2).\mathsf{S}_3$ or $\mathsf{P}\Omega^+_8(3).\mathsf{S}_3$,

(viii) Fi_{22}, Fi_{23}, or Fi_{24}.

If Γ is the collinearity graph of a Fischer space, then its 3-clique extension and sometimes also its 2-coclique extension are also collinearity graphs of Fischer spaces. It is this construction that is ruled out by the condition that $O_2(G)$ and $O_3(G)$ are contained in $Z(G)$. In Cuypers & Hall [248] the classification is redone, without this hypothesis.

Fischer graphs

We already mentioned most of the examples. Here we give the parameters and some further detail. Recall that a Fischer graph is the noncollinearity graph of a Fischer space.

(i) In S_n ($n \geq 2$) the set D of involutions is the set of transpositions (except for S_6, where there are two possibilities). The corresponding graph is $\overline{T(n)}$. The parameters are:

$$v = \binom{n}{2}, \quad k = \binom{n-2}{2}, \quad \lambda = \binom{n-4}{2}, \quad \mu = \binom{n-3}{2},$$
$$r = 1, \quad s = -(n-3), \quad f = \tfrac{1}{2}n(n-3), \quad g = n-1.$$

The group has order $n!$ and $(\mathsf{S}_n)' = \mathsf{A}_n$, of order $\frac{1}{2}n!$, is simple for $n \geq 5$.

(ii) In $\mathsf{Sp}_{2n}(2)$ $(n \geq 1)$ the set D of involutions is the set of transvections $t_a : x \mapsto x + (x,a)a$. The corresponding graph is the symplectic graph $\Gamma(\mathsf{Sp}_{2n}(2))$ (note that t_a and t_b commute if and only if $(a,b) = 0$, i.e., $a \perp b$). The parameters are:

$$\begin{array}{llll} v = 2^{2n} - 1, & k = 2^{2n-1} - 2, & \lambda = 2^{2n-2} - 3, & \mu = 2^{2n-2} - 1, \\ r, s = \pm 2^{n-1} - 1, & & f, g = 2^{2n-1} \pm 2^{n-1} - 1. & \end{array}$$

The group has order $2^{n^2} \prod_{i=1}^{n} (2^{2i} - 1)$ and is simple for $n \geq 3$. For $n = 2$ we have $\mathsf{Sp}_4(2) \simeq \mathsf{O}_4^{-,-}(3) \simeq \mathsf{S}_6$ and we find $\overline{T(6)}$ again; it occurs again under (v). For $n = 1$ we have $\mathsf{Sp}_2(2) \simeq \mathsf{S}_3$ and the graph is $\overline{K_3}$.

(iii) In $\mathsf{O}_{2n}^{\pm}(2)$ $(n \geq 3)$ the set D of involutions is the set of transvections $t_a : x \mapsto x + (x,a)a$ with $Q(a) = 1$. The corresponding graph is the graph $NO_{2n}^{\pm}(2)$ (nonsingular points, adjacent when orthogonal). The parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$ are:

$$\begin{array}{ll} v = 2^{2n-1} - \varepsilon\, 2^{n-1}, & \theta_1 = \varepsilon\, 2^{n-2} - 1, \\ k = 2^{2n-2} - 1, & \theta_2 = -\varepsilon\, 2^{n-1} - 1, \\ \lambda = 2^{2n-3} - 2, & m_1 = \dfrac{4}{3}(2^{2n-2} - 1), \\ \mu = 2^{2n-3} + \varepsilon\, 2^{n-2}, & m_2 = \dfrac{1}{3}(2^{n-1} - \varepsilon)(2^n - \varepsilon). \end{array}$$

The group has order $2^{n(n-1)+1}(2^n \mp 1) \prod_{i=1}^{n-1} (2^{2i} - 1)$ and has simple commutator subgroup $\Omega_{2n}^{\pm}(2)$ of index 2.

(If $n = 2$ then $\Omega_4^-(2) \simeq \mathsf{A}_5$ is still simple, but $\Omega_4^+(2) \simeq \mathsf{S}_3 \times \mathsf{S}_3$. In the former case the graph is the Petersen graph, in the latter $K_{3,3}$.)

(iv) In $\mathsf{PSU}_n(2)$ $(n \geq 4$ or $n = 2)$ the set D of involutions is the set of unitary transvections $x \mapsto x + (x,p)p$ with isotropic p. The corresponding graph is the unitary graph $\Gamma(\mathsf{U}_n(2))$ (isotropic points, adjacent when orthogonal). Let $\varepsilon = (-1)^n$. The parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$ are:

$$\begin{array}{ll} v = \dfrac{1}{3}(2^n - \varepsilon)(2^{n-1} + \varepsilon), & \theta_1 = -1 + \varepsilon 2^{n-2}, \\ k = \dfrac{4}{3}(2^{n-2} - \varepsilon)(2^{n-3} + \varepsilon), & \theta_2 = -1 - \varepsilon 2^{n-3}, \\ \lambda = \dfrac{16}{3}(2^{n-4} - \varepsilon)(2^{n-5} + \varepsilon) + 3, & m_1 = \dfrac{4}{9}(2^n - \varepsilon)(2^{n-3} + \varepsilon), \\ \mu = \dfrac{1}{4}k = \dfrac{1}{3}(2^{n-2} - \varepsilon)(2^{n-3} + \varepsilon), & m_2 = \dfrac{8}{9}(2^{n-2} - \varepsilon)(2^{n-1} + \varepsilon), \end{array}$$

so that $v - k - 1 = 2^{2n-3}$. The group has order $\frac{1}{(n,3)} 2^{\binom{n}{2}} \prod_{i=2}^{n} (2^i - (-1)^i)$ and is simple for $n \geq 4$.

(v) In $\mathsf{O}_n^{\varepsilon}(3)$ consider the reflections $t_a : x \mapsto x + \frac{(x,a)}{(a,a)} a$ with nonsingular a. The reflections t_a, t_b commute when $(a,b) = 0$. The group preserves the value of Q and there are two conjugacy classes D_η $(\eta = \pm 1)$ of such reflections, consisting of the t_a with $Q(a) = \eta$. Let $\mathsf{O}_n^{\varepsilon,\eta}(3)$ be the subgroup generated by D_η.

If n is even, say $n = 2m$, the corresponding Fischer graph is $NO^{\varepsilon}_{2m}(3)$. Its parameters (v, k, λ, μ) and spectrum $k^1\ \theta_1^{m_1}\ \theta_2^{m_2}$ are:

$$\begin{aligned} v &= \frac{1}{2}3^{m-1}(3^m - \varepsilon), & \theta_1 &= \varepsilon 3^{m-1}, \\ k &= \frac{1}{2}3^{m-1}(3^{m-1} - \varepsilon), & \theta_2 &= -\varepsilon 3^{m-2}, \\ \lambda &= \frac{1}{2}3^{m-2}(3^{m-1} + \varepsilon), & m_1 &= \frac{1}{8}(3^m - \varepsilon)(3^{m-1} - \varepsilon), \\ \mu &= \frac{1}{2}3^{m-1}(3^{m-2} - \varepsilon), & m_2 &= \frac{9}{8}(3^{2m-2} - 1). \end{aligned}$$

The group $\mathsf{O}^{\varepsilon,-}_{2m}(3)$ has order $\frac{2}{d}3^{m(m-1)}(3^m - \varepsilon)\prod_{i=1}^{m-1}(3^{2i} - 1)$ where $d := (4, 3^m - \varepsilon)$, and has commutator subgroup $\mathsf{O}^{\varepsilon,+}_{2m}(3) \simeq \mathsf{P}\Omega^{\varepsilon}_{2m}(3)$ of index 2; this latter group is simple, except when $m = 2$ and $\varepsilon = +1$, in which case $\mathsf{P}\Omega^{+}_{4}(3) \simeq \mathsf{A}_4 \times \mathsf{A}_4$.

If n is odd, say $n = 2m + 1$, the corresponding Fischer graph is $NO_{2m+1}(3)$. Its parameters are:

$$\begin{aligned} v &= \frac{1}{2}3^{m}(3^m + \eta), & r &= 3^{m-1}, \\ k &= \frac{1}{2}3^{m-1}(3^m - \eta), & s &= -3^{m-1}, \\ \lambda &= \mu, & f &= \frac{v-1}{2} - \frac{1}{4}(3^m - \eta), \\ \mu &= \frac{1}{2}3^{m-1}(3^{m-1} - \eta), & g &= \frac{v-1}{2} + \frac{1}{4}(3^m - \eta). \end{aligned}$$

Here η is chosen such that for $\eta = 1$ (resp. -1) the perp of a vertex is a hyperbolic (resp. elliptic) quadric. This corresponds to the η as used earlier if we fix Q to have discriminant 1.

The group $\mathsf{O}^{+,-}_{2m+1}(3) \simeq \mathsf{PO}_{2m+1}(3)$ has order $3^{m^2}\prod_{i=1}^{m}(3^{2i} - 1)$ and has simple commutator subgroup $\mathsf{O}^{+,+}_{2m+1}(3) \simeq \mathsf{P}\Omega_{2m+1}(3)$ of index 2. (The first $+$ in these group denotations is just a place-holder: there is no ε here. Sometimes people do use the first sign to denote the discriminant of Q. If Q defines $\mathsf{O}^{\varepsilon,\eta}_{2m+1}(q)$, then $-Q$ defines $\mathsf{O}^{-\varepsilon,-\eta}_{2m+1}(q)$.)

Group notation

Notation for the orthogonal groups in characteristic 3 varies.

Given a quadratic form Q on the vector space V of dimension n over $\mathbb{F}_3$, one defines a symmetric bilinear form by $(x, y) = Q(x+y) - Q(x) - Q(y)$, so that $(x, x) = 2Q(x) = -Q(x)$. The reflection corresponding to a nonsingular vector v is $t_v : x \mapsto x - 2\frac{(x,v)}{(v,v)}v = x - \frac{(x,v)}{Q(v)}v = x + (v, v)(x, v)v$, and all authors agree. There are two conjugacy classes D_η of such reflections, and the Fischer groups $O^{\varepsilon,\eta}_n(3)$ are the groups generated by D_η in $O(V, Q)$ of type ε.

Aschbacher ([13], p. 44) lets D_η consist of the t_v with $Q(v) = \eta$. Fischer ([327]) uses $(v, v) = \eta$ instead, so has the opposite sign.

Aschbacher, Fischer and others define and use a discriminant δ for Q to distinguish forms, and they denote their groups $\mathsf{O}^{\delta,\eta}_n(3)$. Aschbacher takes as δ the discriminant of the polar bilinear form f_Q given by $Q(x + y) - Q(x) - Q(y)$, while Fischer lets δ be the discriminant of the diagonal bilinear form d_Q determined by $d_Q(x, x) = Q(x)$. We have $f_Q = -d_Q$. In even dimension the distinction does not change the determinant, but in odd dimension it negates it. If $n = 2m$ is even, then the Witt sign ε equals $(-1)^m\delta$. If Q defines $\mathsf{O}^{\delta,\eta}_{2m+1}(3)$, then $-Q$ defines $\mathsf{O}^{-\delta,-\eta}_{2m+1}(3)$.

(vi) The graphs in the last three cases have parameters

v	k	λ	μ	$\bar{\lambda}$	$\bar{\mu}$	r	s	f	g
3510	693	180	126	2248	2304	63	−9	429	3080
31671	3510	693	351	25000	25344	351	−9	782	30888
306936	31671	3510	3240	246832	247104	351	−81	57477	249458

The groups have orders

$$\begin{aligned}|\mathsf{Fi}_{22}| &= 2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11\cdot 13,\\ |\mathsf{Fi}_{23}| &= 2^{18}\cdot 3^{13}\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 23,\\ |\mathsf{Fi}_{24}| &= 2^{22}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29.\end{aligned}$$

The first two are simple, the third has simple commutator subgroup of index 2. The point stabilizers are $2.\mathsf{PSU}_6(2)$, $2.\mathsf{Fi}_{22}$ and $2.\mathsf{Fi}_{23}$, respectively. These graphs have maximal cliques of size 22, 23 and 24, respectively, and the stabilizers of the maximal cliques are the Mathieu groups M_{22}, M_{23} and M_{24}.

The group Fi_{22} has exactly one other rank 3 representation. It has parameters

v	k	λ	μ	$\bar{\lambda}$	$\bar{\mu}$	r	s	f	g
14080	3159	918	648	8408	8680	279	−9	429	13650

The point stabilizer is $\mathsf{O}_7^{+,+}(3)$ of order $2^9\cdot 3^9\cdot 5\cdot 7\cdot 13$.

Also Fi_{23} has exactly one other rank 3 representation. It has parameters

v	k	λ	μ	$\bar{\lambda}$	$\bar{\mu}$	r	s	f	g
137632	28431	6030	5832	86600	86800	279	−81	30888	106743

The point stabilizer is $\mathsf{O}_8^{+,+}(3).\mathsf{S}_3$ ($= \mathsf{P}\Omega_8^+(3).\mathsf{S}_3$) of order $2^{12}\cdot 3^{12}\cdot 5^2\cdot 7\cdot 13\times 6$.

In both cases the points of the graph are the 3-lines through a fixed point in the next larger graph (belonging to Fi_{23} and Fi_{24}, respectively), where two lines are adjacent when they span a dual affine plane of order 2. The lines through a fixed point in the Fi_{22} graph give the Conway graph for $\mathsf{U}_6(2).2$ (§10.81).

5.3 Hall triple systems

A *Steiner triple system* is a point-line geometry where any two points determine a unique line, and lines have three points. (That is, is a 2-$(v,3,1)$ design.) A *Hall triple system* is a Steiner triple system in which any two intersecting lines are contained in a subsystem isomorphic to $\mathsf{AG}(2,3)$. That is, Hall triple systems are precisely the connected Fischer spaces without 6-point subplanes, or, equivalently, precisely those whose Fischer graph is edgeless.

These systems were first investigated in Hall [396], where it is shown that a Steiner triple system satisfies the defining property for a Hall triple system if and only if for each point x there is an automorphism s_x of order at most 2 fixing only that point. If this is the case, then all s_x are conjugate, and for distinct x, y the product $s_x s_y$ has order 3.

Obvious examples of Hall triple systems are the affine spaces $\mathsf{AG}(n,3)$. The smallest nonaffine example has order 81 (*loc. cit.*).

The order (number of points) of a Hall triple system is a power of 3. A short proof is given in [389].

Commutative Moufang loops of exponent 3

A *quasigroup* is a set Q with binary operation $\circ$ such that in $x \circ y = z$ any two of x, y, z uniquely determine the third. A *loop* is a quasigroup with two-sided identity e (satisfying $e \circ x = x \circ e = x$ for all $x \in Q$). A *Moufang quasigroup* is a quasigroup satisfying the identity $x \circ (y \circ (x \circ z)) = ((x \circ y) \circ x) \circ z$ for all $x, y, z \in Q$. Every Moufang quasigroup is a loop ([506]).

Any Steiner triple system $(X, \mathscr{B})$ defines a commutative idempotent quasigroup $(X, \circ)$ by $x \circ x = x$ and $x \circ y = z$ if $\{x, y, z\} \in \mathscr{B}$. It also defines a loop $(X, *, e)$ if we pick an arbitrary element $e \in X$ and define $x * y = (e \circ x) \circ (e \circ y)$.

Moufang loops with at most 2 generators are associative (Moufang [575]; [149], p. 117). In particular, powers of an element are well defined. A Moufang loop $(X, *, e)$ is said to be *of exponent* 3 if $x^3 = e$ for all $x \in X$.

Hall triple systems are equivalent to commutative Moufang loops of exponent 3 (Bruck, cf. [397]). (Indeed, the above recipe produces a commutative Moufang loop of exponent 3 from any Hall triple system and arbitrarily chosen e. Conversely, a commutative Moufang loop $(X, *, e)$ of exponent 3 becomes a Hall triple system with the lines $\{x, y, x^2 * y^2\}$. This correspondence is 1-1: the isomorphism type of $(X, *, e)$ does not depend on the choice of e since the corresponding Fischer group is transitive on points.) See also [539].

5.4 Cotriangular graphs

A *cotriangular space* is a partial linear space $(X, \mathscr{L})$ with lines of size 3, such that whenever a point x is not on a line L, it is collinear with none or all but one of the points of L. A Fischer space where any two intersecting lines span a subspace isomorphic to the dual affine plane of order 2 is a cotriangular space.

A *cotriangular graph* is a graph in which every nonedge lies in a 3-coclique ('cotriangle') T such that every vertex outside T is adjacent to one or all of the vertices of T. The noncollinearity graph of a cotriangular space is a cotriangular graph. A clique extension of a cotriangular graph is again cotriangular. A cotriangular graph is called *reduced* when $x^\perp = y^\perp$ implies $x = y$. A reduced cotriangular graph is the noncollinearity graph of a unique cotriangular space. A graph Γ is called *coconnected* when its complement $\overline{\Gamma}$ is connected.

Theorem 5.4.1 (Shult [651], cf. [395]) *Let Γ be a finite reduced coconnected cotriangular graph. Then Γ is either (i) $N^\varepsilon_{2n}(2)$ ($n \geq 3$), or (ii) $Sp_{2n}(2)$ ($n \geq 3$), or (iii) $\overline{T(n)}$ ($n \geq 2$, $n \neq 4$).*

Here $N^\varepsilon_{2n}(2)$ is the graph on the nonsingular vectors in a vector space of dimension $2n$ over $\mathbb{F}_2$ provided with a nondegenerate quadratic form of type ε, adjacent when orthogonal, $Sp_{2n}(2)$ is the graph on the nonzero vectors in a vector space of dimension $2n$ over $\mathbb{F}_2$ provided with a nondegenerate symplectic form, adjacent when orthogonal, and $\overline{T(n)}$ is the complement of the triangular graph $T(n)$. Thus, the cotriangular graphs of the theorem are among the Fischer graphs for examples (i)–(iii) in the list of Fischer spaces in §5.1.

This theorem was generalized to the infinite case in Hall [392, 393]. The '1 or 3 neighbors' of the definition of cotriangular was generalized to 'an odd number of neighbors' in Brouwer & Shult [142].

The locally cotriangular graphs are determined in HALL & SHULT [395]. See also HALL [394], Theorem 6.2. The special case of locally Petersen graphs was done in HALL [388]. The special case of locally $K_{3,3}$ or Petersen graphs was done in BLOKHUIS & BROUWER [75].

Cotriangular graphs and 2-ranks

A graph Γ is cotriangular (resp. the collinearity graph of a polar space with lines of size 3) precisely when the adjacency matrix $\overline{A}$ of $\overline{\Gamma}$ has the property that the mod 2 sum of any two rows corresponding to nonadjacent (resp. adjacent) vertices in Γ is again a row of $\overline{A}$.

The graphs that occur in this situation are characterized by their low 2-rank:

Theorem 5.4.2 (PEETERS [611]) *For $n \geq 2$ the strongly regular graphs $Sp_{2n}(2)$, $S^{\varepsilon}_{2n}(2)$, $N^{\varepsilon}_{2n}(2)$ and their complements are uniquely determined by their parameters and the minimality of the 2-rank, which is $2n+1$ for the graphs mentioned, and $2n$ for their complements.*

Here $S^{\varepsilon}_{2n}(2)$ is the graph on the singular vectors in a vector space of dimension $2n$ over $\mathbb{F}_2$ provided with a nondegenerate quadratic form of type ε, adjacent when orthogonal. The parameters are:

Name	v	k	r	s
$Sp_{2n}(2)$	$2^{2n}-1$	$2^{2n-1}-2$	$2^{n-1}-1$	$-2^{n-1}-1$
$S^{+}_{2n}(2)$	$2^{2n-1}+2^{n-1}-1$	$2^{2n-2}+2^{n-1}-2$	$2^{n-1}-1$	$-2^{n-2}-1$
$S^{-}_{2n}(2)$	$2^{2n-1}-2^{n-1}-1$	$2^{2n-2}-2^{n-1}-2$	$2^{n-2}-1$	$-2^{n-1}-1$
$N^{+}_{2n}(2)$	$2^{2n-1}-2^{n-1}$	$2^{2n-2}-1$	$2^{n-2}-1$	$-2^{n-1}-1$
$N^{-}_{2n}(2)$	$2^{2n-1}+2^{n-1}$	$2^{2n-2}-1$	$2^{n-1}-1$	$-2^{n-2}-1$

(We followed the notation used in the literature. Elsewhere in this volume we used the names $\Gamma(\mathsf{Sp}_{2n}(2))$, $\Gamma(\mathsf{O}^{\varepsilon}_{2n}(2))$ and $NO^{\varepsilon}_{2n}(2)$ for these graphs.)

5.5 Locally grid graphs

A *grid graph* $p \times q$ is the Cartesian product of the complete graphs K_p and K_q (with $(x, y) \sim (x', y')$ when either $x = x'$, $y \sim y'$ or $x \sim x'$, $y = y'$).

A graph Γ is *locally grid* when each point neighborhood $\Gamma(x)$ is a grid graph. If Γ is connected, then it follows that there are p, q such that Γ is locally $p \times q$. For example, the Johnson graph $J(p+q, p)$ is locally $p \times q$. HALL [392] observes that classifying locally $3 \times q$ graphs is equivalent to classifying cotriangular Fischer spaces.

Theorem 5.5.1 (HALL [392, 393]) *Let Γ be a locally $3 \times q$ graph. Then there is a partial linear space $(X, \mathscr{L})$ with lines of size 3, and where any two intersecting lines span a subspace isomorphic to the dual affine plane of order 2, such that Γ is the line graph of $(X, \mathscr{L})$: the vertices are the lines of $(X, \mathscr{L})$, adjacent when they meet. Conversely, such line graphs are locally $3 \times q$ for some fixed q when connected.*

For example, the graph $J(m, 3)$ is the line graph of $T(m)$ with lines of the form $\{(ij), (ik), (jk)\}$. As another example, there are precisely two connected locally 3×3 grid graphs, on 16 and 20 vertices, namely $\overline{4 \times 4}$ and $J(6,3)$. The former is the line graph of $\mathsf{PG}(3,2)$ minus a line; the noncollinearity graph of this partial linear space is $3K_4$.

Locally grid graphs have been classified in a few other cases. It is easy to see that the unique connected locally $2 \times q$ graph is the triangular graph $T(q+2)$.

Blokhuis & Brouwer [74] show that there are precisely four connected locally 4×4 grid graphs, on 35, 40, 40 and 70 vertices. The last one is $J(8,4)$, the first one its antipodal quotient. The second is a member of an infinite family constructed in Cameron [175]. The third one is ugly, with a group that is not vertex-transitive.

A *μ-graph* of a graph is the subgraph induced on the set of common neighbors of two vertices at distance 2.

In [74] the locally grid graphs such that all μ-graphs are unions of 4-cycles are characterized (as quotients of a Johnson graph). In [335] certain locally grid graphs are classified where all μ-graphs are hexagons. In [604] two types of locally 5×5 graphs are constructed inside the $\mathsf{O}_6^+(4)$ polar graph. In [9] some locally $n \times n$ graphs are constructed, and certain locally 5×5 graphs are classified.

Grids are (thin) generalized quadrangles. More generally, people have looked at EGQs (extended generalized quadrangles) and at locally polar graphs. An early reference is Buekenhout & Hubaut [156].

5.6 Copolar spaces

5.6.1 Hall's classification

Gamma spaces

A *gamma space* is a partial linear space such that for each point p and line L the point p is collinear to 0, 1, or all points of L. Equivalently, a gamma space is a partial linear space such that the set of points collinear with any given point is a subspace. For an example, see §10.54.

For graphs of Lie type a strong form of this property holds, see [123], Theorem 10.6.3.

Delta spaces

A *delta space* is a partial linear space such that for each point p and line L the point p is collinear to 0, all-but-one, or all points of L. Equivalently, a delta space is a partial linear space such that the set of points not collinear with any given point is a subspace. Examples are the copolar spaces below.

The concepts of gamma space and delta space are due to D. G. Higman (in various talks, maybe there is no publication). In his terminology, a 'strict gamma space' is a gamma space in which the possibility 0 never occurs, that is, a polar space. A 'strict delta space' is a delta space in which the possibility 'all' never occurs. These are the copolar spaces studied below.

Copolar spaces

A *copolar space* is a partial linear space $(X, \mathscr{L})$ such that for each line L and point $x \notin L$, the point x is collinear with either 0 or all-but-one points of L. For $x \in X$, let $x^\perp$ be the set of all points of X collinear with x. The space is called *reduced* when $x^\perp \setminus \{x\} \neq y^\perp \setminus \{y\}$ for distinct points x, y. The space is called of *order* q when all lines have size $q+1$.

A *copolar graph* is a graph Γ that is the noncollinearity graph of a copolar space. Copolar spaces and copolar graphs generalize cotriangular spaces and cotriangular graphs (where lines have size 3).

Theorem 5.6.1 (Hall [390]) *A finite reduced connected copolar space has some fixed order q and is one of the following.*

(1) A single line of size $q+1$.

(2) The vertices and point neighborhoods $\Gamma(x) = x^\perp \setminus \{x\}$ of a Moore graph Γ of diameter 2 (a strongly regular graph with $\lambda = 0$ and $\mu = 1$). For $q+1 \in \{2, 3, 7\}$ there is a unique example. All other examples have $q+1 = 57$, and no such examples are known.

(3) The $\binom{n}{2}$ pairs and $\binom{n}{3}$ sets of three pairs contained in a fixed triple, in a fixed n-set. Here $q+1=3$.

(4) The points outside a nonsingular quadric in a projective space $\mathsf{PG}(d,q)$ with d odd and $q=2$, with the elliptic lines.

(5) The points of a projective space $\mathsf{PG}(d,q)$ provided with a nondegenerate symplectic polarity, with the hyperbolic lines.

Examples (3) and (4) here are examples (i) and (iii) from the list of Fischer spaces in §5.1. The case $q=2$ of (5) is example (ii).

5.6.2 Lax embeddings of the symplectic copolar spaces

Let us denote the copolar spaces of Case (5) in Theorem 5.6.1 by $HSp_{d+1}(q)$.

A *lax embedding* of a point-line geometry $(X, \mathscr{L})$ in a projective space $\mathsf{PG}(d,q)$ is an injection ϕ sending points to points and lines to lines, preserving incidence and such that $\phi(X)$ spans the space $\mathsf{PG}(d,q)$.

We are interested in classifying the lax embeddings of $HSp_4(q)$ in projective spaces $\mathsf{PG}(d,q')$ for $d \geq 3$ and $q' \geq q$. The *standard* examples are obtained from including the projective space $\mathsf{PG}(3,q)$ provided with a nondegenerate symplectic form in a larger space $\mathsf{PG}(3,q')$ by extending the field $\mathbb{F}_q$ to $\mathbb{F}_{q'}$.

Proposition 5.6.2 *If $q \geq 4$, then every lax embedding of $HSp_4(q)$ in $\mathsf{PG}(d,q')$, $d \geq 3$, is standard.*

Proof (sketch). The geometry induced on the set of points of $HSp_4(q)$ not collinear to a fixed point is a dual affine plane, which, by [521], only admits a (canonical) embedding in $\mathsf{PG}(2,q')$, with $\mathbb{F}_q$ a subfield of $\mathbb{F}_{q'}$. Hence the points on any isotropic line are mapped into some line of $\mathsf{PG}(d,q')$. This yields a lax embedding of $\mathsf{PG}(3,q)$ in $\mathsf{PG}(d,q')$, leading to the proposition. □

The cases $q = 2, 3$ remain. If $q = 2$, then $HSp_4(q)$ is the case $n = 6$ of Case (3) of Theorem 5.6.1. Let us denote the copolar space corresponding to n of that case by Ω_n. Then Ω_n admits the following standard lax embedding into the hyperplane H (after coordinatization) with equation $\sum_{i=1}^{n} X_i = 0$ of $\mathsf{PG}(n-1,q')$: The point $\{a,b\}$, $a, b \in \{1, 2, \ldots, n\}$, $a \neq b$ is identified with the point whose coordinates are zero, except on places a and b, where the coordinates are nonzero and opposite. A subspace S of H is called *admissible* if $\langle S, x\rangle \neq \langle S, y\rangle$ as soon as x and y are distinct points of the copolar space. Without going into details, we just state that, using the techniques of [688], one can easily prove the following proposition.

Proposition 5.6.3 *Every lax embedding of Ω_n is the projection from an admissible subspace of the standard lax embedding described above.*

Since some rank 3 graphs are intimately related to $HSp_4(3)$ (see §10.89A and §10.93), we investigate the case $q = 3$ in some more detail. (We shall write 3 for q, and q for q'.)

First a lemma. (Note that the planes of $HSp_4(3)$ are dual affine planes $\mathsf{AG}(2,3)^*$.)

Lemma 5.6.4 *Let Π be the image of the dual affine plane $\mathsf{AG}(2,3)^*$ into the projective plane $\mathsf{PG}(2,q)$, $q = p^e \geq 3$, under a lax embedding. Then either $p = 3$ and Π is canonically contained in a subplane of order 3, or $q \equiv 1 \pmod 3$ and Π is completely determined by any four points of which three form a coclique. In this latter case, the stabilizer in $\mathsf{PGL}_3(q)$ of Π is a group $(3^2.2)\!:\!\mathsf{A}_4$. The stabilizer in $\mathsf{P\Gamma L}_3(q)$ is the same group when $p \equiv 1 \pmod 3$, and $(3^2.2)\!:\!\mathsf{S}_4$ when $p \equiv 2 \pmod 3$.*

Proof. The collinearity graph of Π is $K_{4\times 3}$, with four 3-cocliques. If each of these cocliques is contained in a line of $\mathsf{PG}(2,q)$, then Π is contained in a subplane $\mathsf{PG}(2,3)$ of $\mathsf{PG}(2,q)$, so that q is a power of 3 and Π is canonically embedded.

So we may assume that some coclique $\{p_1, p_2, p_3\}$ forms a triangle in $\mathsf{PG}(2,q)$, and we label $p_1(1,0,0)$, $p_2(0,1,0)$ and $p_3(0,0,1)$. Let u be any other point of Π. If u is on one of the lines p_ip_j, $i \neq j$, $i,j \in \{1,2,3\}$, then all points of Π must be contained in that line, a contradiction. Hence we may label $u(1,1,1)$. Now we label the points of the line of Π containing u and p_1 by $(a_1,1,1)$ and $(b_1,1,1)$, and similarly we have the points $(1,a_2,1)$, $(1,b_2,1)$ and $(1,1,a_3)$, $(1,1,b_3)$. We may assume that $\{(a_1,1,1),(1,a_2,1),(1,1,a_3)\}$ is a coclique. Expressing that $\{(a_1,1,1),(1,b_2,1),(0,0,1)\}$ forms a line we obtain $a_1b_2 = 1$. Likewise $a_1b_3 = a_2b_1 = a_2b_3 = a_3b_1 = a_3b_2 = 1$. Hence $a_1 = a_2 = a_3 = a$, where $a \neq 1$, and $b_1 = b_2 = b_3 = a^{-1}$. The three lines of $\mathsf{PG}(2,q)$ with respective equations $aX_1 = X_2$, $aX_2 = X_3$ and $aX_3 = X_1$ have a point of Π in common, which implies $a^3 = 1$. It follows that $q \equiv 1 \pmod 3$. Finally, $a^p = a^2$ if and only if $p \equiv 2 \pmod 3$. □

The small cases $q = 4, 7$ have some interesting additional properties.

Lemma 5.6.5 *Consider the situation of the previous lemma, with $q \equiv 1 \pmod 3$.*

(i) The set of points off Π lying on at least two, and then on exactly 4, two-secants forms together with these 12 two-secants, an affine plane Π' of order 3.

(ii) For $q = 4$, Π' is a Hermitian unital, and its point set is the complement of that of Π.

(iii) For $q = 7$, the stabilizer of Π acts transitively on the 36 points of $\mathsf{PG}(2,7)$ not in Π that are incident with a line of Π.

Proof. (i) Using the coordinates introduced in the proof of Lemma 5.6.4, the nine points of the affine plane have coordinates $(0,1,-c)$, with $c^3 = 1$, and all permutations thereof.

(iii) Let G be the stabilizer of Π in $\mathsf{PGL}_3(7)$. By Lemma 5.6.4, the stabilizer G_L of a line L of Π acts on L as A_4. Since no nontrivial element of this A_4 fixes at least three points of L, viewed as a line of $\mathsf{PG}(2,7)$, we see that G_L acts transitively on the the four remaining points of L. □

Proposition 5.6.6 *A non-standard lax embedding of $HSp_4(3)$ in $\mathsf{PG}(d,q)$, $d \geq 3$, $q = p^e \geq 3$, exists if and only if $d = 3$ and $q \equiv 1 \pmod 3$, and is for each such q unique up to a collineation. Moreover, for a given such embedding, the stabilizer in $\mathsf{PGL}_4(q)$ of the image of $HSp_4(3)$ is the group $\mathsf{PSp}_4(3) \simeq \mathsf{U}_4(2)$. The stabilizer in $\mathsf{P\Gamma L}_4(q)$ is the same group when $p \equiv 1 \pmod 3$, and the split extension $\mathsf{PSp}_4(3)\!:\!2 \simeq \mathsf{P\Sigma U}_4(2)$ when $p \equiv 2 \pmod 3$.*

Proof. Let Σ be the image of a non-standard lax embedding of $HSp_4(3)$ in $\mathsf{PG}(d,q)$, $d \geq 3$. Since $HSp_4(3)$ has planes isomorphic to $\mathsf{AG}(2,3)^*$, Lemma 5.6.4 yields $q \equiv 1 \pmod 3$.

Consider an isotropic line M of the $\mathsf{Sp}_4(3)$ geometry on Σ. No three points of M are collinear in $\mathsf{PG}(d,q)$, since any triple $M\backslash\{m\}$ is contained as a coclique in the dual affine plane $m^\perp\backslash\{m\}$. Also, if all four points of M were contained in a plane of $\mathsf{PG}(d,q)$, then Σ would be contained in that plane, contradicting $d \geq 3$. For $m \in M$, let π_m be the plane $\langle M\backslash\{m\}\rangle$ of $\mathsf{PG}(d,q)$. It contains the plane $\Pi_m = m^\perp\backslash\{m\}$ of Σ. The union of the point sets of Π_m over $m \in M$ is the point set of Σ, and it follows that $d = 3$. For $m,n \in M$ the three lines of Π_n through m, viewed as lines of $\mathsf{PG}(3,q)$, intersect π_m in the points of the affine plane Π'_m in π_m as described in Lemma 5.6.5 (i). Hence, choosing Π_m without loss of generality in a unique way, Π_n is determined up to a homology in $\mathsf{PG}(3,q)$ with center m and axis π_m. Hence $\Pi_m \cup \Pi_n$ is projectively unique. If p is a third point of M, then using the same argument with Π_p now with respect to both Π_m and Π_n, we conclude that $HSp_4(3)$ has at most one projectively unique embedding in $\mathsf{PG}(3,q)$. An easy but cumbersome explicit computation, which we shall not perform, now shows existence.

The uniqueness of the construction and the last assertion of Lemma 5.6.4 show the other assertions. □

The above embeddings play a role in the construction of certain rank 3 graphs on 7^4 and 3^8 vertices.

The case $q = 7$

Proposition 5.6.7 *Let the copolar space $HSp_4(3)$ be embedded in $\mathsf{PG}(3,7)$. The stabilizer of $HSp_4(3)$ in $\mathsf{PGL}_4(7)$ acts transitively on the 40 points of $HSp_4(3)$ and also on the 360 points off $HSp_4(3)$.*

The point set of $HSp_4(3)$ is a two-character set of $\mathsf{PG}(3,7)$; planes intersect in either 12 points (and the intersection is a dual affine plane $\mathsf{AG}(2,3)^*$), or 5 points (and the intersection contains a unique line of $HSp_4(3)$ plus some point).

Proof. Let G be the stabilizer of $HSp_4(3)$ in $\mathsf{PGL}_4(7)$. By Lemma 5.6.5 and Proposition 5.6.6, G acts transitively on the set of points that are contained in a line of $HSp_4(3)$. The number of such points is clearly equal to four times the number of lines of $HSp_4(3)$, hence to $90 \times 4 = 360$. Consequently, this comprises all points off $HSp_4(3)$.
The second assertion follows by a straightforward count. □

With the point set of $HSp_4(3)$ at infinity of $\mathbb{F}_7^4$, we find a rank 3 graph with parameters $(v,k,\lambda,\mu) = (2401, 240, 59, 20)$. This is the graph of §10.89A .

The case $q = 4$ and a self-conjugate spread in $HSp_4(3)$

The uniqueness in Proposition 5.6.6 implies that for $q = 4^m$ the copolar space $HSp_4(3)$ is always contained in a subspace $\mathsf{PG}(3,4)$. It arises there as the geometry on the nonisotropic points of the $\mathsf{U}_4(2)$ geometry, provided with the tangent lines—these are the lines with exactly one $\mathsf{U}_4(2)$-isotropic point.

Consider $\mathsf{PG}(3,3)$ provided with a nondegenerate symplectic form. A spread is called *hyperbolic* if it consists of hyperbolic lines, and *self-conjugate* if it is invariant under the symplectic polarity. Up to a collinearity, $\mathsf{PG}(3,3)$ has a unique hyperbolic self-conjugate spread. In the current setting it is found by taking the ten tangents meeting a fixed t.i. line of $\mathsf{U}_4(2)$. An explicit example with respect to the standard alternating form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2$ is given by the following (we omit the braces and commas):

$1000 + 0100$	$1010 + 0101$	$0110 + 1002$	$1011 + 0112$	$1101 + 0211$
$0010 + 0001$	$1020 + 0102$	$0120 + 1001$	$1110 + 2011$	$0111 + 1102$

This spread $\mathscr{S}$ has the following property, which is easy to check with the above given coordinates: For each $L \in \mathscr{S}$, the unique nontrivial homology with axes L and $L^\perp$ (i.e., the unique collineation of $\mathsf{PG}(3,3)$ fixing $L \cup L^\perp$ pointwise) stabilizes $\mathscr{S}$ and interchanges M and $M^\perp$ for each $M \in \mathscr{S}$, $M \neq L, L^\perp$. All such homologies generate an elementary abelian 2-group P of order 16, normalized by $\mathsf{A}_5 \leq \mathsf{PSp}_4(3)$, acting naturally on the five conjugate pairs of lines of $\mathscr{S}$.

A rank 3 graph on 6561 vertices

Using the above spread, we construct a rank 3 graph Γ with parameters $(v,k,\lambda,\mu) = (6561, 1440, 351, 306)$. This is the graph of §10.93.

Let Σ and Σ' be two disjoint solids of $\mathsf{PG}(7,3)$, each furnished with a self-conjugate hyperbolic spread, say $\mathscr{S}$ and $\mathscr{S}'$, respectively. Let $\theta : \Sigma \to \Sigma'$ be an isomorphism mapping $\mathscr{S}$ to $\mathscr{S}'$. There are precisely two collineations φ, φ' of $\mathsf{PG}(7,3)$ interchanging $x \in \Sigma$ with $\theta(x) \in \Sigma'$. Let P be the elementary abelian 2-group of Σ stabilizing $\mathscr{S}$. For each $g \in P$, there are precisely two collineations, say φ_g and φ'_g, of $\mathsf{PG}(7,3)$ such that $\varphi_g(x) = g(x)$, for all $x \in \Sigma$, and $\varphi'_g(x) = \theta(g(\theta^{-1}(x)))$, for all $x \in \Sigma'$. The group Q generated by φ, φ' and all φ_g, φ'_g, for $g \in P$, is an elementary abelian 2-group of order 64. It has exactly 45 orbits of size 16, each consisting of four lines. The union X of all these orbits can be described as follows.

Each $L \in \mathscr{S}$ has a *symplectic conjugate* $L^\perp$ in $\mathscr{S}$, which is the line corresponding to L under the corresponding symplectic polarity. Now X is the union of all solids of $\mathsf{PG}(7,3)$ generated by a line $L \in \mathscr{S}$ and its image $\theta(L)$, or the symplectic conjugate of that image. The orbits of Q of size 16 can be recovered from this construction by iterating the following process.

Select $L \in \mathscr{S}$ arbitrarily. Denote $M := \theta(L)$, and let L^* and M^* be the symplectic conjugates of L and M, respectively. Let $\theta' = \theta \cdot g$, with $g \in Q_L$ arbitrary. Then there are exactly two solids ξ and ξ' intersecting each solid $\langle K, \theta'(K)\rangle$, $K \in \mathscr{S}$, in lines L_K and L'_K, respectively. The set of lines L_K, $K \in \mathscr{S}$, is a hyperbolic self-conjugate spread $\mathscr{S}_L$ of ξ; likewise for the set of lines L'_K of ξ'. The associated isomorphism $\theta_L : \xi \to \xi'$ is given by the unique nontrivial collineation σ of $\mathsf{PG}(7,3)$ fixing all points of $\Sigma \cup \Sigma'$. For fixed $K \in \mathscr{L}$, the union of the four lines L_K, L'_K and their symplectic conjugates is an orbit of size 16 for Q. The above construction of X applied to ξ, ξ', $\mathscr{S}_L$ and $\sigma(\mathscr{S}_L)$ yields X again. Hence the process can be iterated, and one obtains a set of 45 orbits of size 16.

The graph Γ has the points of $\mathbb{F}_3^8$ as vertex set, adjacent when the joining line hits X at infinity.

The previous paragraphs also imply that the stabilizer of X in $\mathsf{PGL}_8(3)$ acts transitively on the orbits of Q of size 16. The graph with vertices these orbits, adjacent when they are contained in the union of two solids, is isomorphic to the graph on the singular points of the $\mathsf{U}_4(2)$ geometry (hence isomorphic to the collinearity graph of the $\mathsf{GQ}(4,2)$). It follows that $G = Q : \mathsf{PGU}_4(2)$ acts as a transitive automorphism group of X.

We now show that G acts transitively on the complement X' of X in $\mathsf{PG}(7,3)$. Indeed, it is not hard to see that the group Q acts freely on X'; it hence partitions X' in 40 orbits of size 64. Let $x \in X'$ be arbitrary. Then there are unique points $p_x \in \Sigma$ and $p'_x \in \Sigma'$ such that $x \in \langle p_x, p'_x\rangle$. Now p_x and $\theta^{-1}(p'_x)$ are contained in unique respective members L_x and L'_x of the spread $\mathscr{S}$. For every point z in the same partition class P of X' as x we have $L_z, L'_z \in \{L_x, L'_x, L_x^{\perp}, L_x'^{\perp}\}$. Letting $\mathsf{A}_5 \le G_{\Sigma\cup\Sigma'}$ act on $\Sigma \cup \Sigma'$ we see that the orbit of P under the action of G has size at least 10. Since $\mathsf{PGU}_4(2)$ has only primitive permutation representations on 27, 35, 40 and 45 elements, we see that $\mathsf{PGU}_4(2)$ acts transitively on X'.

We determine the dimension of the maximal subspaces contained in X'. Clearly any 5-space intersects both Σ and Σ' nontrivially. Now we construct solids entirely contained in X'. To that aim, we note that every solid S defines a unique isomorphism from Σ to Σ' by projection from S. A direct counting argument proves that every (linear) isomorphism arises in this way (and it arises precisely twice). If we consider the isomorphism θ followed by a fixed point free member ρ of $\mathsf{A}_5 \le G_{\Sigma\cup\Sigma'}$, then we see that the corresponding solid, say S, completely lies in X'. Such solids give rise to maximum cocliques of Γ of size 81.

Let S and ρ be as in the previous paragraph and let $L \in \mathscr{S}$ be arbitrary. Then $\langle L, S\rangle \cap \Sigma' = \rho\theta(L)$. The 5-dimensional subspace $\langle L, S\rangle$ has $4 + 4 + 16 \cdot 4 = 72$ points in common with X.

A standard count reveals that a hyperplane of $\mathsf{PG}(7,3)$ containing a solid entirely contained in X intersects X in 261 points and contains exactly three solids contained in X. Each other hyperplane intersects X in 234 points. Hence X is a two-character set of $\mathsf{PG}(7,3)$.

Chapter 6

Golay codes, Witt designs, and Leech lattice

We collect preliminary material on codes, designs, geometries and lattices. Then construct the Golay codes, the Witt designs, and the Leech lattice.

6.1 Codes

A *code* is a subset of a metric space, so that there is a concept of distance. Our metric spaces will mostly be vector spaces with given basis.

Let V be a vector space over $\mathbb{F}_q$ with fixed basis $e_1, ..., e_n$.

A *code* C is a subset of V. A *linear code* is a subspace of V. Its *length* is n. A *binary* (*ternary*) code is a code with $q = 2$ (resp. $q = 3$). The vector with all coordinates equal to zero (resp. one) will be denoted by $\mathbf{0}$ (resp. $\mathbf{1}$).

In a binary code, the *complement* of the vector u is $u + \mathbf{1}$.

The *Hamming distance* $d_H(u, v)$ between two vectors $u, v \in V$ is the number of coordinates where they differ: $d_H(u, v) = |\{i \mid u_i \neq v_i\}|$ when $u = \sum u_i e_i$, $v = \sum v_i e_i$. The *weight* of a vector u is its number of nonzero coordinates, i.e., $d_H(u, \mathbf{0})$.

The *minimum distance* $d(C)$ of a code C is $\min\{d_H(u, v) \mid u, v \in C, u \neq v\}$. The *support* of a vector is the set of coordinate positions where it has a nonzero coordinate.

Two codes are called *equivalent* when one is obtained from the other by a permutation of coordinate positions, followed by a permutation of the set of coordinate values, independently for each coordinate position. Equivalent codes have the same size and length and minimum distance.

Parameters

An $(n, M, d)_q$-code is a code of length n, size M and minimum distance at least d. An $[n, k, d]_q$-code is a linear code of length n, dimension k and minimum distance at least d. Its size is q^k. The subscript q is omitted for binary codes. The parameter d may be omitted.

Given an $[n, k, d]_q$-code C, a *shortened code* is an $[n-1, k-1, d]_q$ code obtained by selecting all code words that are 0 at some fixed coordinate position, and dropping that coordinate position.

6.1.1 The Golay codes

The most beautiful and important sporadic structures in algebraic combinatorics are the Golay codes, named after their discoverer, M. J. E. Golay, who published them in the 1-page paper [357].[1] The binary and ternary Golay codes are perfect (defined below). The extended binary Golay code is the basis for the definition of the Leech lattice (§6.3.1), which in turn allows the definition of many sporadic simple groups, including the Fischer-Griess Monster group.

Theorem 6.1.1 *There exist codes, unique up to equivalence, with the indicated values of* n, q, $|C|$ *and* $d(C)$*:*

	n	q	$\|C\|$	$d(C)$	*name of* C
(i)	23	2	4096	7	*binary Golay code*
(ii)	24	2	4096	8	*extended binary Golay code*
(iii)	11	3	729	5	*ternary Golay code*
(iv)	12	3	729	6	*extended ternary Golay code*

If they contain $\mathbf{0}$*, these codes are linear (with dimensions* 12, 12, 6, 6*).*

Below we first construct some examples of codes with these parameters, then we study their properties, and we finish showing uniqueness. The binary part of this theorem will be proved in full. For some details in the ternary case we refer to the literature.

6.1.2 The Golay codes — constructions

We give four constructions of the extended binary Golay code, and a construction of the binary and ternary Golay codes.

A construction of the extended binary Golay code

This code is the lexicographically first code with word length $n = 24$ and minimum distance 8: write down the numbers 0, 1, ..., $2^{24} - 1$ in binary and consider them as binary vectors of length 24. Cross out each vector that has distance less than 8 to a previous non-crossed out vector. The 4096 vectors not crossed out form the extended binary Golay code.

Proof: just do it. Some work may be saved by observing (Levenshtein [753]) that any lexicographically minimal binary code with a number of vectors that is a power of two is linear so that all one needs are the 12 base vectors. These turn out to be

[1] The ternary Golay code was discovered independently, and a year earlier, by Juhani Virtakallio (pseudonym Jukka) as a football pool system (Veikkaaja 27/1947 and subsequent issues). See [409] and [207], §15.3.

```
000000000000000011111111
000000000000111100001111
000000000011001100110011
000000000101010101010101
000000001001011001101001
000000110000001101010110
000001010000010101100011
000010010000011000111010
000100010001000101111000
001000001000100100011101
010000010001010001001110
100000010001011100100100
```

Construction from the icosahedron

Let A be the adjacency matrix of (the 1-skeleton of) the icosahedron (with 12 vertices, regular of valency 5). Then the rows of the 12×24 matrix $(I\ J{-}A)$ generate the extended binary Golay code.

Construction as quadratic residue codes

For $(n, q) = (11, 3)$ or $(23, 2)$ consider the linear code generated over $\mathbb{F}_q$ by the n vectors c_i $(1 \le i \le n)$ with coordinates

$$(c_i)_j = \begin{cases} 1 & \text{if } j - i \text{ is a nonzero square mod } n, \\ 0 & \text{otherwise.} \end{cases}$$

This yields the ternary and binary Golay codes, and shows that these have an automorphism that permutes the 11 or 23 coordinate positions cyclically.

Two Hamming codes

Let H be the $[8, 4, 4]$ extended binary Hamming code consisting of the 8 rows of $\begin{pmatrix} 0 & \mathbf{0}^\top \\ \mathbf{1} & F \end{pmatrix}$ (where $F = \text{circ}(0110100)$ is the incidence matrix of the Fano plane $\mathsf{PG}(2, 2)$) and their complements.

Let H^* be the code obtained by replacing F by $F^* = \text{circ}(0001011)$. Then $H \cap H^* = \{\mathbf{0}, \mathbf{1}\}$.

Let $C = \{(a + x, b + x, a + b + x) \mid a, b \in H, x \in H^*\}$. Then C has word length 24, dimension 12 and minimum distance 8 as one easily checks. Hence C is the extended binary Golay code. This representation shows an automorphism with cycle structure $1^3 7^3$.

Hexacode and Miracle Octad Generator

Up to equivalence, there is a unique $[6, 3, 4]_4$ code, known as the *hexacode*. A generator matrix (over $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$) is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \omega & \omega^2 \\ 0 & 0 & 1 & 1 & \omega^2 & \omega \end{bmatrix}.$$

The weight enumerator (see below) is $1 + 45x^4 + 18x^6$. This code is self-dual (for the sesquilinear form $f(x, y) = \sum_i \bar{x}_i y_i$).

The extended binary Golay code can be defined in terms of the hexacode as follows: codewords are binary 4×6 matrices M that satisfy:

(i) The six column sums and the sum of the top row all have the same parity.

(ii) Let $n = (0, 1, \omega, \omega^2)$. Then nM is a codeword in the hexacode.

This description is due to CURTIS [246], and known as the Miracle Octad Generator or MOG.

6.1.3 Properties and uniqueness

We study properties of (arbitrary) codes with parameters as in Theorem 6.1.1.

The codes (i) and (iii) are *perfect*, i.e., the balls with radius $\frac{1}{2}(d(C) - 1)$ around the code words partition the vector space.

(Proof by counting: $|\text{ball}| = 1 + \binom{23}{1} + \binom{23}{2} + \binom{23}{3} = 2048 = 2^{11}$ in case (i), and $|\text{ball}| = 1 + 2\binom{11}{1} + 4\binom{11}{2} = 243 = 3^5$ in case (iii).)
Except for the repetition codes (with $|C| = q$, $d(C) = n$), there are no other perfect codes C with $d(C) > 1$ (TIETÄVÄINEN [689], VAN LINT [523]).[2]

From now on, assume that C contains $\mathbf{0}$. The *weight enumerators* $A(x) := \sum a_i x^i$, where a_i is the number of code words of weight i, are:

(i) $1 + 253x^7 + 506x^8 + 1288x^{11} + 1288x^{12} + 506x^{15} + 253x^{16} + x^{23}$
(ii) $1 + 759x^8 + 2576x^{12} + 759x^{16} + x^{24}$
(iii) $1 + 132x^5 + 132x^6 + 330x^8 + 110x^9 + 24x^{11}$
(iv) $1 + 264x^6 + 440x^9 + 24x^{12}$

(Proof: For cases (i) and (iii) use the fact that the codes are perfect. E.g. in case (iii) the ball around $\mathbf{0}$ covers the vectors of weight at most 2. The $2^3\binom{11}{3}$ vectors of weight 3 must be covered by balls around codewords of weight 5, so that $a_5 = 2^3.\binom{11}{3}/\binom{5}{3} = 132$. Next $a_6 = (2^4.\binom{11}{4} - 132.\binom{5}{4} - 132.\binom{5}{3}.2)/\binom{6}{4} = 132$. Etc. For cases (ii) and (iv), use that dropping any coordinate yields a case (i) or (iii) code.)

The codes (ii) and (iv) are *self-dual*, i.e., with the standard inner product $(u, v) = \sum u_i v_i$ one has $C = C^\perp$ for these codes. In particular codes (ii) and (iv) are linear.

(Proof: If $v \in C$, then also $C - v$ contains $\mathbf{0}$, hence has the same weight enumerator as C. In the binary case this means that all distances are divisible by 4 so that all inner products vanish. In the ternary case, $(u, v) = (u - v, u - v) - (u, u) - (v, v) = 0$. That shows $C \subseteq C^\perp$. But $C^\perp$ is linear. Since $|C|$ is 2^{12} and 3^6 in the two cases, the span $\langle C \rangle$ has dimension at least 12 resp. 6, so that $C^\perp$ has dimension at most 12 resp. 6, and equality holds.)

The codes (i) and (iii) are linear.

(Proof: Given one of the extended codes one may *puncture* it by deleting one coordinate position. This produces (i) and (iii) from (ii), (iv). Conversely, given (i) one may construct (ii) by *extending* it, i.e., adding a *parity check* bit so as to make the weight of all code words even. After adding the check bit all

[2] More generally, if the alphabet size q is not necessarily a prime power, nonexistence of perfect codes is known for $d \geq 7$. There are partial results for $d = 5$.

distances are even, and $d(C) \geq 8$. This shows that every code (i) is linear. For codes (iii) (normalized by multiplying certain coordinate positions by -1 such that the normalized code contains $\mathbf{1}$) one may construct (iv) by adding a check trit so as to make the sum of all coordinates a multiple of three, as was shown by DELSARTE & GOETHALS [277]. Hence every code (iii) is linear.)

The code (ii) is unique up to equivalence.

(Proof: Let C be a code as in (ii). From the weight enumerator we see that $\mathbf{1} \in C$. Let u be a weight 12 vector in C. The code C_u obtained from C by throwing away all coordinate positions where u has a 1, has word length 12 and dimension 11 and hence must be the even weight code (consisting of all vectors of even weight). This means that we can pick a basis for C consisting of u and 11 vectors v_j with $(u + \mathbf{1}, v_j) = 2$ so as to get a generator matrix of the form $\begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top & 1 \\ \mathbf{1} & I & K & \mathbf{1} \end{pmatrix}$, where I is an identity matrix of order 11. A little reflection shows that K is the incidence matrix of a 2-(11,5,2) biplane (see §6.2). This shows uniqueness of C given the uniqueness of the 2-(11,5,2) biplane, and the latter is easily verified by hand.)

Finally, the code (i) is unique up to equivalence.

(Proof: the unique code (ii) has a group that is transitive on the 24 positions.)

We omit the uniqueness proof in the ternary case.

The supports of the code words of minimal nonzero weight form Steiner systems $S(4, 7, 23)$, $S(5, 8, 24)$, $S(4, 5, 11)$ and $S(5, 6, 12)$, respectively. (See §6.2.)

6.1.4 The Mathieu group M_{24}

M_{24} is by definition the automorphism group of the extended binary Golay code C, i.e., the group of permutations of the 24 coordinate positions preserving the code. For a beautiful discussion of this and related groups, see CONWAY [213].

Using the automorphisms visible in a few different constructions of the extended binary Golay code C it is not difficult to see

Theorem 6.1.2 M_{24} *has order* $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 16 \cdot 3$ *and acts* 5*-transitively on the* 24 *coordinate positions.* □

Let a *point* be a coordinate position, and an *octad* be the support of a code word of weight 8.

Theorem 6.1.3 *Let H be the subgroup of M_{24} fixing an octad B (setwise) and a point $x \notin B$. Then $H \simeq \mathsf{A}_8 \simeq \mathsf{PGL}_4(2)$.* □

Theorem 6.1.4 M_{24} *is transitive on trios (partitions of the point set into* 3 *octads), sextets (partitions of the point set into six* 4*-sets, such that the union of any two is an octad) and dodecads (vectors in C of weight* 12*).* □

6.1.5 More uniqueness results

Theorem 6.1.5 *(a) Let $C^{(i)}$ be a binary code containing* **0** *with word length $24-i$, minimum distance 8, and size at least 2^{12-i}. If $0 \le i \le 3$ then $C^{(i)}$ is the i times shortened extended binary Golay code.*

(b) Let $C_0^{(i)}$ be a binary code containing **0** *with word length $23-i$, minimum distance 7, and size at least 2^{12-i}. If $0 \le i \le 3$ then $C_0^{(i)}$ is the i times shortened binary Golay code.*

The weight enumerators are (for $i > 0$) given by

i	n	dim	weight enumerator
1	23	11	$1+506x^8 \ +1288x^{12}+253x^{16}$
2	22	10	$1+330x^8+616x^{12}+77x^{16}$
3	21	9	$1+210x^8+280x^{12}+21x^{16}$
1	22	11	$1+176x^7+330x^8+672x^{11}+616x^{12}+176x^{15}+77x^{16}$
2	21	10	$1+120x^7+210x^8+336x^{11}+280x^{12}+56x^{15}+21x^{16}$
3	20	9	$1+80x^7+130x^8+160x^{11}+120x^{12}+16x^{15}+5x^{16}$

Adding a parity check bit to $C_0^{(i)}$ we find $C^{(i)}$, and for $i > 0$ the latter is the even weight subcode of $C_0^{(i)}$.

(c) Let C_{00} be a binary self-dual code with word length 22 and minimum distance 6. Then C_{00} is the once truncated binary Golay code. □

It is true that a binary code with word length 20 and minimum distance 8 has size at most 256 ([344]), but there are many codes achieving this ([141]).

6.2 Designs

A t-(v,k,λ) *design* is a set of v *points* together with a collection of subsets of size k (called *blocks*) such that each set of t points is in precisely λ blocks.

A *Steiner system* $S(t,k,v)$ is such a design with $\lambda = 1$.

A *projective plane* $\mathsf{PG}(2,n)$ is a Steiner system $S(2,n+1,n^2+n+1)$. (We shall not suppose that the plane is Desarguesian.)

An *affine plane* $\mathsf{AG}(2,n)$ is a Steiner system $S(2,n,n^2)$.

A *BIBD* (balanced incomplete block design) is a 2-(v,k,λ) design.

A *square design*, or *symmetric design*, or *SBIBD*, is a 2-(v,k,λ) design with equally many points as blocks. A *biplane* is such a design with $\lambda = 2$.

A *parallel class* is a set of blocks partitioning the point set. The design is *resolvable* when the set of blocks has a partition into parallel classes. For example, $\mathsf{AG}(2,n)$ is resolvable.

A necessary condition for the existence of a t-(v,k,λ) design is that $\binom{k-i}{t-i}$ divides $\lambda\binom{v-i}{t-i}$ for $0 \le i \le t$ (since the number of blocks on a given i-set is an integer). Wilson [736] showed for $t = 2$, and Keevash [487] showed for all t, that if t, k, λ are fixed, and the divisibility condition is satisfied, and v is sufficiently large, then a t-(v,k,λ) design exists.

The number of nonisomorphic designs increases quickly with v: there are 1, 1, 2, 80, 11084874829 Steiner triple systems $STS(v)$ (that is, $S(2,3,v)$) for $v = 7, 9, 13, 15, 19$, and $(v/e^2 + o(v))^{v^2/6}$ such systems for large v ([482], [488]).

Given a t-(v,k,λ) design one may delete one point and all blocks not containing that point and obtain a $(t-1)$-$(v-1,k-1,\lambda)$ design (called the *derived* design).

On the other hand, deleting a point and all blocks containing it one obtains a $(t-1)$-$(v-1,k,\frac{v-k}{k-t}\lambda)$ design (called the *residual* design).

A t-(v,k,λ) design is also an i-(v,k,λ_i) design for $0 \le i \le t$, with $\lambda_i = \lambda(v-t+1)\cdots(v-i)/(k-t+1)\cdots(k-i)$.

For a t-(v,k,λ) design, the number of blocks containing a point set X and disjoint from a point set Y (where $X \cap Y = \emptyset$) can be expressed in the parameters t, v, k, λ, $|X|$, $|Y|$ when $|X \cup Y| \le t$. Let us call these numbers $\mu(|X|,|Y|)$.

6.2.1 The Witt designs

We are mostly interested in the systems $S(5,8,24)$ and $S(5,6,12)$ and derived designs.

These designs are generally known as the *Witt designs* because of Witt [740, 741]. An earlier construction was given in Carmichael [187].

For $S(5,8,24)$ we have: $\lambda_5 = 1$, $\lambda_4 = 5$, $\lambda_3 = 21$, $\lambda_2 = 77$, $\lambda_1 = 253$, $\lambda_0 = 759$. The 'intersection' triangle here gives the numbers $\mu(|X|,|Y|)$ with $|X \cup Y|$ constant in each row and $|X|$ increasing in each row, where $X \cup Y$ is contained in a block.

$$\begin{array}{c}
759 \\
253 \quad 506 \\
77 \quad 176 \quad 330 \\
21 \quad 56 \quad 120 \quad 210 \\
5 \quad 16 \quad 40 \quad 80 \quad 130 \\
1 \quad 4 \quad 12 \quad 28 \quad 52 \quad 78 \\
1 \quad 0 \quad 4 \quad 8 \quad 20 \quad 32 \quad 46 \\
1 \quad 0 \quad 0 \quad 4 \quad 4 \quad 16 \quad 16 \quad 30 \\
1 \quad 0 \quad 0 \quad 0 \quad 4 \quad 0 \quad 16 \quad 0 \quad 30
\end{array}$$

Given a block B_0 of $S(5,8,24)$, let n_i be the number of blocks B such that $|B_0 \cap B| = i$. Then $n_8 = 1$, $n_4 = 280$, $n_2 = 448$, $n_0 = 30$ and all other n_i are zero.

For $S(5,6,12)$ we have: $\lambda_5 = 1$, $\lambda_4 = 4$, $\lambda_3 = 12$, $\lambda_2 = 30$, $\lambda_1 = 66$, $\lambda_0 = 132$. Our intersection triangle becomes

$$\begin{array}{c}
132 \\
66 \quad 66 \\
30 \quad 36 \quad 30 \\
12 \quad 18 \quad 18 \quad 12 \\
4 \quad 8 \quad 10 \quad 8 \quad 4 \\
1 \quad 3 \quad 5 \quad 5 \quad 3 \quad 1 \\
1 \quad 0 \quad 3 \quad 2 \quad 3 \quad 0 \quad 1
\end{array}$$

Given a block B_0 of $S(5,6,12)$, let n_i be the number of blocks B such that $|B_0 \cap B| = i$. Then $n_6 = 1$, $n_4 = 45$, $n_3 = 40$, $n_2 = 45$, $n_0 = 1$ (and $n_5 = n_1 = 0$). In particular the complement of a block is again a block.

Note that the above intersection numbers are a consequence of the parameters alone (and may thus be used in uniqueness proofs).

As we shall see, there exist unique designs $S(5,8,24)$, $S(4,7,23)$, $S(3,6,22)$, $S(2,5,21)$, $S(1,4,20)$, $S(5,6,12)$, $S(4,5,11)$, $S(3,4,10)$, $S(2,3,9)$, $S(1,2,8)$. The system $S(2,5,21)$ is the projective plane of order 4, $S(2,3,9)$ the affine plane of order 3, $S(3,4,10)$ the Möbius plane of order 3. (In view of the derivation $S(t,k,v) \to S(t-1,k-1,v-1)$ it suffices to construct $S(5,8,24)$ and $S(5,6,12)$, and we shall find these as the supports of the code words of minimal nonzero weight in the extended Golay codes. Uniqueness will come as a corollary of the uniqueness of the Golay codes.)

Theorem 6.2.1 *There is a unique Steiner system* $S(5,8,24)$.

Proof. (i) Existence: the words of weight 8 in the extended binary Golay code C cover each 5-set at most once since $d(C) = 8$, and exactly once since $\binom{24}{5} = 759 \cdot \binom{8}{5}$.

(ii) Uniqueness: Let $\mathscr{S}$ be such a system, and let C_1 be the binary linear code spanned by (the characteristic functions of) its blocks. From the intersection numbers we know that C_1 is self-orthogonal (i.e., $C_1 \subseteq {C_1}^{\perp}$) with all weights divisible by 4. In order to show that $|C_1| \ge 2^{12}$ (so that $|C_1| = 2^{12}$), fix three independent coordinate positions, say 1, 2, 3, and look at the subcode C_2 of C_1 consisting of the vectors u with $u_1 = u_2 = u_3$. Then $\dim C_1 = 2 + \dim C_2$. Thus, in order to prove $\dim C_1 \ge 12$ it suffices to show that the code generated by the blocks of $S(5,8,24)$ containing three given points has dimension at least 10. In other words, we must show that the code generated by the lines of the projective plane $\mathsf{PG}(2,4)$ (which is nothing but $S(2,5,21)$) has dimension at least 10, but that is the result of the next theorem.

The blocks of an $S(5,8,24)$ assume all possible 0-1 patterns on sets of cardinality at most 5 so that ${C_1}^{\perp}$ has minimum weight at least 6. Since C_1 has all weights divisible by 4 and $C_1 \subseteq {C_1}^{\perp}$ it follows that $d(C_1) = 8$. Now apply Theorem 6.1.1 to see that C_1 is the extended binary Golay code. Since that has $a_8 = 759$, $\mathscr{S}$ is the set of its weight 8 vectors. □

Theorem 6.2.2 *The binary code spanned by the lines of the projective plane* $\mathsf{PG}(2,4)$ *has dimension* 10.

Proof. Let *abcde* be a line in $\mathsf{PG}(2,4)$. The set of ten lines consisting of all five lines on a, three more lines on b, and one more line on each of c, d, is linearly independent, so the dimension is at least 10. But the previous proof (or a simple direct argument showing that the extended code cannot be self-dual) shows that it is at most 10. □

Theorem 6.2.3 *There is a unique Steiner system* $S(4,7,23)$.

Proof. The proof is very similar to that of the uniqueness of $S(5,8,24)$. Let C_0 be the code spanned by the blocks and add a parity bit to obtain a self-orthogonal code C of word length 24. As before one identifies C as the extended binary Golay code, then C_0 as the (perfect) binary Golay code, then the blocks of $S(4,7,23)$ as the words of weight 7 in this code. □

Theorem 6.2.4 *There is a unique Steiner system* $S(3,6,22)$.

Proof. Inspired by LANDER [508] (esp. pp. 54 and 71), we first construct D as the binary linear code spanned by the lines of $\mathsf{PG}(2,4)$, extended by a parity check bit. Then D has word length 22, and $\dim D = 10$. The code D is self-orthogonal and hence there are three codes D_i of dimension 11 such that $D \subseteq D_i \subseteq D^\perp$ $(i = 1,2,3)$. But D can be identified with the subcode of the extended binary Golay code C defined by $u_1 = u_2 = u_3$, and the three codes D_i are found as subcodes defined by $u_2 = u_3$, $u_1 = u_3$ and $u_1 = u_2$, respectively. (More precisely, our codes are obtained from the subcodes of C just mentioned by dropping the first three coordinate positions and adding a parity bit; note that $\mathbf{1} \in D$.) Now 3-transitivity of M_{24} tells us that the three codes D_i are equivalent; each has 77 words of weight 6. Given any Steiner system $S(3,6,22)$, its blocks must span one of the codes D_i, and the blocks of the Steiner system are recovered as the supports of the code words of weight 6 in this code. □

Starting from $S(5,8,24)$ and taking successive derived or residual designs we find designs with the following parameters:

5-(24,8,1)
4-(23,7,1) 4-(23,8,4)
3-(22,6,1) 3-(22,7,4) 3-(22,8,12)
2-(21,5,1) 2-(21,6,4) 2-(21,7,12) 2-(21,8,28)

Up to now we have seen uniqueness of the three largest Steiner systems (and used the uniqueness of $S(2,5,21) = \mathsf{PG}(2,4)$—an easy exercise). Such strong results are not available for the remaining six designs.

(In fact, observe that a 2-(21,7,3) design exists—e.g., the residual of an SBIBD 2-(31,10,3). Taking 4 copies of such a design, independently permuting the point sets in each case, produces large numbers of nonisomorphic designs with parameters 2-(21,7,12), so this structure is certainly not determined by its parameters alone.)

Let $\mathscr{D}$ be a collection of k-subsets of an n-set such that (the characteristic vectors of) any two k-subsets have Hamming distance at least 8. Then for each of the cases listed below we have $|\mathscr{D}| \le b$ with b as given in the table, and when equality holds then the system is known to be unique, except in five cases. For $(n,k,b) = (19,5,12)$ there are precisely two nonisomorphic systems, corresponding to the two Latin squares of order 4. For $(n,k,b) = (18,5,9)$ there are precisely three nonisomorphic systems. For the three cases $(n,k,b) = (19,6,28)$, $(20,7,80)$, $(21,8,210)$ no information is available. In all cases other than these three, the block intersection numbers are as shown ([116]).

$k \setminus n$	18	19	20	21	22	23	24	intersections
5	9	12	16	21				1
6		*28*	40	56	77			0,2
7			*80*	120	176	253		1,3
8				*210*	330	506	759	0,2,4

Also the systems with $(n,k,b) = (22,10,616)$, $(22,11,672)$, $(23,11,1288)$, and $(24,12,2576)$ are unique ([141]).

6.2.2 Substructures of $S(5,8,24)$

Sextets

A *tetrad* is a 4-subset of the point set of $S(5,8,24)$.

Proposition 6.2.5 *Let T_0 be a fixed tetrad. Then T_0 determines a unique sextet, i.e., partition of the 24-set into six tetrads T_i such that $T_i \cup T_j$ is a block for all i, j $(i \neq j)$.*

Proof. Since $\lambda_4 = 5$ there are five blocks B_i on T_0 $(1 \leq i \leq 5)$ and with $T_i := B_i \setminus T_0$ we have $T_i \cup T_j = B_i + B_j$ $(0 \neq i \neq j \neq 0)$. Since $\lambda_5 = 1$, the six tetrads T_i are pairwise disjoint. □

The embedding of S(5,6,12)

A *dodecad* is the support of a vector of weight 12 in C.

Proposition 6.2.6 *Let D_0 be a fixed dodecad. The 132 octads meeting D_0 in six points form the blocks of a Steiner system $S(5,6,12)$ on D_0.*

Proof. Each 5-set in D_0 is in a unique block of $S(5,8,24)$, and this block must meet D_0 in six points. □

A *Hadamard 3-design* is a 3-$(4n, 2n, n-1)$ design. If H is a Hadamard matrix of order $4n$ having a row **1**, then the $8n-2$ rows different from $\pm\mathbf{1}$ in $\binom{H}{-H}$ give the (± 1-characteristic vectors of the) blocks of a Hadamard 3-design.

Proposition 6.2.7 *Let D_0 be a fixed dodecad and $x \notin D_0$. The 22 octads meeting D_0 in six points and containing x form the blocks of a Hadamard 3-design 3-$(12, 6, 2)$. There is a natural 1-1 correspondence between the $\frac{1}{2}.132 = 66$ pairs of disjoint blocks of the $S(5,6,12)$ on D_0 and the $\binom{12}{2} = 66$ pairs of points not in D_0.*

Proof. Given a pair of points x,y outside D_0, there are precisely two octads on $\{x, y\}$ meeting D_0 in six points, and these give disjoint blocks in the $S(5,6,12)$ (for: if these octads are B, B' then $B' = B + D_0$). Varying y we find 11 pairs of disjoint blocks, blocks from different pairs having precisely 3 points in common. □

Labeling the lines of PG(3,2) with triples from a 7-set

The isomorphism $\mathsf{PGL}_4(2) \simeq \mathsf{A}_8$ can be seen inside M_{24}. A useful consequence is that the 35 lines of $\mathsf{PG}(3,2)$ can be labeled with the 35 triples from a 7-set in such a way that intersecting lines correspond to triples that meet in a singleton.

Proposition 6.2.8 *Let B_0 be a fixed octad. The 30 octads disjoint from B_0 form a self-complementary*[3] 3-$(16, 8, 3)$ *design, namely the design of the points and affine hyperplanes in $\mathsf{AG}(4,2)$, the 4-dimensional affine space over $\mathbb{F}_2$.* □

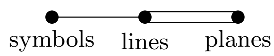
Proposition 6.2.9 *Let B_0 be a fixed octad, $x \in B_0$, $y \notin B_0$, Z the complement of $B_0 \cup \{y\}$. Then there is a natural 1-1 correspondence between the $\binom{7}{3} = 35$ triples in $B_0 \setminus \{x\}$ and the 35 lines in the $\mathsf{PG}(3,2)$ defined on Z. Triples meeting in a singleton correspond to intersecting lines.*

[3] A design $(X, \mathscr{B})$ is called *self-complementary* if for each $B \in \mathscr{B}$ also $X \setminus B \in \mathscr{B}$.

Proof. A line in the $\mathsf{PG}(3,2)$ on Z is a set $T \setminus \{y\}$ where T is a 4-set such that three of the blocks on it are disjoint from B_0. Of the remaining two blocks on T, precisely one contains the point x, and if B is this one then $B \cap B_0 \setminus \{x\}$ is the triple corresponding to the given line. □

Remark. For a discussion of this correspondence, cf. Jordan [469, n° 426, 516], Moore [571], Dickson [291], Conwell [219], Edge [304], Wagner [717, p. 424] and Hall [387].

Remark. Using this correspondence we find a description of the Neumaier geometry. Let Σ be a set of 7 symbols, and let a 1-1 correspondence between the triples from Σ and the lines of $\mathsf{PG}(3,2)$ be given, such that triples meeting in a singleton correspond to intersecting lines. Construct a geometry with three types: the 7 symbols of Σ, the 35 lines of $\mathsf{PG}(3,2)$, and the 15 planes of $\mathsf{PG}(3,2)$, where the incidence is natural, and each symbol is incident with each plane. This defines a geometry with diagram



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Remark. This also yields the '15+35' construction of the Hoffman-Singleton graph (§10.19).

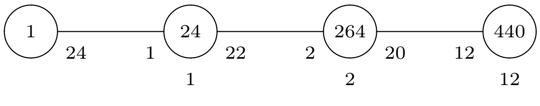
6.2.3 Near polygons

A *near polygon* is a partial linear space such that for each point x and each line L there is a unique point y on L closest to x in the collinearity graph. A *quad* in a near polygon is a geodetically closed sub near polygon of diameter 2. A *near hexagon* is a near polygon of diameter 3.

Near polygons were introduced in [655]. For properties and classification of near polygons, see [174], [143], [122], [260].

Example: the extended ternary Golay code

The partial linear space with as points the vectors of the extended ternary Golay code and as lines the cosets of 1-dimensional subspaces spanned by a vector of weight 12 is a near hexagon with 3 points per line and 12 lines per point and diagram (as distance-transitive graph)

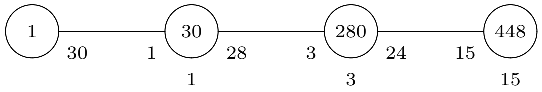


$v = 729$

It has quads (3×3 grids).

Example: the Witt design $S(5,8,24)$

The partial linear space with as points the 759 blocks of the Steiner system $S(5,8,24)$ and as lines the partitions of the point set of the design into three pairwise disjoint blocks, is a near hexagon with 3 points per line and 15 lines per point and diagram (as distance transitive graph)



$v = 759$

It has quads ($\mathsf{Sp}(4,2)$ generalized quadrangles). A quad in the near polygon corresponds to a *sextet* in the design: a partition of the point set into six 4-sets such that the union of any two of them is a block. Distances 0, 1, 2, 3 in the near polygon correspond to intersections of size 8, 0, 4, 2, respectively.

6.2.4 The geometry of the projective plane of order 4

There is a unique projective plane of order 4. It has 5 points on each line and 5 lines on each point (by definition of 'order'), and 21 points and 21 lines (more generally, a projective plane of order q has q^2+q+1 points and as many lines). Its geometry is closely related to the structure of the Witt designs, and we discuss it in some detail.

Hyperovals

A *hyperoval* in a projective plane is a set of points intersecting any line in either 0 or exactly 2 points. In $\mathsf{PG}(2,4)$ each hyperoval contains six points (in general a hyperoval of a projective plane of order q contains $q+2$ points) and may be constructed as follows. Select four points arbitrarily but such that no three are on a line, and cross out all points on each line containing two of these. Then exactly two points remain. Add these to the four previously selected points and these six points form a hyperoval. A simple count reveals that there are $\frac{21\cdot20\cdot16\cdot9}{6\cdot5\cdot4\cdot3} = 168$ hyperovals in $\mathsf{PG}(2,4)$.

Baer subplanes

A *Baer subplane* B of a projective plane P is a proper subset of points and lines such that the induced incidence relation renders it a projective plane with the property that every line of P contains at least one point of B and every point of P is on at least one line of B. In the finite case the order of B is necessarily equal to the square root of the order of P, and every subplane with that order is a Baer subplane. In $\mathsf{PG}(2,4)$ each Baer subplane is a *Fano plane* (see §4.1.1) and may be constructed as follows. Select four points arbitrarily but such that no three are on a line, and add the intersection points of all pairs of lines spanned by two of the selected points. A simple count reveals that there are $\frac{21\cdot20\cdot16\cdot9}{7\cdot6\cdot4\cdot1} = 360$ Baer subplanes in $\mathsf{PG}(2,4)$.

Unitals

A *unital* is an $S(2, q+1, q^3+1)$ Steiner system, for a certain natural number q. An *embedded* unital U is a set of q^3+1 points in a projective plane P of order q^2 such that each line of P intersects U in either 1 (tangent line) or exactly $q+1$ (secant line) points. It follows that each point of U is incident with a unique tangent line and that U together with the subsets induced by the secant lines is a unital. A *Hermitian* unital is a unital in a classical projective plane $\mathsf{PG}(2,F)$ over a field F such that its points correspond precisely to the set of isotropic

1-spaces of a nondegenerate Hermitian form on F^3. In $\mathsf{PG}(2,4)$ every unital is Hermitian, contains 9 points and may be constructed as the set of points on the lines of a triangle, excluding the vertices of the triangle. A simple count reveals that there are $\frac{21\cdot 20\cdot 16}{4\cdot 3\cdot 2\cdot 1} = 280$ unitals in $\mathsf{PG}(2,4)$.

Going down

One can see $\mathsf{PG}(2,4)$ and its hyperovals and Baer subplanes inside the Witt design $S(5,8,24)$. Let Y be a 21-set, and $X = Y \cup \{\infty_1, \infty_2, \infty_3\}$ be a 24-set, and $\mathscr{S}$ the collection of 759 blocks of an $S(5,8,24)$ on X. Write the blocks using their characteristic vectors, with $\infty_1, \infty_2, \infty_3$ as the first three coordinates. The 21 blocks starting $111\ldots$ give the 21 lines of a $\mathsf{PG}(2,4)$ on Y. The $56+56+56 = 168$ blocks starting 110, 101, or 011 give the 168 hyperovals on Y. The $120+120+120 = 360$ blocks starting 100, 010, or 001 give the 360 Baer subplanes on Y. Let C be the extended binary Golay code spanned by the characteristic vectors of the blocks in $\mathscr{S}$. The 280 vectors of weight 12 in C starting 111 give the 280 unitals on Y.

Often, geometric questions about $\mathsf{PG}(2,4)$ can be answered quickly by using this representation. For example, $\mathsf{PG}(2,4)$ does not contain three pairwise disjoint hyperovals since their sum would be a vector of weight more than 16 but less than 24 in the extended binary Golay code C, and there is no such vector.

Going up

On the other hand, it is possible (but a bit cumbersome) to construct $S(5,8,24)$ from the above data in $\mathsf{PG}(2,4)$ ([697, 529]). The main step is partitioning the 168 hyperovals into three sets of 56 and the 360 Baer subplanes into three sets of 120.

One way to do this is via the group. The above discussion shows that $\mathsf{PGL}_3(4)$ is transitive on hyperovals and on Baer subplanes. Its index 3 subgroup $\mathsf{PSL}_3(4)$ has three orbits on hyperovals and on Baer subplanes, and provides the needed partition.

On the other hand, from the description in terms of the extended binary Golay code C (and the fact that C is self-orthogonal) it is clear that meeting in an even number of points is an equivalence relation with three classes on the hyperovals, and meeting in an odd number of points is an equivalence relation with three classes on the Baer subplanes. This can be verified directly, without use of C, from the geometry of $\mathsf{PG}(2,4)$:

Proposition 6.2.10 *Let $\mathscr{H}$ be the set of hyperovals of $\mathsf{PG}(2,4)$ and let $\mathscr{B}$ be the set of Baer subplanes of $\mathsf{PG}(2,4)$. There are partitions $\{\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3\}$ of $\mathscr{H}$ and $\{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}$ of $\mathscr{B}$ into three classes such that*

(i) hyperovals intersect in an even number of points if and only if they belong to the same class;

(ii) Baer subplanes intersect in an odd number of points if and only if they belong to the same class;

(iii) for $H \in \mathscr{H}_i$ and $B \in \mathscr{B}_j$ the intersection size $|B \cap H|$ is even if and only if $i = j$.

Proof. The proof is elementary but tedious. A simple count shows that there are $1, 3, 12, 48, 168$ hyperovals on $4, 3, 2, 1, 0$ given points, no three collinear. For a fixed hyperoval H it follows that there are $1, 0, 0, 40, 45, 72, 10$ hyperovals intersecting H in precisely $6, 5, 4, 3, 2, 1, 0$ points, respectively, so that there are precisely 56 hyperovals intersecting H in an even number of points.

For a fixed Baer subplane B, a simple count yields $1, 0, 0, 56, 77, 168, 42, 16$ Baer subplanes intersecting B in precisely $7, 6, 5, 4, 3, 2, 1, 0$ points, respectively. Hence $1 + 0 + 77 + 42 = 120$ intersect B in an odd number of points.

Fix a Baer subplane B in $\mathsf{PG}(2, 4)$. We show that there are 7, 42, 7 hyperovals meeting it in 4, 2, 0 points, respectively. Also, that the mutual intersection sizes of these 56 hyperovals are even. That will prove that meeting in an even number of points is an equivalence relation on the hyperovals, with classes of size 56.

Given 4 points of B no three on a line there is a unique hyperoval containing these. Hence 7 hyperovals intersect B in exactly 4 points.

Given two points p_1, p_2 of B, let q_1, q_2 be two points off B but on different lines of B through p_1 and such that no three points among p_1, p_2, q_1, q_2 are collinear (there are two possible choices for $\{q_1, q_2\}$). The hyperoval determined by p_1, p_2, q_1, q_2 intersects B in just $\{p_1, p_2\}$, and every hyperoval intersecting B in exactly two points arises this way. Hence there are 42 hyperovals intersecting B in exactly 2 points.

Given a point p of B, the points off B on the lines of B through p form a hyperoval disjoint from B; we claim every hyperoval H disjoint from B arises in this way: since the lines disjoint from H form a dual hyperoval (as can be easily checked), at most 4 can be contained in B; hence B contains at least three secants to H which must necessarily be concurrent and the claim follows. This accounts for 7 hyperovals disjoint from B.

In total we have $7 + 42 + 7$ hyperovals intersecting B in an even number of points. It is an elementary verification that each pair of such hyperovals intersects in an even number of points itself. Hence we have a set of 56 hyperovals pairwise intersecting in an even number of points.

Completely similar the reciprocal to the previous paragraph can be proved: Fix a hyperoval H in $\mathsf{PG}(2, 4)$ and exhibit all Baer subplanes intersecting H in an even number of points. Clearly $\binom{6}{4} = 15$ intersect H in exactly 4 points. Consider two points p_1, p_2 of H. In order to include p_1, p_2 in a Baer subplane B it is necessary and sufficient to select two lines through each of them (distinct from the line p_1p_2). To avoid further intersection points with H, it is necessary and sufficient to make the selection so that each points of $H \setminus \{p_1, p_2\}$ is on exactly one selected line. This can be done in 6 ways, giving rise to $6 \cdot \binom{6}{2} = 90$ Baer subplanes intersecting H in precisely 2 points. Finally, for each point p outside H, the set of points off H but on a secant through p constitutes a Fano plane, as is easily checked, and no other disjoint Fano planes exist. Hence there are 15 such and in total we have $15 + 90 + 15 = 120$ Baer subplanes intersecting H in an even number of points. Again it is readily seen that all these subplanes intersect each other in an odd number of points. □

We have the following construction/theorem.

Theorem 6.2.11 *Let $\mathscr{H}_i \subseteq \mathscr{H}$ and $\mathscr{B}_i \subseteq \mathscr{B}$, $i = 1, 2, 3$, be the partition classes of hyperovals and Baer subplanes, respectively, as defined in the previous proposition. Let $\mathscr{L}$ be the set of lines of* $\mathsf{PG}(2, 4)$ *and let X be the set of points*

of $\mathsf{PG}(2,4)$ *enriched with three new elements* $\infty_1, \infty_2, \infty_3$ *(so* $|X| = 24$*). Define the following* 8*-subsets of* X *and call them blocks of* X*:*
(*i*) $L \cup \{\infty_1, \infty_2, \infty_3\}$, *for every* $L \in \mathscr{L}$*;*
(*ii*) $H \cup \{\infty_i, \infty_j\}$, *for every* $H \in \mathscr{H}_k$, *for all* i, j, k *with* $\{i, j, k\} = \{1, 2, 3\}$*;*
(*iii*) $B \cup \{\infty_i\}$, *for every* $B \in \mathscr{B}_i$, *for all* $i \in \{1, 2, 3\}$*;*
(*iv*) $(L \cup M) \setminus (L \cap M)$, *for all distinct* $L, M \in \mathscr{L}$.
Then X *endowed with these* 8*-subsets is an* $S(5, 8, 24)$.

Moreover, if U *is a Hermitian unital in* $\mathsf{PG}(2,4)$, *then the set* $Y = U \cup \{\infty_1, \infty_2, \infty_3\}$ *endowed with the blocks of* X *that intersect* Y *in at least 5 elements, is an* $S(5, 6, 12)$.

Proof. The fact that X endowed with its blocks is an $S(5, 8, 24)$ is an easy exercise. The second statement follows from the observation that, if a block of X intersects Y in at least 5 elements, then it has precisely 6 elements in common with Y, which is equivalent to verifying that

- If a line intersects U in at least 2 points, then it shares exactly 3 points with it.
- If a hyperoval intersects U in at least 3 points, then it shares exactly 4 points with it.
- If a Baer subplane intersects U in at least 4 points, then it shares exactly 5 points with it.

All these follow easily from the above construction of any Hermitian unital as the set of points on a given triangle, except for the vertices of the triangle. □

As a Hermitian unital of $\mathsf{PG}(2,4)$ endowed with the secant lines is just an affine plane of order 3, we deduce the following independent construction of $S(5, 6, 12)$.

Theorem 6.2.12 *Let* $\mathsf{AG}(2,3)$ *be the affine plane of order* 3*, let* $\mathscr{M}$ *be its set of lines, and denote by* $\{p_1, p_2, p_3, p_4\}$ *the set of directions (points at infinity). Let* Y *be the set of points of* $\mathsf{AG}(2,3)$ *enriched with three new elements* $\infty_1, \infty_2, \infty_3$ *(so* $|Y| = 12$*). If, for two intersecting lines* L, M *in* $\mathsf{AG}(2,3)$*, we denote by* $L\Delta M = (L \cup M) \setminus (L \cap M)$*, and we denote by* $p(L)$ *the direction of* L*, then we define the following* 6*-subsets of* Y *and call them blocks of* Y*:*
(*i*) $L \cup \{\infty_1, \infty_2, \infty_3\}$, *for every* $L \in \mathscr{M}$*;*
(*ii*) $(L\Delta M) \cup \{\infty_i, \infty_j\}$, *for every intersecting pair* $L, M \in \mathscr{M}$ *such that the sets* $\{p(L), p(M)\}$ *and* $\{p_i, p_j\}$ *either coincide or are disjoint,* $i, j \in \{1, 2, 3\}$*;*
(*iii*) $L \cup M \cup \{\infty_i\}$, *for all intersecting pairs* $L, M \in \mathscr{M}$ *such that either* $\{p(L), p(M), p_i\} = \{p_1, p_2, p_3\}$, *or* $\{p(L), p(M)\} = \{p_i, p_4\}$, $i \in \{1, 2, 3\}$*;*
(*iv*) $L \cup M$, *for disjoint pairs* $L, M \in \mathscr{M}$.
Then Y *endowed with these* 6*-subsets is an* $S(5, 6, 12)$.

Remarks

(1) The sets $L\Delta M$, for intersecting lines in $\mathsf{AG}(2,3)$ can also be defined as conics; the elements ∞_i, $i = 1, 2, 3$, can then be identified with the conjugate pairs of points at infinity in a quadratic extension plane $\mathsf{AG}(2,9)$, and the

equivalence classes are defined by the relation 'having the same points at infinity in $\mathsf{AG}(2,9)$'.

(2) The geometric construction can also be used to prove uniqueness. For example, let us prove uniqueness of $S(3,6,22)$. Since $S(2,5,21)$ is unique as a projective plane of order 4, we may without loss of generality view $S(3,6,22)$ as $\mathsf{PG}(2,4) \cup \{\infty\}$, where the blocks are the lines completed with ∞, and 56 subsets of size 6 in $\mathsf{PG}(2,4)$. These subsets do not intersect any line in at least 3 points, hence they are hyperovals. They do not mutually intersect in 3 points, and by the numbers, all hyperovals intersecting a given one (that is a block of $S(3,6,22)$) in two points, are also blocks of $S(3,6,22)$. It follows that the hyperovals that are blocks exactly constitute one equivalence class. This shows uniqueness. Likewise, uniqueness of $S(4,7,23)$ and $S(5,8,24)$ is shown, as well as uniqueness of $S(3,4,10)$, $S(4,5,11)$ and $S(5,6,12)$.

6.3 Lattices

A *lattice* is a discrete additive subgroup of $\mathbb{R}^n$. (Or, equivalently, a finitely-generated free $\mathbb{Z}$-module with positive definite symmetric bilinear form.)

Determinant

Assume that the lattice Λ has dimension n, i.e., spans $\mathbb{R}^n$. Let $\{a_1, ..., a_n\}$ be a $\mathbb{Z}$-basis of Λ. Let A be the matrix with the vectors a_i as rows. If we choose a different $\mathbb{Z}$-basis $\{b_1, ..., b_n\}$, so that $b_i = \sum s_{ij} a_j$, and B is the matrix with the vectors b_i as rows, then $B = SA$, with $S = (s_{ij})$. Since S is integral and invertible, it has determinant ± 1. It follows that $|\det A|$ is uniquely determined by Λ, independent of the choice of basis.

Volume and Gram matrix

$\mathbb{R}^n/\Lambda$ is an n-dimensional torus, compact with finite volume. Its volume is the volume of the fundamental domain, which equals $|\det A|$.

If Λ' is a sublattice of Λ, then $\mathrm{vol}(\mathbb{R}^n/\Lambda') = \mathrm{vol}(\mathbb{R}^n/\Lambda).|\Lambda/\Lambda'|$.

Let G be the matrix (a_i, a_j) of inner products of basis vectors for a given basis. Then $G = AA^\top$, so $\mathrm{vol}(\mathbb{R}^n/\Lambda) = \sqrt{\det G}$.

Dual lattice

The *dual* Λ^* of a lattice Λ is the lattice of vectors having integral inner products with all vectors in Λ: $\Lambda^* = \{x \in \mathbb{R}^n \mid (x,r) \in \mathbb{Z} \text{ for all } r \in \Lambda\}$.

It has a basis $\{a_1^*, ..., a_n^*\}$ defined by $(a_i^*, a_j) = \delta_{ij}$. Now $A^* A^\top = I$, so $A^* = (A^{-1})^\top$ and Λ^* has Gram matrix $G^* = G^{-1}$.

It follows that $\mathrm{vol}(\mathbb{R}^n/\Lambda^*) = 1/\mathrm{vol}(\mathbb{R}^n/\Lambda)$. We have $\Lambda^{**} = \Lambda$.

Integral lattice

The lattice Λ is called *integral* when the inner products of lattice vectors are all integral. For an integral lattice Λ one has $\Lambda \subseteq \Lambda^*$.

The lattice Λ is called *even* when (x,x) is an even integer for each $x \in \Lambda$. An even lattice is integral. An integral lattice that is not even is called *odd*.

Roots are lattice vectors x with $(x, x) = 2$.

Unimodular lattice

The *discriminant* (or *determinant*) $\operatorname{disc} \Lambda$ of a lattice Λ is defined by $\operatorname{disc} \Lambda = \det G$. When Λ is integral, we have $\operatorname{disc} \Lambda = |\Lambda^*/\Lambda|$.

A lattice is called *self-dual* or *unimodular* when $\Lambda = \Lambda^*$, i.e., when it is integral with discriminant 1. An even unimodular lattice is called *Type II*, the remaining unimodular lattices are called *Type I*.

If there is an even unimodular lattice in $\mathbb{R}^n$, then n is divisible by 8. (This follows by studying the associated theta series and modular forms.)

6.3.1 The Leech lattice

The *Leech lattice* Λ is the unique even unimodular lattice in $\mathbb{R}^{24}$ without roots. For lots of information, see CONWAY & SLOANE [217].

Theorem 6.3.1 (CONWAY [212]) *There exists a unique even unimodular lattice without roots in* $\mathbb{R}^{24}$. *It has* 196560 *vectors of norm (squared length)* 4.

Proof (very brief sketch). For the construction, take the lattice spanned by the vectors $\frac{1}{\sqrt{8}}(\mp 3, \pm 1^{23})$ with ∓ 3 in any position, and the upper signs in a code word of the extended binary Golay code.

For the vectors of norm 4 one finds the shapes $4^2\,0^{22}$, $3\,1^{23}$, $2^8\,0^{16}$ (omitting the $\frac{1}{\sqrt{8}}$) with frequencies $2^2\binom{24}{2} = 1104$, $2^{12} \cdot 24 = 98304$ and $2^7 \cdot 759 = 97152$, respectively.

Uniqueness is proved using theta functions and the theory of modular forms.

Given a lattice Λ, define

$$\theta_\Lambda(z) = \sum_{x \in \Lambda} q^{\frac{1}{2}(x,x)}$$

where $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$.

One has

$$\theta_{\Lambda^*}(z) = \det(\Lambda)^{\frac{1}{2}} \left(\frac{i}{z}\right)^{\frac{n}{2}} \theta_\Lambda(-\frac{1}{z}).$$

For the Leech lattice one has $\Lambda = \Lambda^*$ and $\det(\Lambda) = 1$, so that $\theta_\Lambda(z)$ is a modular form of weight 12.

The space of modular forms of weight 12 has dimension 2, and the two conditions: unique vector of norm 0, no vectors of norm 2, determine $\theta_\Lambda(z)$ uniquely. Thus, any even unimodular lattice without roots in $\mathbb{R}^{24}$ must have the same weight enumerator as the Leech lattice.

Some more work gives the desired conclusion. □

One can replace the requirement 'unimodular' by giving three counts.

Proposition 6.3.2 ([735], Theorem 5.1) *Let* Λ *be an even integral lattice in* $\mathbb{R}^{24}$ *with* a_i *vectors of squared norm* i, *where* $a_2 = 0$, $a_4 = 196560$, $a_6 = 16773120$, $a_8 = 398034000$. *Then* Λ *is isomorphic to the Leech lattice.* □

The automorphism group of the Leech lattice (fixing the zero vector) is $2.\mathsf{Co}_1$ of order $2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$. It is transitive on the vectors of squared norm 4 and on those of squared norm 6. The stabilizer of a vector of squared norm 4 is Co_2 of order $2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. The stabilizer of a vector of squared norm 6 is Co_3 of order $2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$.

6.3.2 The mod 2 Leech lattice

Let V be the $\mathbb{F}_2^{24}$ obtained as $\Lambda/2\Lambda$. The $2^{24} = 1 + \frac{1}{2}a_4 + \frac{1}{2}a_6 + \frac{1}{48}a_8$ vectors of V each have a representative of squared norm at most 8, unique up to sign when it has squared norm less than 8, while vectors in Λ of squared norm 8 fall into classes of 48 congruent mod 2Λ ([217], p. 332).

Let X be the image in V of the set of vectors of squared norm 4. Then $|X| = 98280$ and each hyperplane of $\mathsf{P}V$ meets X in either 49128 or 51176 points. We find a rank 4 strongly regular graph with parameters $(v, k, \lambda, \mu) = (16777216, 98280, 4600, 552)$ with group $2^{24}.\mathsf{Co}_1$ ([129], [627]).

6.3.3 The complex Leech lattice

Let $\theta = \sqrt{-3}$ and $\omega = (-1 + \sqrt{-3})/2$, so that $\omega^3 = 1$ and $\theta = \omega - \bar{\omega}$ is a prime in $\mathbb{Z}[\omega]$. Let C be the extended ternary Golay code (as subset of $\{-1, 0, 1\}^{12}$). The *complex Leech lattice* is the lattice L in $\mathbb{Z}[\omega]^{12}$ consisting of the vectors

$$\mathbf{0} + \theta c + 3x, \quad \mathbf{1} + \theta c + 3y, \quad -\mathbf{1} + \theta c + 3z$$

with $c \in C$, $x, y, z \in \mathbb{Z}[\omega]^{12}$, and $\sum x_i \equiv 0$, $\sum y_i \equiv 1$, $\sum z_i \equiv -1 \pmod{\theta}$.

Now L is a lattice, with minimal squared norm 18.

If we view L as 24-dimensional real lattice, and scale by a factor $\frac{1}{3}\sqrt{2}$, we get the Leech lattice. (For example, by Proposition 6.3.2.)

See also [522], [217] (pp. 200, 293), [734], [735] (§5.6.10).

The automorphism group of L is $6.\mathsf{Suz}$ of order $2^{14} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. The central 6 arises from the scalars $(-\omega)^i$. The quotient $L/\theta L$ is isomorphic to 3^{12}.

Let V be $\mathbb{F}_3^{12}$ obtained as $L/\theta L$. Then $2.\mathsf{Suz}$ acts on V, and Suz has precisely two orbits on $\mathsf{P}V$, of sizes 32760 and 232960, respectively. This leads to a rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (531441, 65520, 8559, 8010)$ and automorphism group $3^{12}.2.\mathsf{Suz}.2$. See also §10.100 and Table 11.6.

Chapter 7

Cyclotomic constructions

We look at graphs defined by a difference set in a usually abelian group. Difference sets in a vector space that are invariant under multiplication by scalars are equivalent to two-weight codes and to two-character subsets of a projective space.

7.1 Difference sets

Given an abelian group G and a subset D of G such that $D = -D$ and $0 \notin D$, we can define a graph Γ with vertex set G by letting $x \sim y$ whenever $y - x \in D$. This graph is known as the *Cayley graph* on G with *difference set* D.[1]

If A is the adjacency matrix of Γ, and χ is a character of G, then $(A\chi)(x) = \sum_{y \sim x} \chi(y) = \sum_{d \in D} \chi(x + d) = (\sum_{d \in D} \chi(d))\chi(x)$. It follows that the spectrum of Γ consists of the numbers $\sum_{d \in D} \chi(d)$, where χ runs through the characters of G. In particular, the trivial character χ_0 yields the eigenvalue $|D|$, the valency of Γ.

7.1.1 Two-character projective sets

Let V be a vector space of dimension m over the finite field $\mathbb{F}_q$. Let X be a subset of size n of the point set of the projective space $\mathsf{P}V$. Define a graph Γ with vertex set V by letting $x \sim y$ whenever $\langle y - x \rangle \in X$. This graph has $v = q^m$ vertices, and is regular of valency $k = (q-1)n$. It is the Cayley graph on V with difference set $D = \{x \in V \mid \langle x \rangle \in X\}$.

Let q be a power of the prime p, let $\zeta = e^{2\pi i/p}$ be a primitive p-th root of unity, and let $\mathrm{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$ be the trace function. Let V^* be the dual vector space to V, that is the space of linear forms on V. Then the characters χ are of the form $\chi_a(x) = \zeta^{\mathrm{tr}(a(x))}$, with $a \in V^*$. Now

$$\sum_{\lambda \in \mathbb{F}_q} \chi_a(\lambda x) = \begin{cases} q & \text{if } a(x) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

[1] About the terminology: in the area of design theory a difference set D in a group G is a set such that $\{gD \mid g \in G\}$ is a symmetric (i.e., square) design. A *partial difference set* is a set such that the Cayley graph for this difference set is a strongly regular graph.

Hence Γ has the eigenvalues $(q-1)|X|$ and $\sum_{d\in D}\chi_a(d) = q.|H_a\cap X|-|X|$ (for $a \neq 0$), where H_a is the hyperplane $\{\langle x\rangle \mid a(x)=0\}$ in $\mathsf{P}V$. Consequently, Γ will be strongly regular precisely when $|H_a\cap X|$ takes only two different values.

The above construction of the graph Γ is often described as 'take the vector space V with the subset X of PV *at infinity*'.

7.1.2 Projective two-weight codes

This can be formulated in terms of coding theory (Delsarte [275]). To the set X corresponds a linear code C of word length n and dimension m. Each $a \in V^*$ gives rise to the vector $(a(x))_{x\in X}$ indexed by X, and the collection of all these vectors is the code C.[2,3] A code word a of weight w corresponds to a hyperplane H_a that meets X in $n-w$ points, and hence to an eigenvalue $q(n-w)-n = k-qw$ of Γ.

If X is a *two-character set*, that is, if the size of hyperplane intersections $H\cap X$ takes only two different values, then C is a *two-weight code*, that is, the weight $\mathrm{wt}(c)$ of nonzero code words $c \in C$ takes only two different values.

A survey of two-weight codes was given by Calderbank & Kantor [169]. Additional families and examples were given in [112], [284], [283], [53], [511], [287], [230], [235], [228], [288], [229], [627].

The code C obtained above is called *projective*: no two coordinate positions are dependent. That is, the dual code $C^\perp$ has minimum distance at least 3. The more general case of a code C with dual $C^\perp$ of minimum distance at least 2 corresponds to a multiset X. Brouwer & van Eupen [127] gives a 1-1 correspondence between arbitrary projective codes and two-weight codes.

7.1.3 Delsarte duality

Suppose X is a subset of the point set of $\mathsf{P}V$ that meets hyperplanes in either n_1 or n_2 points. We find a subset Y of the point set of the dual space $\mathsf{P}V^*$ consisting of the hyperplanes that meet X in n_1 points. Also Y is a two-character set. If each point of $\mathsf{P}V$ is on n_1' or n_2' hyperplanes in Y, then $(n_1-n_2)(n_1'-n_2') = q^{m-2}$. It follows that the difference of the weights in a projective two-weight code is a power of the characteristic. (This is a special case of the duality for translation association schemes. See [276], §2.6, and [123], §2.10B.)

A strongly regular graph invariant for a regular abelian translation group is called *self-dual* when it is isomorphic to its dual, and *formally self-dual* when it has the same parameters as its dual (so that $\{k,l\} = \{f,g\}$). For formally self-dual graphs/codes, $w_2 - w_1 = n_1 - n_2 = q^{\frac{1}{2}m-1}$. This is the most common situation. Different examples are for example i-subspaces of $\mathsf{P}V$ (with $n_1 - n_2 = q^{i-1}$) or the third De Lange set (cf. §7.3.3 below), which can be seen as a 39-set in $\mathsf{PG}(3,8)$ such that all planes meet it in either 3 or 7 points, so that $(q, n_1-n_2) = (8,4)$.

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[2] More precisely, each $a \in V^*$ gives rise to the vector $(a(u_x))_{x\in X}$ indexed by X, where u_x is some fixed vector in V spanning the projective point $x = \langle u_x\rangle$. Different choices for these representatives u_x yield equivalent codes.

[3] More precisely, the dimension of C is the dimension of the span $\langle X\rangle$ of X.

7.1.4 Parameters

Let V be a vector space of dimension m over $\mathbb{F}_q$. Let X be a subset of size n of its hyperplane at infinity $\mathsf{P}V$. Construct the graph Γ by taking V as vertex set, where two vertices u, v are adjacent when $\langle v - u \rangle \in X$. This graph has $v = q^m$ vertices, and is regular of valency $k = (q-1)n$. As we saw, the other eigenvalues are $q|H \cap X| - n$ where H runs through the hyperplanes of $\mathsf{P}V$.

We obtain a strongly regular graph when $|H \cap X|$ takes precisely two values, say n_1 and n_2, with $n_1 > n_2$. Let f_1 and f_2 be the number of hyperplanes meeting X in n_1 and n_2 points, respectively. Then f_1 and f_2 satisfy

$$f_1 + f_2 = \frac{q^m - 1}{q - 1},$$
$$f_1 n_1 + f_2 n_2 = n\frac{q^{m-1} - 1}{q - 1},$$
$$f_1 n_1(n_1 - 1) + f_2 n_2(n_2 - 1) = n(n-1)\frac{q^{m-2} - 1}{q - 1}$$

and it follows that

$$(q^m - 1)n_1 n_2 - n(q^{m-1} - 1)(n_1 + n_2 - 1) + n(n-1)(q^{m-2} - 1) = 0,$$

so that in particular $n \,|\, (q^m - 1)n_1 n_2$.

The strongly regular graph Γ has parameters

$$\begin{aligned} v &= q^m, & r &= qn_1 - n,\\ k &= (q-1)n, & s &= qn_2 - n,\\ \lambda &= \mu + r + s, & f &= (q-1)f_1,\\ \mu &= k + rs = \frac{(n-n_1)(n-n_2)}{q^{m-2}}, & g &= (q-1)f_2. \end{aligned}$$

If X spans $\mathsf{P}V$, then the code C constructed above has parameters $[n, m, w_1]_q$ and weight enumerator $1 + fx^{w_1} + gx^{w_2}$, where $w_1 = n - n_1$, $w_2 = n - n_2$, and

$$f = \frac{1}{w_2 - w_1}(w_2(q^m - 1) - nq^{m-1}(q-1)).$$

7.1.5 Complements and imprimitivity

If Γ is the graph corresponding to the subset X of $\mathsf{P}V$, then $\overline{\Gamma}$ corresponds to the complementary subset $\overline{X} = \mathsf{P}V \setminus X$. For the parameters n, n_1, n_2 we find $\overline{n} = \frac{q^m-1}{q-1} - n$, $\overline{n_i} = \frac{q^{m-1}-1}{q-1} - n_j$, so that $\overline{w_i} = q^{m-1} - w_j$, where $\{i, j\} = \{1, 2\}$.

The graph Γ is disconnected if and only if X is a proper subspace of $\mathsf{P}V$. In particular, the code C has dimension m precisely when Γ is connected.

7.1.6 Divisibility

From $(n_1 - n_2)(n_1' - n_2') = q^{m-2}$ and $f_1(n_1 - n_2) = n\frac{q^{m-1}-1}{q-1} - n_2\frac{q^m-1}{q-1}$, it follows that $(n_1 - n_2) \,|\, q^{m-2}$ and $(n_1 - n_2) \,|\, (n - n_2)$, so that w_1 and w_2 are divisible by $w_2 - w_1$.

Now w_1 and w_2 are divisible by p, except perhaps when $n_1 - n_2 = 1$. If $m \geq 3$, this latter case occurs only when X is a point or the complement of a point, so that $n = 1$, $\mu = 0$ or $n = \frac{q^m-1}{q-1} - 1$, $k = \mu$ ([103]).

(Indeed, since q divides $\mu q^{m-2} = w_1 w_2$, it must divide one of w_1, w_2, if the other does not have a factor p. Let A be an $(m-2)$-space, and count the g_i hyperplanes on A meeting X in n_i points $(i = 1, 2)$. From $g_1 + g_2 = q+1$ and $(n_1 - a)g_1 + (n_2 - a)g_2 = n - a$, where $a = |A \cap X|$, we see $g_1 = (n_1 - n_2)g_1 = (n-a) - (n_2 - a)(q+1) = w_2 - q(n_2 - a)$ and $g_2 = -w_1 + q(n_1 - a)$. Let $\{i, j\} = \{1, 2\}$, where $q|w_i$. Then $q|g_j$, so that $g_j \in \{0, q\}$ and $a \in \{n_1 - \frac{w_i}{q}, n_2 - \frac{w_i}{q}\}$. If $m > 3$, then we are done by induction on m. If $m = 3$, then A is a single point, so $a \in \{0, 1\}$ and $n_2 = \frac{n-n_i}{q}$. If $i = 1$, then $n - 1 = (q+1)n_2$ so that all lines on a point of X meet X in n_1 points, and $n_2 = 0$, X is a single point. If $i = 2$, then $n = (q+1)n_2$ so that all lines on a point outside X meet X in n_2 points, and $n_1 = q+1$, X is the complement of a point.)

7.1.7 Field change

If $q = r^e$, then from an $[n, k]_q$ code we find a $[\frac{q-1}{r-1}n, ke]_r$ code by choosing a basis of $\mathbb{F}_q$ over $\mathbb{F}_r$. To weights w of the q-ary code there correspond weights $\frac{q}{r}w$ of the r-ary code. The corresponding graphs are the same.

7.1.8 Unions and differences

Let Z be an arbitrary subset of $\mathsf{PG}(m-1, q)$ with hyperplane intersections of size n_i for f_i hyperplanes. Then, as above, $\sum f_i = \frac{q^m-1}{q-1}$, and $\sum f_i n_i = n\frac{q^{m-1}-1}{q-1}$, and $\sum f_i n_i(n_i - 1) = n(n-1)\frac{q^{m-2}-1}{q-1}$. When at most three distinct n_i occur, the f_i are determined (since the coefficient determinant is nonzero), and we can conclude that Z is in fact a two-character set when one of these f_i vanishes.

Consider the situation where X and Y are disjoint, and $|X \cap H| \in \{a, a+d\}$ and $|Y \cap H| \in \{b, b+d\}$ for all hyperplanes H. Put $c = a+b$. Then $|(X \cup Y) \cap H| \in \{c, c+d, c+2d\}$ for all hyperplanes H, and we can read off from the parameters whether $c + 2d$ actually occurs. If $|X \cup Y| = n$, then $c + 2d$ does not occur precisely when $c(c+d)\frac{q^m-1}{q-1} - (2c - 1 + d)n\frac{q^{m-1}-1}{q-1} + n(n-1)\frac{q^{m-2}-1}{q-1} = 0$.

Let $\mathscr{F} = \mathscr{F}(\alpha, d, m, q)$ be the collection of two-character sets X in $\mathsf{PG}(m-1, q)$ with hyperplane intersection sizes $\alpha|X|$ and $\alpha|X| + d$, where d may be negative. If

$$\alpha^2(q^m - 1) - 2\alpha(q^{m-1} - 1) + (q^{m-2} - 1) = 0,$$

then $\mathscr{F}$ is closed under disjoint unions and under taking differences $X \setminus Y$ when $Y \subseteq X$. For example, if m is even, then $\frac{1}{2}m$-subspaces and hyperbolic quadrics belong to the same collection $\mathscr{F}$, and we find the examples under C below.

7.1.9 Geometric examples

We give some examples of two-character sets in projective spaces $\mathsf{P}V$, where V is an m-dimensional vector space over $\mathbb{F}_q$.

A. Subspaces

Let W be an i-dimensional subspace of V, where $0 < i < m$. Then $X = \mathsf{P}W$ is a two-character set of size $n = \frac{q^i-1}{q-1}$ with hyperplane intersection sizes $n_1 = \frac{q^i-1}{q-1}$ and $n_2 = \frac{q^{i-1}-1}{q-1}$, so that $n_1 - n_2 = q^{i-1}$.

B. Partial spreads

For $m = 2d$, let X be the union of t pairwise disjoint d-subspaces of $\mathsf{P}V$ ($1 \le t \le q^d$). Then X is a two-character set of size $n = t\frac{q^d-1}{q-1}$ with hyperplane intersection sizes $n_1 = q^{d-1} + n_2$ and $n_2 = t\frac{q^{d-1}-1}{q-1}$, so that $n_1 - n_2 = q^{d-1}$.

C. Quadrics

For $m = 2d$, let X be the point set of a nondegenerate hyperbolic ($\varepsilon = 1$) or elliptic ($\varepsilon = -1$) quadric. Then X has size $n = \frac{q^{2d-1}-1}{q-1} + \varepsilon q^{d-1}$ with hyperplane intersection sizes $\{n_1, n_2\} = \{\frac{q^{2d-2}-1}{q-1}, \frac{q^{2d-2}-1}{q-1} + \varepsilon q^{d-1}\}$, so that $n_1 - n_2 = q^{d-1}$. The corresponding graphs are the affine polar graphs $VO^{\varepsilon}(m, q)$.

For $\varepsilon = 1$, this example has the same parameters as the partial spread construction (Ex. B) with $t = q^{d-1} + 1$. Since the union condition is satisfied one can take (for $m = 2d$) the disjoint union of pairwise disjoint d-spaces and nondegenerate hyperbolic quadrics, where possibly a number of pairwise disjoint d-spaces contained in some of the hyperbolic quadrics is removed ([134]).

Also for $\varepsilon = -1$ the union condition is satisfied. In particular, if $m = 4$, one can take the disjoint union of pairwise disjoint nondegenerate elliptic quadrics (or arbitrary ovoids). Since $\mathsf{PG}(3, q)$ has a partition into $q + 1$ ovoids, this gives two-character sets with intersection numbers $n_1 = j(q + 1)$, $n_2 = n_1 - q$ for $1 \le j \le q$.

D. Nonisotropic points

For odd q and $m = 2d$, consider a nondegenerate quadric Q of type $\varepsilon = \pm 1$ in V. Let X be the set of nonisotropic projective points $x = \langle v \rangle$ where $Q(v)$ is a nonzero square. Then X has size $n = \frac{1}{2}(q^{2d-1} - \varepsilon q^{d-1})$ and $n_1, n_2 = \frac{1}{2}q^{d-1}(q^{d-1} \pm 1)$ (independent of ε), so that $n_1 - n_2 = q^{d-1}$. The corresponding graphs are the affine nonisotropics graphs $VNO^{\varepsilon}(m, q)$.

E. Quadric minus quadric over overfield

Let $r = q^e$ where $e > 1$, and write $F_1 = \mathbb{F}_r$, $F = \mathbb{F}_q$. Let V_1 be a vector space of dimension d over F_1, where d is even, and write V for V_1 regarded as a vector space of dimension de over F. Let $\mathrm{tr}\colon F_1 \to F$ be the trace map. Let $Q_1\colon V_1 \to F_1$ be a nondegenerate quadratic form on V_1. Then $Q = \mathrm{tr} \circ Q_1$ is a nondegenerate quadratic form on V. Let $X = \{x \in PV \mid Q(x) = 0 \text{ and } Q_1(x) \ne 0\}$. Write $\varepsilon = 1$ ($\varepsilon = -1$) if Q is hyperbolic (elliptic). The set X is a two-character set in $\mathsf{P}V$, has size $n = \frac{q^{e-1}-1}{q-1}(q^{de-e} - \varepsilon q^{de/2-e})$, and hyperplane intersection sizes $\{n_1, n_2\} = \{a, a + \varepsilon q^{de/2-1}\}$, with $a = \frac{q^{e-1}-1}{q-1}(q^{de-e-1} - \varepsilon q^{de/2-e})$, so that $n_1 - n_2 = q^{de/2-1}$ (Brouwer [112]).

For example, when $q = e = 2$, $d = 4$, $\varepsilon = -1$, this yields a 68-set in $\mathsf{PG}(7, 2)$ with hyperplane intersections of sizes 28 and 36. This construction was generalized in Hamilton [410].

F. Hermitian quadrics

Let $q = r^2$ and let V be provided with a nondegenerate Hermitian form. Let X be the set of isotropic projective points. Then X has size $n = (r^m - \varepsilon)(r^{m-1} + \varepsilon)/(q-1)$ where $\varepsilon = (-1)^m$, and $n - n_2 = r^{2m-3}$, $n_1 - n_2 = r^{m-2}$.

If we view V as a vector space of dimension $2m$ over $\mathbb{F}_r$, the same set X now has $n = (r^m - \varepsilon)(r^{m-1} + \varepsilon)/(r-1)$, $n - n_2 = r^{2m-2}$, $n_1 - n_2 = r^{m-1}$, as expected, since the form is a nondegenerate quadratic form in $2m$ dimensions over $\mathbb{F}_r$. Thus, the graphs that one gets here are also graphs one gets from quadratic forms, but the codes here are defined over a larger field.

G. Baer subspaces

Let $q = r^2$ and let m be odd. Then $\mathsf{PG}(m-1, q)$ has a partition into pairwise disjoint Baer subspaces $\mathsf{PG}(m-1, r)$. Each hyperplane hits all of these in a $\mathsf{PG}(m-3, r)$, except for one which is hit in a $\mathsf{PG}(m-2, r)$. Let X be the union of u such Baer subspaces, $1 \le u < \frac{r^m+1}{r+1}$. Then $n = |X| = u\frac{r^m-1}{r-1}$, $n_1 = n_2 + r^{m-2}$ and $n_2 = u\frac{r^{m-2}-1}{r-1}$, so that $n_1 - n_2 = r^{m-2}$.

H. Maximal arcs and hyperovals

A *maximal arc* in a projective plane $\mathsf{PG}(2, q)$ is a two-character set with intersection numbers $n_1 = a$, $n_2 = 0$, for some constant a $(1 < a < q)$. Clearly, maximal arcs have size $n = qa - q + a$, and necessarily $a \,|\, q$. For $a = 2$ these objects are called *hyperovals*, and exist for all even q. Denniston [281] constructed maximal arcs for all even q and all divisors a of q. Ball, Blokhuis & Mazzocca [35] showed that there are no maximal arcs in $\mathsf{PG}(2, q)$ when q is odd.

These arcs show that the difference between the intersection numbers need not be a power of q.

I. Two-character subsets of the plane

Penttila & Royle [616] determined all two-character sets in each of the four projective planes of order 9. They say that a two-character set in a projective plane has *standard parameters* when q is a square and $n_1 - n_2 = \sqrt{q}$. (It follows that the set has size $n = n_2(q + \sqrt{q} + 1)$ or $n = n_1(q - \sqrt{q} + 1)$.) For $q = 9$ only standard parameters are feasible and the number of nonisomorphic examples in $\mathsf{PG}(2, 9)$ is given in the table below.

n	n_2	n_1	#	comments
13	1	4	1	Baer subplane
28	1	4	2	unital
26	2	5	3	e.g., union of two Baer subplanes
35	2	5	7	sporadic
39	3	6	22	e.g., union of three Baer subplanes
42	3	6	6	sporadic

J. Caps

The dual code $C^{\perp}$ has minimum distance at least 4 if and only if X is a *cap*, that is, does not have three collinear points.

Characterizing two-weight projective codes C with dual distance (minimum distance of $C^{\perp}$) at least 4 is equivalent to characterizing two-character projective sets that are caps. There are strong parameter conditions, and Calderbank, Beukers, Bremner and others solved the corresponding Diophantine equations in a series of papers [164], [64], [106], [107], [705]. The final result was:

Theorem 7.1.1 (Tzanakis & Wolfskill [706]) *Let C be a q-ary two-weight $[n, m]$-code with weights w_1, w_2 and dual distance at least 4. Then we have one of the cases in Table 7.1 below.*

q	m	n	w_1	w_2	comment
q	2	2	1	2	two points
2^e	3	$q+2$	q	$q+2$	hyperoval
q	4	q^2+1	q^2-q	q^2	ovoid
3	5	11	6	9	ternary Golay code
3	6	56	36	45	Hill [427]
4	6	78	56	64	Hill [428]
4	7	430	320	352	unknown
2	m	2^{m-1}	2^{m-2}	2^{m-1}	hyperplane complement

Table 7.1: Two-character sets that are caps

This table with examples already occurs in [332], p. 72.

7.1.10 Small two-weight codes

For $m = 2$ any subset of $\mathsf{PG}(m-1, q)$ is met by any hyperplane in either 0 or 1 points. One finds q-ary projective two-weight codes $[n, 2]_q$ with weights $n-1$ and n for $2 \le n \le q+1$, and primitive strongly regular graphs with parameters $\mathrm{LS}_n(q)$ (cf. §8.4.2) for $2 \le n \le q-1$. For $n = 2$ these are the grid graphs $q \times q$.

Bouyukliev, Fack, Willems & Winne [103] enumerated the two-weight codes with $m \ge 3$, $q \le 4$, $n \le 68$ or $m = 4$, $q = 5$, $n \le 39$ (and also give the automorphism group sizes). In the table below, the codes are $[n, m, w_1]_q$ codes. The weight enumerators are $1 + f_1 z^{w_1} + f_2 z^{w_2}$. The column # gives the number of nonequivalent such codes. The corresponding strongly regular graphs have the parameters given above. In particular, $v = q^m$ and $k = (q-1)n$ and $\mu = w_1 w_2 / q^{m-2}$.

q	m	n	wt. enum.	#	v	k	λ	μ	example
2	4	5	$1+10z^2+5z^4$	1	16	5	0	2	$VO_4^-(2)$
2	4	6	$1+6z^2+9z^4$	1	16	6	2	2	$VO_4^+(2)$
2	6	14	$1+14z^4+49z^8$	1	64	14	6	2	$q=8$
2	6	18	$1+45z^8+18z^{12}$	1	64	18	2	6	$q=4$
2	6	21	$1+21z^8+42z^{12}$	2	64	21	8	6	$H_2(2,3)$
2	6	27	$1+36z^{12}+27z^{16}$	5	64	27	10	12	$VO_6^-(2)$
2	6	28	$1+28z^{12}+35z^{16}$	7	64	28	12	12	$VO_6^+(2)$
2	8	30	$1+30z^8+225z^{16}$	1	256	30	14	2	$q=4$
2	8	45	$1+45z^{16}+210z^{24}$	2	256	45	16	6	$H_2(2,4)$

continued...

q	m	n	wt. enum.	#	v	k	λ	μ	example
2	8	51	$1+204z^{24}+51z^{32}$	1	256	51	2	12	$q=4$
2	8	60	$1+60z^{24}+195z^{32}$	12	256	60	20	12	$q=4$
2	8	68	$1+187z^{32}+68z^{40}$	41	256	68	12	20	$VO_8^-(2)\setminus VO_4^-(4)$
3	4	8	$1+16z^3+64z^6$	1	81	16	7	2	two skew lines
3	4	10	$1+60z^6+20z^9$	1	81	20	1	6	$VO_4^-(3)$
3	4	12	$1+24z^6+56z^9$	2	81	24	9	6	$VNO_4^+(3)$
3	4	15	$1+50z^9+30z^{12}$	2	81	30	9	12	$VNO_4^-(3)$
3	4	16	$1+32z^9+48z^{12}$	4	81	32	13	12	$VO_4^+(3)$
3	4	20	$1+40z^{12}+40z^{15}$	4	81	40	19	20	five skew lines
3	5	11	$1+132z^6+110z^9$	1	243	22	1	2	dual ternary Golay
3	5	55	$1+220z^{36}+22z^{45}$	1	243	110	37	60	its Delsarte dual
3	6	56	$1+616z^{36}+112z^{45}$	1	729	112	1	20	Hill cap
4	3	6	$1+45z^4+18z^6$	1	64	18	2	6	hyperoval
4	3	7	$1+21z^4+42z^6$	1	64	21	8	6	Baer subplane
4	3	9	$1+36z^6+27z^8$	1	64	27	10	12	unital
4	4	10	$1+30z^4+225z^8$	1	256	30	14	2	two skew lines
4	4	15	$1+45z^8+210z^{12}$	2	256	45	16	6	three skew lines
4	4	17	$1+204z^{12}+51z^{16}$	1	256	51	2	12	$VO_4^-(4)$
4	4	20	$1+60z^{12}+195z^{16}$	7	256	60	20	12	four skew lines
4	4	25	$1+75z^{16}+180z^{20}$	19	256	75	26	20	$VO_4^+(4)$
4	4	30	$1+90z^{20}+165z^{24}$	68	256	90	34	30	six skew lines
4	4	34	$1+153z^{24}+102z^{28}$	84	256	102	38	42	two ovoids
4	4	35	$1+105z^{24}+150z^{28}$	231	256	105	44	42	seven skew lines
4	4	40	$1+120z^{28}+135z^{32}$	481	256	120	56	56	eight skew lines
5	4	12	$1+48z^5+576z^{10}$	1	625	48	23	2	two skew lines
5	4	18	$1+72z^{10}+552z^{15}$	1	625	72	25	6	three skew lines
5	4	24	$1+96z^{15}+528z^{20}$	7	625	96	29	12	four skew lines
5	4	26	$1+520z^{20}+104z^{25}$	1	625	104	3	20	$VO_4^-(5)$
5	4	30	$1+120z^{20}+504z^{25}$	38	625	120	35	20	five skew lines
5	4	36	$1+144z^{25}+480z^{30}$	547†	625	144	43	30	$VO_4^+(5)$
5	4	39	$1+468z^{30}+156z^{35}$	8	625	156	29	42	[312], [104]

Table 7.2: Small two-weight codes and graphs

Minihypers and the Griesmer bound

Part of the literature in this area is formulated in terms of 'minihypers'. A subset X of $\mathsf{PG}(m-1,q)$ is called an $\{n,c;\ m-1,q\}$-*minihyper* if $|X|=n$ and $|X\cap H|\geq c$ for each hyperplane H, with equality for at least one hyperplane.[5] In the above we have been looking at $\{n,n_2;\ m-1,q\}$-minihypers.

Put $v_i=\frac{q^i-1}{q-1}$. If X is the disjoint union of e_0 points (1-spaces), e_1 lines (2-spaces), ..., then X is a $\{\sum_{i=0}^{m-2}e_iv_{i+1},\ \sum_{i=0}^{m-2}e_iv_i;\ m-1,q\}$-minihyper. Many classification theorems for minihypers give sufficient conditions for a minihyper X to be such a union. See, e.g., [406], [407], [361], [673].

The *Griesmer bound* on the length of an $[n,m,d]_q$ code says that $n\geq\sum_{i=0}^{m-1}\lceil\frac{d}{q^i}\rceil$. Suppose $1\leq d\leq q^{m-1}$. Then one can uniquely write $d=q^{m-1}-\sum_{i=0}^{m-2}e_iq^i$ with $0\leq e_i\leq q-1$ for all i. Hamada [404, 405] showed that the $[n,m,d]_q$ codes with equality in the Griesmer bound are precisely the codes that correspond to $\mathsf{P}V\setminus X$, where X is a $\{\sum_{i=0}^{m-2}e_iv_{i+1},\ \sum_{i=0}^{m-2}e_iv_i;\ m-1,q\}$-minihyper.

†Iliya Bouyukliev, pers. comm.

[5] The word 'minihyper' is supposed to suggest 'with prescribed minimal size for hyperplane intersections'. Early publications also used 'min·hyper'.

7.1.11 Sporadic two-weight codes

Most known examples of projective two-weight codes arise from well-known geometric objects, and come in infinite families. Below a table with some sporadic two-weight codes and corresponding graphs.

q	m	n	w_1	w_2-w_1	comments
2	9	73	32	8	Fiedler & Klin [326]; [496]
2	9	219	96	16	Delsarte dual of previous
2	10	198	96	16	Kohnert [496]
2	11	276	128	16	$2^{11}.\mathsf{M}_{24}$, see §10.84
2	11	759	352	32	Delsarte dual of previous; [355]
2	12	$65i$	$32i$	32	Kohnert [496] $(12 \le i \le 31, i \ne 19)$
2	24	98280	47104	2048	Rodrigues [627], see §6.3.2
4	5	$11i$	$8i$	8	Dissett [292] $(7 \le i \le 14, i \ne 8)$
4	6	78	56	8	Hill [428]
4	6	429	320	32	Delsarte dual of previous
4	6	147	96	16	[112]; Cossidente et al. [228]
4	6	210	144	16	Cossidente et al. [228]
4	6	273	192	16	Ex. B; De Wispelaere & Van Maldeghem [287]
4	6	315	224	16	[112]; Cossidente et al. [228]
4	6	525	384	16	Liebeck [517] $2^{12}.\mathsf{HJ}$, see §10.92
4	6	585	432	16	Chen quasi-twisted
8	4	39	32	4	De Lange [510]
8	4	273	224	16	Delsarte dual of previous
16	3	78	72	4	De Resmini & Migliori [284]
3	5	11	6	3	dual of the ternary Golay code
3	5	55	36	9	Delsarte dual of previous
3	6	56	36	9	Games graph (see §10.75), Hill cap [427]
3	6	84	54	9	Gulliver [369]; [540]
3	6	98	63	9	Gulliver [369]; [540]
3	6	154	99	9	Van Eupen [310]; [370]
3	8	$82i$	$54i$	27	Kohnert [496] $(8 \le i \le 12)$
3	8	$41i$	$27i$	27	Kohnert [496] $(26 \le i \le 39)$
3	8	1435	945	27	De Lange [510]
3	12	7592	5022	81	Schmidt & White [637]
3	12	32760	21627	243	$3^{12}.2.\mathsf{Suz}.2$, see §10.100
9	3	35	30	3	De Resmini [283]
9	3	42	36	3	Penttila & Royle [616]
9	4	287	252	9	De Lange [510]
81	3	3285	3240	9	Lane-Harvard & Penttila [509]
5	4	39	30	5	Dissett [292]; [103]
5	6	1890	1500	25	Liebeck [517] $5^6.4.\mathsf{HJ}$, see §10.95
25	3	$21i$	$20i$	5	Lane-Harvard & Penttila [509] $(i = 10\text{–}12,15)$
125	3	829	820	5	Batten & Dover [53]
125	3	7461	7400	25	Delsarte dual of previous
343	3	3189	3178	7	Batten & Dover [53]
343	3	28701	28616	49	Delsarte dual of previous
13	4	595	546	13	Chen quasi-twisted

Table 7.3: Sporadic two-weight codes and graphs

7.2 Cyclic codes

An $[n,k]_q$ code is a linear code of length n and dimension k over the field $\mathbb{F}_q$. Its size is q^k. This code is *cyclic* if it is invariant under the map $(c_1, c_2, \ldots, c_n) \mapsto (c_n, c_1, \ldots, c_{n-1})$ that cyclically permutes the coordinate positions. Let x be a variable, and represent the codeword $c = (c_1, c_2, \ldots, c_n)$ by the polynomial $c(x) = \sum_i c_i x^{i-1}$. The code C is cyclic precisely when $\{c(x) \mid c \in C\}$ is an ideal in the ring $R = \mathbb{F}_q[x]/(x^n - 1)$.

In this ring every ideal is generated by a single element, so every cyclic code has the representation $g(x)R$ for some *generator polynomial* $g(x)$. W.l.o.g. $g(x) \,|\, (x^n - 1)$. Now if $x^n - 1 = g(x)h(x)$, then $c \in C$ if and only if $c(x)h(x) = 0$ in R, and $h(x)$ is called the *check polynomial* of C. It has degree k.

The code C is called *irreducible* when its check polynomial is irreducible, that is, when the ideal of the code is minimal nonzero.

7.2.1 Trace representation of an irreducible cyclic code

Let C be irreducible. Let $F_0 = \mathbb{F}_q$ and $F = \mathbb{F}_{q^k}$. Let $\mathrm{tr}\colon F \to F_0$ be the trace. Let $\alpha \in F$ be a root of $h(x)$. Then C can be represented as $C = \{c(\xi) \mid \xi \in F\}$, where $c(\xi) = (c_0(\xi), \ldots, c_{n-1}(\xi))$ and $c_i(\xi) = \mathrm{tr}(\xi\alpha^{-i})$.

Indeed, this latter code is linear and cyclic, and if $h(x) = \sum h_i x^i$ then the coefficient of x^j in $c(x)h(x)$ is $\sum_i c_{j-i}h_i = \mathrm{tr}(\xi\alpha^{-j}h(\alpha)) = 0$. Thus, the check polynomial of the code divides $h(x)$, and hence equals $h(x)$.

If $\alpha^t = 1$ for some $t < n$, then the code words in C are periodic with period t. We shall assume that this is not the case, so that α is a primitive n-th root of unity. It follows that $\gcd(q,n) = 1$, and that k is the order of q mod n (since $h(x) = \prod_{i=0}^{k-1}(x - \alpha^{q^i})$).

Let $\beta = \alpha^{-1}$. The code C here is one as in §7.1.2 corresponding to the (multi)set $X = \{\langle\beta^i\rangle \mid 0 \le i \le n-1\}$. It is projective when there are no repeated points, i.e., when $\beta^i \notin F_0$ for $1 \le i \le n-1$, i.e., when $\gcd(q-1,n) = 1$. Now $n \,|\, \frac{q^k-1}{q-1}$ and X is the orbit of a suitable power of the Singer cycle on $\mathsf{PG}(k-1,q)$. In this situation, C is an irreducible cyclic two-weight code if and only if X is a two-character projective set.

7.2.2 Wolfmann's theorem

Wolfmann [742] shows that every two-weight projective cyclic code is either irreducible or the direct sum of two one-weight irreducible cyclic codes, where the latter case can occur only for $q > 2$. For examples of the latter possibility, see [713], [714].

7.2.3 Irreducible cyclic two-weight codes

In the case of a vector space that is a field F, one conjectures that one knows all examples of difference sets that are subgroups of the multiplicative group F^* containing the multiplicative group of the base field.

Conjecture 7.2.1 (Schmidt & White [637], Conj. 4.4; cf. [340], Conj. 1.2)
Let F be a finite field of order $q = p^f$. Suppose $1 < e \,|\, (q-1)/(p-1)$ and let D be the subgroup of F^* of index e. If the Cayley graph on F with difference set D is strongly regular, then one of the following holds:

(i) (subfield case) D is the multiplicative group of a subfield of F.

(ii) (semiprimitive case) There exists a positive integer l such that $p^l \equiv -1 \pmod e$.

(iii) (exceptional case) $|F| = p^f$, and (e, p, f) takes one of the following eleven values: $(11, 3, 5)$, $(19, 5, 9)$, $(35, 3, 12)$, $(37, 7, 9)$, $(43, 11, 7)$, $(67, 17, 33)$, $(107, 3, 53)$, $(133, 5, 18)$, $(163, 41, 81)$, $(323, 3, 144)$, $(499, 5, 249)$.

In each of the mentioned cases the graph is strongly regular. See also below.

Since F^* has a partition into cosets of D, the point set of the projective space PF is partitioned into isomorphic copies of the two-intersection set $X = \{\langle d\rangle \mid d \in D\}$.

7.3 Cyclotomy

More generally, the difference set D can be be a union of cosets of a subgroup of F^*, for some finite field F. Let $F = \mathbb{F}_q$ where $q = p^f$, p is prime, and let $e \,|\, q-1$, say $q = em+1$. Let $K \subseteq \mathbb{F}_q^*$ be the subgroup of the e-th powers (so that $|K| = m$). Let α be a primitive element of $\mathbb{F}_q$. For $J \subseteq \{0, 1, \ldots, e-1\}$ put $u := |J|$ and $D := D_J := \bigcup\{\alpha^j K \mid j \in J\} = \{\alpha^{ie+j} \mid j \in J,\ 0 \le i < m\}$. Define a graph $\Gamma = \Gamma_J$ with vertex set $\mathbb{F}_q$ and edges (x, y) whenever $y - x \in D$. Note that Γ will be undirected if q is even or $e|(q-1)/2$.

As before, the eigenvalues of Γ are the sums $\sum_{d\in D} \chi(d)$ for the characters χ of F. Their explicit determination requires some theory of Gauss sums. Let us write $A\chi = \theta(\chi)\chi$. Clearly, $\theta(1) = mu$, the valency of Γ. Now assume $\chi \ne 1$. Then $\chi = \chi_g$ for some g, where

$$\chi_g(\alpha^j) = \exp(\frac{2\pi i}{p}\mathrm{tr}(\alpha^{j+g}))$$

and tr: $\mathbb{F}_q \to \mathbb{F}_p$ is the trace function. If μ is any multiplicative character of order e (say, $\mu(\alpha^j) = \zeta^j$, where $\zeta = \exp(\frac{2\pi i}{e})$), then

$$\sum_{i=0}^{e-1} \mu^i(x) = \begin{cases} e & \text{if } \mu(x) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\theta(\chi_g) = \sum_{d\in D} \chi_g(d) = \sum_{j\in J}\sum_{y\in K} \chi_{j+g}(y) = \frac{1}{e}\sum_{j\in J}\sum_{x\in\mathbb{F}_q^*} \chi_{j+g}(x) \sum_{i=0}^{e-1} \mu^i(x) =$$

$$= \frac{1}{e}\sum_{j\in J}(-1 + \sum_{i=1}^{e-1}\sum_{x\ne 0} \chi_{j+g}(x)\mu^i(x)) = \frac{1}{e}\sum_{j\in J}(-1 + \sum_{i=1}^{e-1} \mu^{-i}(\alpha^{j+g})G_i)$$

where G_i is the Gauss sum $\sum_{x\ne 0} \chi_0(x)\mu^i(x)$.

In a few cases these sums can be evaluated.

Proposition 7.3.1 (Stickelberger and Davenport & Hasse; see [553])

Suppose $e > 2$ *and* p *is semiprimitive* $\bmod e$, *i.e., there exists an* l *such that* $p^l \equiv -1 \pmod e$. *Choose* l *minimal and write* $f = 2lt$. *Then*

$$G_i = (-1)^{t+1}\varepsilon^{it}\sqrt{q},$$

where

$$\varepsilon = \begin{cases} -1 & \textit{if } e \textit{ is even and } (p^l+1)/e \textit{ is odd} \\ +1 & \textit{otherwise.} \end{cases}$$

Under the hypotheses of this proposition, we have

$$\sum_{i=1}^{e-1} \mu^{-i}(\alpha^{j+g})G_i = \sum_{i=1}^{e-1} \zeta^{-i(j+g)}(-1)^{t+1}\varepsilon^{it}\sqrt{q} = \begin{cases} (-1)^t\sqrt{q} & \text{if } r \neq 1, \\ (-1)^{t+1}\sqrt{q}(e-1) & \text{if } r = 1, \end{cases}$$

where $r = r_{g,j} = \zeta^{-j-g}\varepsilon^t$ (so that $r^e = \varepsilon^{et} = 1$), and hence

$$\theta(\chi_g) = \frac{u}{e}(-1+(-1)^t\sqrt{q}) + (-1)^{t+1}\sqrt{q}\,.\,\#\{j \in J \mid r_{g,j} = 1\}.$$

If we abbreviate the cardinality in this formula with $\#$ then: If $\varepsilon^t = 1$ then $\# = 1$ if $g \in -J \pmod e$, and $\# = 0$ otherwise. If $\varepsilon^t = -1$ (then e is even and p is odd) then $\# = 1$ if $g \in \frac{1}{2}e - J \pmod e$, and $\# = 0$ otherwise. We proved:

Theorem 7.3.2 ([54], [146]) *Let* $q = p^f$, p *prime,* $f = 2lt$ *and* $e \mid p^l + 1 \mid q - 1$. *Let* $u = |J|$, $1 \le u \le e - 1$. *Then the graphs* Γ_J *are strongly regular with eigenvalues*

$$\begin{array}{rl} k = \frac{q-1}{e}u & \textit{with multiplicity } 1, \\ \frac{u}{e}(-1+(-1)^t\sqrt{q}) & \textit{with multiplicity } q-1-k, \\ \frac{u}{e}(-1+(-1)^t\sqrt{q}) + (-1)^{t+1}\sqrt{q} & \textit{with multiplicity } k. \end{array}$$

The above construction can be generalized.

7.3.1 The Van Lint-Schrijver graphs

Van Lint & Schrijver [524] use the above setup in case e is an odd prime, and p primitive mod e (so that $l = (e-1)/2$ and $f = (e-1)t$), and notice that the group G consisting of the maps $x \mapsto ax^{p^i} + b$, where $a \in K$ and $b \in F$ and $i \ge 0$ acts as a rank 3 group on F. Thus one obtains rank 3 graphs for $u = 1$, and strongly regular graphs for arbitrary u.

7.3.2 The Hill graph

The cap of size 78 in $\mathbb{F}_4^6$ found by Hill [428] corresponds to a strongly regular graph with parameters $(4096, 234, 2, 14)$. It is obtained from the above setup for $q = 2^{12}$, $e = 35$, and $J = \{0, 7\}$.

7.3.3 The De Lange graphs

De Lange [510] found that one gets strongly regular graphs in the following three cases (that are not semiprimitive).

p	f	e	J
3	8	20	$\{0,1,4,8,11,12,16\}$
3	8	16	$\{0,1,2,8,10,11,13\}$
2	12	45	$\{0,5,10\}$

This latter graph can be viewed as a graph with vertex set $\mathbb{F}_q^3$ for $q = 16$ such that each vertex has a unique neighbor in each of the $q^2+q+1 = 273$ directions.

7.3.4 Generalizations

The examples given by De Lange and by Ikuta & Munemasa [453, 454] ($p = 2$, $f = 20$, $e = 75$, $J = \{0,3,6,9,12\}$ and $p = 2$, $f = 21$, $e = 49$, $J = \{0,1,2,3,4,5,6\}$) and the sporadic cases of the Schmidt-White Conjecture 7.2.1 were generalized by Feng & Xiang [322], Ge, Xiang & Yuan [340], Momihara [568], Wu [745], Momihara & Xiang [570], and Momihara [569], who find several further infinite families of strongly regular graphs. The first two papers use results on Gauss sums for the case when $\langle p \rangle$ does not contain -1 but has index 2 or 4 in $(\mathbb{Z}/e\mathbb{Z})^*$. Momihara [568] uses relative Gauss sums. Wu [745] treats the case of higher even index. Momihara [569] generalizes [44] (but has a typo in the stated values for λ, μ; for example, the sporadic graph found on the last page has parameters $(v, k, \lambda, \mu) = (q^2, r(q+1), -q+r(r+3), r(r+1))$, where $q = 7^7$ and $r = 35(q-1)/58$).

For more on Gauss sums, see the monograph [62].

7.3.5 Amorphic association schemes

An association scheme $(X, \{R_0, \ldots, R_d\})$ with d classes is called *amorphic* if every fusion $(X, \{S_0, \ldots, S_e\})$ (where $R_0 = S_0$ is the identity relation, the S_i partition $X \times X$, and each S_i is the union of some R_j) is again an association scheme. In an amorphic association scheme all relations are strongly regular graphs. The setting of Theorem 7.3.2 yields amorphic association schemes with e classes. For a survey, see Van Dam & Muzychuk [253].

7.3.6 Self-complementary graphs and Peisert graphs

A graph is called *self-complementary* when it is isomorphic to its complement. For example, the path P_4 (with 4 vertices and 3 edges) is self-complementary. Mathon [548] found all self-complementary strongly regular graphs on at most 49 vertices. For earlier work, see Rosenberg [631].

A graph is called *symmetric* when its group is transitive on its vertices and edges. Of course a self-complementary symmetric graph is strongly regular. Peisert [613] classified the self-complementary symmetric graphs. These turn out to be (i) the Paley graphs, (ii) the graphs Γ_J constructed above for $q = p^{2t}$, where $p \equiv 3 \pmod 4$, $e = 4$, and $J = \{0,1\}$ so that $u = 2$, $l = 1$, and (iii) one graph on 23^2 vertices. We call these the Paley graphs of order q, the *Peisert graphs* of order q, and the *sporadic Peisert graph*. For (i) and (iii), see §7.4.4 and §10.70. In case (ii), the full automorphism group has size $fq(q-1)/4$ for $q = p^f$, $q \neq 3^2, 7^2, 3^4$ and is 2, 3, 6 times as large in the three exceptional cases.

7.4 One-dimensional affine rank 3 groups

Let q be a prime power, say $q = p^r$, where p is prime. Consider the 1-dimensional semilinear group $G = \Gamma L(1, q)$ acting on the nonzero elements of $\mathbb{F}_q$. It consists of the maps $t_{a,i} \colon x \mapsto ax^\sigma$, where $a \neq 0$ and $\sigma = p^i$.

Foulser & Kallaher ([330], §3) determined which subgroups H of G have precisely two orbits. We need some preparation.

7.4.1 Divisibility

For a prime p, let $p^a||x$ mean that $p^a|x$ and $p^{a+1} \nmid x$.

Lemma 7.4.1 *Let x, s, t, a be integers with $x > 1$, $s, t > 0$ and $a \geq 0$. Let u be an odd prime such that $u|x^s - 1$ and $u \nmid t$. Then $u^a||(x^{stu^a} - 1)/(x^s - 1)$.*

Proof. Since

$$\frac{x^{stu^a} - 1}{x^s - 1} = \frac{x^{stu^a} - 1}{x^{stu^{a-1}} - 1} \cdots \frac{x^{stu^2} - 1}{x^{stu} - 1} \frac{x^{stu} - 1}{x^{st} - 1} \frac{x^{st} - 1}{x^s - 1}$$

it suffices to consider the case $a = 1$, $t = 1$ and the case $a = 0$. Write $x^s = ku+1$. Then $(x^{se} - 1)/(x^s - 1) = ((1 + ku)^e - 1)/(ku) = \sum_{i=1}^{e} \binom{e}{i} (ku)^{i-1}$, and this is congruent u (mod u^2) for $e = u$, and congruent t (mod u) for $e = t$. □

For $u = 2$ one has $((1+2k)^2 - 1)/(2k) = 2+2k$, which has additional factors 2 when k is odd.

Lemma 7.4.2 *Let x, s, t, a be integers with $x > 1$ and $s, t, a > 0$. If x and t are odd and $2^{b+1}||x^s + 1$, then $2^{a+b}||(x^{st2^a} - 1)/(x^s - 1)$.* □

Lemma 7.4.3 *Let $x > 1$ and $s, m > 0$ be integers such that each prime divisor of m divides $x^s - 1$. Then $m \,|\, (x^{ms} - 1)/(x^s - 1)$.* □

We shall write $\mathrm{ord}_m x$ for the order of x in the multiplicative group (of order $\phi(m)$) of residues mod m, coprime with m.

7.4.2 Subgroups of $\Gamma L(1, q)$ with two orbits

Let $q = p^r$, where p is prime, and let H be a subgroup of $\Gamma L(1, q)$. It acts on $\mathbb{F}_q^*$. In this section we determine in what cases this action has precisely two orbits. All results are due to Foulser & Kallaher [330].

Lemma 7.4.4 *Let H be a subgroup of $\Gamma L(1, q)$. Then $H = \langle t_{b,0} \rangle$ for suitable b, or $H = \langle t_{b,0}, t_{c,s} \rangle$ for suitable b, c, s, where $s|r$ and $c^{(q-1)/(p^s-1)} \in \langle b \rangle$.*

Proof. The subgroup of all elements $t_{a,0}$ in H is cyclic and has a generator $t_{b,0}$. If this was not all of H, then $H/\langle t_{b,0} \rangle$ is cyclic again, and has a generator $t_{c,s}$ with $s|r$. Since ${t_{c,s}}^i = t_{c^j,is}$ where $j = 1 + p^s + p^{2s} + \cdots + p^{(i-1)s}$, it follows for $i = r/s$ that $c^{(q-1)/(p^s-1)} \in \langle b \rangle$. □

Theorem 7.4.5 *$H = \langle t_{b,0} \rangle$ has two orbits if and only if q is odd and H consists precisely of the elements $t_{a,0}$ with a a square in $\mathbb{F}_q^*$.*

Proof. Let b have multiplicative order m. Then $m|(q - 1)$, and $\langle t_{b,0} \rangle$ has d orbits, where $d = (q - 1)/m$. □

Let b have order m and put $d = (q - 1)/m$. Choose a primitive element $\omega \in \mathbb{F}_q^*$ with $b = \omega^d$. Let $c = \omega^e$.

Theorem 7.4.6 *$H = \langle t_{b,0}, t_{c,s}\rangle$ (where $s|r$ and $d \,|\, e(q-1)/(p^s-1)$) has two orbits of different lengths n_1, n_2, where $n_1 < n_2$, $n_1 + n_2 = q-1$, if and only if (0) $n_1 = m_1 m$, where (1) the prime divisors of m_1 divide $p^s - 1$, and (2) $v := (q-1)/n_1$ is an odd prime, and $p^{m_1 s}$ is a primitive root mod v, and (3) $\gcd(e, m_1) = 1$, and (4) $m_1 s(v-1)|r$.*

Proof. Let $P_0, \ldots, P_{d-1}$ be the orbits (of size m each) of $\langle t_{b,0}\rangle$. Then $t_{c,s}$ permutes the P_i. The group H will have two orbits of lengths n_1, n_2 precisely when $\langle t_{c,s}\rangle$ has two orbits on $\{P_0, \ldots, P_{d-1}\}$ of lengths m_1, m_2, where $n_1 = m_1 m$, $n_2 = m_2 m$.

Recall that ${t_{c,s}}^i = t_{c^j,is}$ where $j = (p^{is}-1)/(p^s-1)$. The element ${t_{c,s}}^i$ fixes P_k (where $\omega^k \in P_k$) if and only if $d \,|\, ej + k(p^{is}-1)$. Let $g = \gcd(d, p^{is}-1)$. There are fixed P_k only when $g|ej$, and if this is the case there are precisely g fixed sets P_k.

For $i = m_1$ the element ${t_{c,s}}^i$ fixes precisely m_1 of the P_k, and we find $m_1 = \gcd(d, p^{m_1 s}-1) \,|\, ej = e(p^{m_1 s}-1)/(p^s-1)$. In particular, $m_1|d$.

(1) For $i < m_1$ the element ${t_{c,s}}^i$ fixes no P_k, so $\gcd(d, p^{is}-1) \nmid e(p^{is}-1)/(p^s-1)$. Let k_1 (resp. k_2) be the products of the prime powers u^a in m_1 where u does (resp. does not) divide p^s-1. Then $m_1 = k_1 k_2$, and $k_1|(p^{k_1 s}-1)/(p^s-1)$ by Lemma 7.4.3. In order to show (1) we have to show that $m_1 = k_1$. If not, then $k_1 < m_1$ and we can use the nondivisibility for $i = k_1$. Since $k_1|m_1|d$, we can write $\gcd(d, p^{k_1 s}-1) = k_1 k_3$, where $k_3|k_2$ since $\gcd(d, p^{k_1 s}-1) \,|\, \gcd(d, p^{m_1 s}-1) = m_1 = k_1 k_2$. It follows that the primes in k_3 are not in p^s-1, so that $k_3|(p^{k_1 s}-1)/(p^s-1)$, contradicting the nondivisibility.

(2) Since $v = (q-1)/n_1 = d/m_1$, this is an integer, and $m_2 = (v-1)m_1$, so $v > 2$. The element ${t_{c,s}}^{im_1}$ fixes precisely m_1 of the P_k for $1 \le i \le v-2$, but fixes them all for $i = v-1$. It follows that $\gcd(d, p^{im_1 s}-1) = m_1$ for $1 \le i \le v-2$, and $\gcd(d, p^{(v-1)m_1 s}-1) = d$. If $\gcd(m_1, v) = 1$, this says that $\mathrm{ord}_v p^{m_1 s} = v-1$, so that $v-1 \le \phi(v)$, and v is prime, as desired. Let u be a prime factor of $\gcd(m_1, v)$, so that d contains more factors u than m_1. Then $u|p^s-1$ by (1), and if $u \ne 2$ then by Lemma 7.4.1 $(v-1)m_1$ contains more factors u than m_1, so that $u|v-1$, a contradiction. Hence $u = 2$. Since $p^{2m_1 s}-1$ contains more factors 2 than $p^{m_1 s}-1$, we have $v = 3$, contradicting $u \,|\, v$.

From $\mathrm{ord}_v p^{m_1 s} = v-1$ it follows immediately that $\gcd(m_1 s, v-1) = 1$, so that m_1, s and d are all odd. We saw that $\gcd(m_1, v) = 1$.

(3) Let u be a prime factor of $\gcd(m_1, e)$ and $i = m_1/u$. Then i is odd, and all prime factors of i divide p^s-1. By Lemma 7.4.3, $\gcd(d, p^{is}-1) \,|\, \gcd(d, p^{m_1 s}-1) = m_1 = ui \,|\, e(p^{is}-1)/(p^s-1)$, contradicting nondivisibility for i.

(4) The orbit size $m_2 = m_1(v-1)$ divides the order of $t_{c,s}$ in its action on the P_k, which is r/s.

That proved the necessity of (0)–(4). Conversely, assume (0)–(4). We investigate the number of fixed sets P_k under the action of ${t_{c,s}}^i$ for different i.

First, look at $i = m_1 w$ with $1 \le w < v-1$. By (1) and Lemma 7.4.3, $m_1 \,|\, (p^{m_1 s}-1)/(p^s-1)$, and by (2) $v \nmid p^{is}-1$, and since $d = vm_1$ it follows that $\gcd(d, p^{is}-1) = m_1$. It follows that for these i the element ${t_{c,s}}^i$ fixes precisely m_1 of the sets P_k.

Next, look at $i = m_1(v-1)$. We have $\gcd(d, p^{is}-1) = d \,|\, (p^{is}-1)/(p^s-1)$ so the element ${t_{c,s}}^i$ fixes all sets P_k.

Finally, consider the case $m_1 \nmid i$. Let u be a prime with $u^a || m_1$ and $u^b || i$ with $b < a$. (Then u is odd since, as we saw, (2) implies that m_1 is odd; also, by (3), $u \nmid e$.) Now $u^{b+1} \mid \gcd(d, p^{is}-1)$ and $u^b || e(p^{is}-1)/(p^s-1)$ so that there are no fixed sets P_k for these i.

Since $d = m_1 + (v-1)m_1$, it follows that $t_{c,s}$ has precisely two orbits (of lengths m_1 and $(v-1)m_1$) on $\{P_0, \ldots, P_{d-1}\}$. □

That settled the case of two orbits of different lengths. Next consider that of two orbits of equal length. As before, let b have order m and put $d = (q-1)/m$. Choose a primitive element $\omega \in \mathbb{F}_q^*$ with $b = \omega^d$. Let $c = \omega^e$.

Theorem 7.4.7 *$H = \langle t_{b,0}, t_{c,s}\rangle$ (where $s|r$ and $d \mid e(q-1)/(p^s-1)$) has exactly two orbits of the same length $(q-1)/2$ if and only if (0) $(q-1)/2 = m_1 m$, (1) the prime divisors of $2m_1$ divide p^s-1, (2) no odd prime divisor of m_1 divides e, (3) $m_1 s|r$, (4) one of the following cases applies: (i) m_1 is even, $p^s \equiv 3 \pmod 8$, and e is odd, (ii) $m_1 \equiv 2 \pmod 4$, $p^s \equiv 7 \pmod 8$, and e is odd, (iii) m_1 is even, $p^s \equiv 1 \pmod 4$, and $e \equiv 2 \pmod 4$, (iv) m_1 is odd and e is even.*

Proof. As before, let $P_0, \ldots, P_{d-1}$ be the orbits (of size m) of $\langle t_{b,0}\rangle$. The group H will have two orbits of equal length $(q-1)/2$ precisely when $\langle t_{c,s}\rangle$ has two orbits on $\{P_0, \ldots, P_{d-1}\}$ of equal length $m_1 = d/2$, where $(q-1)/2 = m_1 m$.

Recall that ${t_{c,s}}^i = t_{c^j, is}$ where $j = (p^{is}-1)/(p^s-1)$. The element ${t_{c,s}}^i$ fixes P_k if and only if $d \mid ej + k(p^{is}-1)$. Let $g = \gcd(d, p^{is}-1)$. There are fixed P_k only when $g|ej$, and if this is the case there are precisely g fixed sets P_k.

For $i = m_1$ all P_k are fixed, so $d \mid p^{m_1 s}-1$ and $d \mid e(p^{m_1 s}-1)/(p^s-1)$. We shall use twice below that if u is an odd divisor of m_1, then all factors 2 in d are in $e(p^{(m_1/u)s}-1)/(p^s-1)$, since $(p^{m_1 s}-1)/(p^{(m_1/u)s}-1)$ is odd.

(1) Since $q-1$ is even, p is odd. Let k_1 (resp. k_2) be the products of the prime powers u^a in m_1 where u does (resp. does not) divide p^s-1. Then $m_1 = k_1 k_2$, and $k_1 \mid (p^{k_1 s}-1)/(p^s-1)$ by Lemma 7.4.3. Since $2k_1 \mid d$ and $2k_1 \mid p^{k_1 s}-1$, we can write $\gcd(d, p^{k_1 s}-1) = 2k_1 k_3$, where $k_3 | k_2$ since $\gcd(d, p^{k_1 s}-1) \mid \gcd(d, p^{m_1 s}-1) = 2m_1 = 2k_1 k_2$. It follows that the primes in k_3 are not in p^s-1, so that $\gcd(d, p^{k_1 s}-1) = 2k_1 k_3 \mid e(p^{k_1 s}-1)/(p^s-1)$, since k_2 is odd. This shows that for $i = k_1$ the element ${t_{c,s}}^i$ has fixed points, and therefore $k_1 = m_1$.

(2) Let u be an odd prime factor of $\gcd(m_1, e)$. By part (1), $2u \mid p^s-1$. Let $i = m_1/u$. By Lemma 7.4.3, $i \mid (p^{is}-1)/(p^s-1)$, so that $d = 2ui \mid p^{is}-1$. Then $\gcd(d, p^{is}-1) = 2ui \mid e(p^{is}-1)/(p^s-1)$ contradicting nondivisibility.

(3) The orbit size m_1 divides the order of $t_{c,s}$ in its action on the P_k, which is r/s.

(4) Since $d \mid e(p^{m_1 s}-1)/(p^s-1)$ and d is even, e must be even if m_1 is odd (case (iv)). Let m_1 be even. Write $2^a || m_1$ with $a \geq 1$, so that $2^{a+1} || d$. Let $2^{b+1} || (p^s+1)$ and $2^c || e$ and $2^h || (p^s-1)$. Then $b + h \geq 2$ since $8 \mid (p^{2s}-1)$. Since $\gcd(d, p^{(m_1/2)s}-1) \nmid e(p^{(m_1/2)s}-1)/(p^s-1)$, the LHS has a single factor 2 more than the RHS. If $a \geq 2$, then $2^{a-1+b+c} || e(p^{(m_1/2)s}-1)/(p^s-1)$, and $\gcd(d, p^{(m_1/2)s}-1) = d$ (since $p^{(m_1/2)s}-1$ is divisible by $m_1/2 = d/4$ and by $2^{a-1+b+h}$), so $a + b + c = a + 1$, and we have case (i) or (iii). If $a = 1$, so that $m_1/2$ is odd, then $c + 1 = \min(2, h)$. Now if $b = 0$ then $c = 1$, case (iii). If $b = 1$ then $c = 0$, case (i). If $b \geq 2$, we have case (ii).

That proved the necessity of (0)–(4). Conversely, assume (0)–(4). By (1), p is odd. We investigate the number of fixed sets P_k under the action of ${t_{c,s}}^i$ for different i.

First, look at $i = m_1$. We want to show that all P_k are fixed, that is, that (a) $\gcd(d, p^{m_1 s} - 1) = d$ and (b) $d | e(p^{m_1 s} - 1)/(p^s - 1)$. By (1) and Lemma 7.4.3 we have $m_1 | (p^{m_1 s} - 1)/(p^s - 1)$. Since $d = 2m_1$ and p is odd, this implies (a). For (b) we only have to check the powers of 2. If e is even, then it provides the needed extra factor 2. Otherwise, by (4), m_1 is even and $4 | p^s + 1$, and we are done by Lemma 7.4.2.

Next, look at $i = m_1/u$ where u is prime. We want to show that no P_k is fixed, that is, that $\gcd(d, p^{is} - 1) \nmid e(p^{is} - 1)/(p^s - 1)$. If u is odd, then this nondivisibility follows from (1) and (2) and Lemma 7.4.1. If $u = 2$, nondivisibility follows from (4). It follows that the orbit of each P_k has size m_1. □

7.4.3 One-dimensional affine rank 3 groups

Let $q = p^r$ be a prime power, where p is prime. Consider the group $G = \mathrm{A\Gamma L}(1, q)$ consisting of the semilinear maps $x \mapsto ax^\sigma + b$ on $\mathbb{F}_q$. Let T be the subgroup of size q consisting of the translations $x \mapsto x + b$. The previous section provides a classification of the rank 3 subgroups of G that contain T.

The graphs from Theorem 7.4.5 are the Paley graphs, discussed further below in §7.4.4. The (rank 3) Van Lint-Schrijver graphs from §7.3.1 are the special case of Theorem 7.4.6 where $s = 1$, $e = 0$, $m_1 = 1$. The Peisert graphs from §7.3.6 are the special case of Theorem 7.4.7 where $s = 1$, $e = 1$, $m_1 = 2$, $d = 4$.

Muzychuk [581] determined all graphs Γ with vertex set $\mathbb{F}_q$ such that $G \cap \mathrm{Aut}\,\Gamma$ acts as a rank 3 group on Γ, and finds that these are the Paley graphs, the Van Lint-Schrijver graphs (and complements), and the Peisert graphs.

7.4.4 Paley graphs

Construction

The Paley graph of order q has as vertex set the finite field $\mathbb{F}_q$ of order q, where $q \equiv 1 \pmod 4$, and two vertices are adjacent when their difference is a square in the field.

Parameters

The Paley graph $P(q)$ of order $q = 4t + 1$ is strongly regular with parameters $(v, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)$. It has eigenvalues k and $(-1 \pm \sqrt{q})/2$ with multiplicities 1 and $(q - 1)/2$ (twice).

Automorphism group

Let $q = p^e$, where p is prime. The full group of automorphisms of $P(q)$ consists of the maps $x \mapsto ax^\sigma + b$ where $a, b \in \mathbb{F}_q$, a a nonzero square, and $\sigma = p^i$ with $0 \le i < e$ (Carlitz [186]). It has order $eq(q - 1)/2$.

The Paley graph $P(q)$ is self-complementary. The map $x \mapsto ax$, where a is a nonsquare, maps $P(q)$ to its complement.

The subgraph Π of $P(q)$ induced on the neighbors of 0 has full automorphism group consisting of the maps $x \mapsto ax^{\pm\sigma}$ where a is a nonzero square and $\sigma = p^i$ with $0 \le i < e$ (Muzychuk & Kovács [582]). It has order $e(q-1)$ for $q > 9$. For

$q = 5$ and $q = 9$ the group is only half as large because in the first case $x \mapsto x^{-1}$ is the identity, while in the second case $x \mapsto x^{-1}$ is the field automorphism $x \mapsto x^3$.

When $q = p$ is prime, the Paley graph has a regular cyclic group of automorphisms. There are no other such primitive strongly regular graphs ([489], [108]). See also [109].

Independence and chromatic numbers

Since the Paley graphs are self-complementary, bounds for cliques are equivalent to bounds for cocliques.

The Hoffman upper bound for the size of cliques and cocliques is $\sqrt{q}$. If q is a square then the subfield of size $\sqrt{q}$ is a clique of this size, and BLOKHUIS [72] showed that every clique or coclique of size $\sqrt{q}$ is the affine image of a subfield. The translates of this subfield form a partition into cliques (hence a partition into cocliques for the complementary graph). One finds that if q is a square, then the independence number α and the chromatic number χ are given by $\alpha = \chi = \sqrt{q}$.

If q is prime, say $q = p$, the best upper bound known is $\alpha \le \frac{1}{2}(\sqrt{2p-1}+1)$ ([413], see also [290]). Equality holds for $p = 5, 13, 41$. For nonsquare prime powers $q = p^{2e+1}$ a similar bound (a bit more than $\sqrt{q/2}$) was proved in [194].

Concerning lower bounds, one has $\alpha > (\frac{1}{2}+o(1)) \log_2 q$ ([209]). For infinitely many primes q the smallest quadratic nonresidue is $\Omega(\log q \log\log\log q)$ ([362]), and this is a lower bound for α.

James Shearer [645] computed the independence numbers of the Paley graphs of order p, p a prime, $p < 7000$. Geoffrey Exoo [313] extended that table beyond order 16000. Below we present a small table. The upper bounds on χ come from an actual coloring, the lower bounds from $\chi \ge \lceil v/\alpha \rceil$. The chromatic numbers for $q = 125, 173$, and $q \ge 197$ are due to G. Exoo. For $q < 16000$, the values of α grow roughly like $\frac{1}{10}(\log_2 q)^2$.

q	5	9	13	17	25	29	37	41	49	53	61	73	81	89	97
α	2	3	3	3	5	4	4	5	7	5	5	5	9	5	6
χ	3	3	5	6	5	8	10	9	7	11	13	15	9	18	17

q	101	109	113	121	125	137	149	157	169	173	181	193	197
α	5	6	7	11	7	7	7	7	13	8	7	7	8
χ	21	19	17	11	18	20	22	23	13	22	26	28	25

q	229	233	241	257	269	277	281	293	313	317	14797	15461
α	9	7	7	7	8	8	7	8	8	9	27	19
χ	26 or 27	34	35	37	34	35	41	37	40	36		

Table 7.4: Independence and chromatic numbers of small Paley graphs

p-rank

Let $q = p^e$. We have $\mathrm{rk}_p(A + \frac{1}{2}I - bJ) = (\frac{p+1}{2})^e$ for all b. See also §9.3.

Locally Paley graphs

For $q = 5$, the Paley graph is a pentagon, and the unique connected locally pentagon graph is the icosahedron. For $q = 9$ the Paley graph is the 3×3 grid, and there are precisely two connected locally 3×3 graphs, namely the Johnson graph $\binom{6}{3}$ on 20 vertices, and the complement of the 4×4 grid on 16 vertices ([123], p. 258). For all $q \neq 9$ there is a unique connected locally $P(q)$ graph, namely the Taylor extension of $P(q)$. This graph is distance-transitive, with intersection array $\{q, (q-1)/2, 1; 1, (q-1)/2, q\}$ (an antipodal 2-cover of the complete graph K_{q+1}) and with full automorphism group $2 \times P\Sigma L(2, q)$, cf. [123], p. 15 and p. 228.

That these are all locally Paley graphs was shown for $13 \leq q \leq 41$ in [160], and for $q > 41$ in [119] under a hypothesis that was proved in [582].

Ramsey numbers

The *Ramsey number* $R(m, n)$ is the minimum number of vertices v_0 such that any graph of size $v \geq v_0$ contains a clique of size m or a coclique of size n. It follows that if $P(q)$ has independence number α, then $R(\alpha+1, \alpha+1) > q$ and (using the above locally Paley graphs) $R(\alpha+2, \alpha+2) > 2q+2$. Using $q = 5, 17, 101, 281$ one finds $R(3,3) \geq 6$, $R(4,4) \geq 18$, $R(6,6) \geq 102$, $R(7,7) \geq 205$, $R(8,8) \geq 282$, $R(9,9) \geq 565$, and these are the sharpest bounds known today. See also [644].

Quasi-randomness

The Paley graphs $P(q)$ are fully deterministic, but exhibit the behavior one expects from random graphs. This is caused by the large eigenvalue gap: the other eigenvalues are much smaller in absolute value than the valency. Chung, Graham & Wilson [196] discuss a number of equivalent properties, each implying quasi-random behavior, where the Paley graphs satisfy these properties. See also [69], [84] and §8.17.2.

Name

Jones [467] has an extensive historical discussion about the naming of these graphs.

7.4.5 Power residue difference sets

Consider the graph with as vertex set the finite field $\mathbb{F}_q$, where two vertices are adjacent when their difference is an e-th power. W.l.o.g. $e|(q-1)$, and in order to get an undirected graph we require that q is even or $e|(q-1)/2$. Of course we get the Paley graphs for $e = 2$, so assume $e > 2$.

Below we give a small table of the cases with $q \leq 2^{10}$ where this yields a connected strongly regular graph.

All of these are of the shape $q = r^{2t}$, where r is a prime power, and $e \mid r+1$, special cases of Theorem 7.3.2, with the single exception of $(q, e) = (243, 11)$. In particular, in all cases here except $(q, e) = (243, 11)$ one can take u disjoint copies of these graphs and get strongly regular graphs of valency uk for $1 \leq u \leq e-1$, or take e disjoint copies and get K_q.

Other examples exist, like $(q, e) = (3^{12}, 35)$. See also Conjecture 7.2.1.

q	e	k	λ	μ	comment	q	e	k	λ	μ	comment
16	3	5	0	2	$VO_4^-(2)$	529	4	132	41	30	
25	3	8	3	2	5×5	529	6	88	27	12	
49	4	12	5	2	7×7	529	8	66	23	6	
64	3	21	8	6	$H_2(2,3)$	529	12	44	21	2	23×23
81	4	20	1	6	$VO_4^-(3)$	625	3	208	63	72	
81	5	16	7	2	9×9	625	6	104	3	20	$VO_4^-(5)$
121	3	40	15	12		625	13	48	23	2	25×25
121	4	30	11	6		729	4	182	55	42	
121	6	20	9	2	11×11	729	7	104	31	12	$H_3(2,3)$
169	7	24	11	2	13×13	729	14	52	25	2	27×27
243	11	22	1	2	§10.55	841	3	280	99	90	
256	3	85	24	30		841	5	168	47	30	
256	5	51	2	12	$VO_4^-(4)$	841	6	140	39	20	
289	3	96	35	30		841	10	84	29	6	
289	6	48	17	6		841	15	56	27	2	29×29
289	9	32	15	2	17×17	961	4	240	71	56	
361	4	90	29	20		961	8	120	35	12	
361	5	72	23	12		961	16	60	29	2	31×31
361	10	36	17	2	19×19	1024	3	341	120	110	
529	3	176	63	56		1024	11	93	32	6	$H_2(2,5)$

Table 7.5: Strongly regular power residue graphs

7.5 Icosahedrals

7.5.1 Orbits of A_5 on the projective line and plane

For $\tau = \frac{1}{2}(1+\sqrt{5})$, consider the set $S = \{(0,\pm 1,\pm\tau),(\pm\tau,0,\pm 1),(\pm 1,\pm\tau,0)\}$ in $\mathbb{R}^3$. All inner products (x,y) for $y \neq \pm x$ equal $\pm\tau$, and we see that S is the set of 12 vertices of an icosahedron. Its isometry group $2\times\mathsf{A}_5$ has a matrix representation with entries in $\mathbb{Z}[\tau,\frac{1}{2}]$.

Let q be a prime power. The group $\mathsf{L}_2(q)$ has subgroups A_5 if and only if $q \equiv 0$, 1, or 4 (mod 5). Indeed, this is necessary, since $|\mathsf{A}_5| = 60$ must divide $|\mathsf{L}_2(q)|$, and suffices, since for these q the field $\mathbb{F}_q$ contains an element τ satisfying $\tau^2 = \tau + 1$ and the above construction produces a 6-set in $\mathsf{PG}(2,q)$ stabilized by a subgroup A_5 of $\mathsf{O}_3(q) \simeq \mathsf{L}_2(q)$ if q is odd. Finally, $\mathsf{L}_2(4) \simeq \mathsf{L}_2(5) \simeq \mathsf{A}_5$.

Such a 6-set in $\mathsf{PG}(2,q)$ stabilized by A_5 is called an *icosahedral.*

Let q be odd. Then the plane $\mathsf{PG}(2,q)$ is partitioned into the $q+1$ points of a conic, $\frac{1}{2}q(q+1)$ exterior points, and $\frac{1}{2}q(q-1)$ interior points. Consider the action of $\mathsf{A}_5 < \mathsf{O}_3(q)$. There are unique A_5-orbits of sizes 6, 10, and 15, and at most one orbit of sizes 12 and 20. The following table gives the conditions on q for each possible quadratic character of the points in these orbits.

orbitsize	isotropic	exterior	interior
6	0 (mod 5)	1 (mod 5)	-1 (mod 5)
10	3 (mod 6)	1 (mod 6)	-1 (mod 6)
15	-	1 (mod 4)	-1 (mod 4)
12	1 (mod 10)	-	-
20	1 (mod 6)	-	-

The remaining orbits have sizes 30 or 60. For $q = 5,9,11,19,29,59$, the group A_5 is transitive on the conic. For $q = 25,31,41,49,71,79,89$ the group A_5 has two orbits on the conic (of sizes $6+20$, $12+20$, $12+30$, $20+30$, $12+60$, $20+60$, $30+60$, respectively). It follows that in these cases one finds a rank 3 graph on q^2 vertices with one of these orbits at infinity.

For $q = 16,64,125$ the group A_5 has orbits of sizes $5+12$, $5+60$, $6+60+60$ on the conic, where in the last case these are fused to $6+120$ in S_5. Again this leads to rank 3 graphs.

See also §10.89D and Theorem 11.4.3.

7.5.2 Orbits of S_4 on the projective line

One can similarly look at S_4-orbits on the projective line $\mathsf{PG}(1,q)$. The group $\mathsf{L}_2(q)$ contains (two conjugacy classes of) subgroups S_4 precisely when $q \equiv \pm 1 \pmod 8$. The group $\mathsf{PGL}_2(q)$ contains (a single conjugacy class of) subgroups S_4 precisely when q is not a power of 2.

The orbit sizes of S_4 on $\mathsf{PG}(1,q)$ are uniquely determined by the fact that their sum is $q+1$ and the sizes are among 4, 6, 8, 12, 24, where only 24 may be repeated and 4, 8 do not occur together. It follows that 4 occurs for $q \equiv 3, 9 \pmod{24}$, 6 occurs for $q \equiv 1 \pmod 4$, 8 occurs for $q \equiv 1 \pmod 6$, and 12 occurs for $q \equiv 1, 11, 17, 19 \pmod{24}$.

The group S_4 is transitive for $q = 3, 5, 7, 11, 23$. It has two orbits (with sizes determined by the above) for $q = 9, 13, 17, 19, 27, 29, 31, 47$. It follows that in these cases one finds a rank 3 graph on q^2 vertices with one of these orbits at infinity. For $q = 7, 23$ the single S_4-orbit splits into two A_4-orbits. Again that leads to rank 3 graphs. See also Theorem 11.4.4.

7.6 Bent functions

Bent functions are maximally nonlinear Boolean functions. They have applications e.g. in coding theory and cryptography.

Given $F\colon \mathbb{Z}_2^m \to \mathbb{R}$, let its *Hadamard transform* be the map $F^*\colon \mathbb{Z}_2^m \to \mathbb{R}$ defined by $F^*(w) = \sum_x (-1)^{(x,w)} F(x)$, where $(x,w) = \sum_i x_i w_i$. Then $F^{**} = 2^m F$. Given $f\colon \mathbb{Z}_2^m \to \mathbb{Z}_2$, let its *Walsh transform* be F^*, where F is defined by $F(x) = (-1)^{f(x)}$.

A function $f\colon \mathbb{Z}_2^m \to \mathbb{Z}_2$ is called a *bent function* when the equation $f(x+a) - f(x) = b$ has 2^{m-1} solutions x for all nonzero a and all b. For $F(x) = (-1)^{f(x)}$ this means that $\sum_x F(x)F(x+a) = 0$ for all nonzero a.

Proposition 7.6.1 *Equivalent are*
(i) f is a bent function,
(ii) $|F^*(w)| = 2^{m/2}$ for all w,
(iii) the matrix $(F(x+y))_{x,y}$ is a Hadamard matrix.

Proof. That (i) $\Leftrightarrow$ (iii) is clear from the definitions. We prove (i) $\Leftrightarrow$ (ii). If $\sum_x F(x)F(x+a) = 0$ for $a \neq 0$, then $F^*(w)^2 = \sum_{x,y} (-1)^{(x+y,w)} F(x)F(y) = \sum_{x,a} (-1)^{(a,w)} F(x)F(x+a) = \sum_x F(x)^2 = 2^m$. Conversely, if $|F^*(w)| = 2^{m/2}$ for all w, then $2^{2m} \sum_x F(x)F(x+a) = \sum_{v,w,x} F^*(v)F^*(w)(-1)^{(a,w)}(-1)^{(v+w,x)}$ $= 2^m \sum_w (-1)^{(a,w)} F^*(w)^2 = 2^{2m} \sum_w (-1)^{(a,w)} = 0$ if $a \neq 0$. □

It follows that m is even. For $m = 2, 4, 6, 8, 10$ the number of bent functions is 2, 8, 896, 5425430528, 99270589265934370305785861242880 (according to OEIS [661] (A004491); the last number is from Langevin & Leander [512]).

Let $V = \mathbb{F}_2^m$ be the m-dimensional vector space over $\mathbb{F}_2$. The first order Reed-Muller code $RM(1,m)$ is the code C (with vectors indexed by V) generated by the all-one vector $\mathbf{1}$ together with the characteristic vectors of the hyperplanes in V. Now C has length 2^m, dimension $m+1$, and minimum weight 2^{m-1}. The bent functions are the vectors at maximal distance from C. A bent function has distance $2^{m-1} \pm 2^{m/2-1}$ from each vector in C.

Given a function $f\colon \mathbb{Z}_2^m \to \mathbb{Z}_2$, let $D = D_f = \{x \in \mathbb{Z}_2^m \mid f(x) = 1\}$. Let $\Gamma = \Gamma_D$ be the graph on $\mathbb{Z}_2^m$ defined by the difference set D, so that $u \sim v$ when $v - u \in D$, i.e., when $f(v-u) = 1$. (Then Γ has loops if $f(0) = 1$.)

Proposition 7.6.2 (cf. [60], [61]) *The spectrum of* Γ *consists of* $|D|$ *and the numbers* $-\frac{1}{2}F^*(a)$ *for* $a \neq 0$. *Suppose* $f(0) = 0$. *The graph* Γ *is strongly regular with* $\lambda = \mu$ *if and only if* f *is a bent function, and in that case has parameters* $v = 2^m$, $k = 2^{m-1} + \varepsilon 2^{m/2-1}$, $\lambda = \mu = 2^{m-2} + \varepsilon 2^{m/2-1}$, *and eigenvalues* $r, s = \pm 2^{m/2-1}$ *and multiplicities* $f, g = 2^{m-1} \mp 2^{m/2-1} - \frac{1}{2}(1 \pm \varepsilon)$, *where* $\varepsilon \in \{\pm 1\}$.

Proof. For $\chi(d) = (-1)^{(a,d)}$ we find $\sum_{d \in D} \chi(d) = \frac{1}{2}\sum_x (-1)^{(a,x)}(1 - F(x)) = -\frac{1}{2}F^*(a)$ if $a \neq 0$. The two nontrivial eigenvalues of a strongly regular graph have the same absolute value precisely when $\lambda = \mu$. □

Of course the parameters here are those of $\overline{VO^{\varepsilon}_m(2)}$. The above large numbers show that there are many nonisomorphic examples.

There is a large literature, with many generalizations.

Chapter 8

Combinatorial constructions

This chapter collects constructions related to some combinatorial setting, where the starting point is not a group. It discusses e.g. Hadamard and conference matrices, Latin squares and various designs, partial geometries, two-graphs, and spherical designs.

8.1 Regular Hadamard matrices with constant diagonal

A *Hadamard matrix* is a matrix H of order n with entries ± 1 such that $HH^\top = nI$. It is called *symmetric* when $H = H^\top$. It is called *regular* when all row sums are equal. If J denotes the all-1 matrix of order n, then all row sums are a if and only if $HJ = aJ$. (It follows that $JH = aJ$ and $a^2 = n$.) The matrix $H = (h_{ij})$ has *constant diagonal* when $h_{ii} = e$ for all i and some fixed $e \in \{\pm 1\}$. Abbreviate the phrase 'regular symmetric Hadamard matrix with constant diagonal' with RSHCD.

Let H be a RSHCD with parameters n, a, e. Then $a^2 = n$ so that $a = \pm\sqrt{n}$. The matrix $-H$ is a RSHCD with parameters $n, -a, -e$, so that there are the two essentially distinct cases $ae > 0$ and $ae < 0$. Put $ae = \varepsilon\sqrt{n}$ with $\varepsilon \in \{\pm 1\}$, and call H *of type* ε. If $n > 1$, then $4|n$, so $2|a$, say $a = 2t$. Then $A = \frac{1}{2}(J - eH)$ is the adjacency matrix of a strongly regular graph (complete for $(n, \varepsilon) = (4, -1)$) with parameters

$$v = 4t^2,\ k = 2t^2 - \varepsilon t,\ \lambda = \mu = t^2 - \varepsilon t,$$

$$r = t,\ s = -t,\ f = 2t^2 - t - (1-\varepsilon)/2,\ g = 2t^2 + t - (1+\varepsilon)/2.$$

And $J - I - A = \frac{1}{2}(J + eH - 2I)$ is the adjacency matrix of the complementary strongly regular graph with parameters

$$v = 4t^2,\ k = 2t^2 + \varepsilon t - 1,\ \lambda = t^2 + \varepsilon t - 2,\ \mu = t^2 + \varepsilon t,$$

$$r = t - 1,\ s = -t - 1,\ f = 2t^2 + t - (1+\varepsilon)/2,\ g = 2t^2 - t - (1-\varepsilon)/2.$$

Conversely, graphs with these parameters yield RSHCDs.

We see that A is the incidence matrix of a square $(4t^2, 2t^2 \pm t, t^2 \pm t)$-design (and moreover is symmetric with zero diagonal). Designs with these parameters are known as *Menon designs*.

In [497] it is shown that the sum of the absolute values of the eigenvalues (the 'energy') of a graph on n vertices is at most $n(\sqrt{n}+1)/2$, with equality precisely in the case of a graph corresponding to a RSHCD of negative type. See also [379].

8.1.1 Examples

Constructions for RSHCDs were discussed in [719] and [137]. However, not all details given there are correct, so we resurvey this area.

Let R be the set of pairs (n, ε) for which an RSHCD of order n and type ε exists.

Section 8D of Brouwer & Van Lint [137] is about RSHCDs. It is the first place that kept track of the sign ε involved. It contains the recursive construction

$$(m, \delta), (n, \varepsilon) \in R \;\Rightarrow\; (mn, \delta\varepsilon) \in R$$

and six direct constructions:

(i) $(4, \pm 1), (36, \pm 1) \in R$.

(ii) If there exists a Hadamard matrix of order m, then $(m^2, 1) \in R$ ([355], Theorem 4.4).

(iii) If both $a-1$ and $a+1$ are odd prime powers, and $4|a$, then $(a^2, 1) \in R$ ([355], Theorem 4.3).

(iv) If $a+1$ is a prime power, and there exists a symmetric conference matrix of order a, then $(a^2, 1) \in R$ ([720], Corollary 17).

(v) If there is a set of $t-2$ mutually orthogonal latin squares of order $2t$, then $(4t^2, 1) \in R$.

(vi) Suppose we have a Steiner system $S(2, K, V)$ with $V = K(2K-1)$. If we form the block graph, and add an isolated point, we get a graph in the switching class of a regular two-graph. The corresponding Hadamard matrix is symmetric with constant diagonal, but not regular. If this Steiner system is invariant under a regular abelian group of automorphisms (which necessarily has orbits on the blocks of sizes V, V, and $2K-1$), then by switching with respect to a block orbit of size V we obtain a strongly regular graph with parameters

$$v = 4K^2,\; k = K(2K-1),\; \lambda = \mu = K(K-1)$$

showing that $(4K^2, 1) \in R$. Steiner systems $S(2, K, K(2K-1))$ are known for $K = 3, 5, 6, 7$ or 2^t, but only for $K = 2, 3, 5, 7$ are systems known that have a regular abelian group of automorphisms. Thus we find $(196, 1) \in R$. The required switching set also exists when the design is resolvable: take the union of K parallel classes. Resolvable designs are known for $K = 3$ or 2^t. ([100], Theorem 2.2.)

See also Bose & Shrikhande [99], Goethals & Seidel [355], §4, and Wallis [719], §5.3.

More recent constructions:

(vii) In JØRGENSEN & KLIN [471] it is shown that $(100, -1) \in R$.

(viii) In HAEMERS [379] it is shown that if there exists a Hadamard matrix of order m, then $(m^2, -1) \in R$.

(ix) In MUZYCHUK & XIANG [583] it is shown that $(4m^4, 1) \in R$ for all m.

(x) In HAEMERS & XIANG [386] it is shown that $(4m^4, -1) \in R$ for all m.

8.1.2 Errata

Nathann Cohen and Dima Pasechnik and others implemented a large number of constructions for strongly regular graphs in `SageMath` (cf. [208], [633]), and encountered flaws in various descriptions.

Ad (iii)

In [137] the condition $4|a$ was omitted from (iii) above. But it seems necessary. (Here [137] referred to [719], which gives the result without this condition in Theorem 5.11, and Corollary 5.12, and in the table on p. 454. It says 'we strengthen a theorem of Goethals and Seidel', but the proof is wrong.)

After correction, (iii) becomes a special case of (ii).

Ad (iv)

Many of the parameter sets that would be produced by (iii) without the condition $4|a$ are also produced by (iv). In this way one finds e.g. $(676, 1) \in R$ and $(900, 1) \in R$. Now in [208] the authors found that also (iv) was wrong, or at least could not be reproduced. The reference for (iv) was [719], Corollary 5.16 which uses the construction of [719], Theorem 5.15. There is a typo in that theorem: the expression given for H misses a minus-sign in front of the C in the bottom-right entry. In [720] the expression is correct. So, construction (iv) stands. (The construction uses Szekeres difference sets, and if one tries to find those in the original Szekeres paper [674] one may stumble over another sign typo: in (4.2) the $-$ should be a $+$.)

Ad (vi)

In the Handbook of Combinatorial Designs the chapter on Hadamard matrices [239] contains (Theorem 1.44, p. 277) the statement

If there is a BIBD$(u(2u-1), 4u^2-1, 2u+1, u, 1)$, then there is a regular graphical Hadamard matrix of order u^2.

with a reference to [650]. Here 'graphical' means 'symmetric with constant diagonal'. However, that reference constructs the Hadamard matrix by observing that the block graph is strongly regular with parameters $(v, k, \lambda, \mu) = (4u^2-1, 2u^2, u^2, u^2)$ and bordering its $(-1, 1)$-adjacency matrix with a constant border, so that the resulting Hadamard matrix is not regular. In [355], Theorem 4.5 and also in [719], Theorem 5.14 this same result is shown without the 'regular'. In [719], p. 454, construction GV is mistakenly starred.

In [386] the statement $(196, \pm 1) \in R$ is attributed to [456], p. 258. As we saw, $(196, 1) \in R$ was shown in [137] as application of [355], Theorem 4.5. It is still unknown whether $(196, -1) \in R$. The proof of Theorem 8.2.26 (iii) in [456] is wrong. For [386], §5 this means that the smallest open case again is $n = 196$.

8.2 Conference matrices and conference graphs

Conference matrices

A *conference matrix* of order n is an $n \times n$ matrix C with diagonal entries 0 and off-diagonal entries ± 1 such that $C^\top C = (n-1)I$. This property does not change if we multiply some rows or columns by -1. Let a *normalized* conference matrix be such a matrix where the off-diagonal entries of the first row and column are all $+1$. Let S be the matrix obtained from a normalized conference matrix by deleting the first row and column. It is called the *core* of C.

Theorem 8.2.1 (DELSARTE, GOETHALS & SEIDEL [751]) *If* $n > 1$ *then* n *is even. If* $n \equiv 2 \pmod 4$*, then* $S = S^\top$*. If* $n \equiv 0 \pmod 4$*, then* $S = -S^\top$*.*

Proof. Since $C^\top C = (n-1)I$ also $CC^\top = (n-1)I$, and rows are mutually orthogonal. Rows 1 and 2 of C agree in $(n-2)/2$ positions, so n is even, say $n = 2m$. Normalize C, and compare rows 2 and 3 in positions 4 up to n. Let $n_{\varepsilon\eta}$ be the number of these positions where row 2 has entry ε and row 3 η, where $\varepsilon, \eta \in \{+, -\}$. If $C_{23} = 1 = -C_{32}$, then the orthogonality of rows 1 and 2 gives $n_{+-} + n_{++} = m-2$; the orthogonality of rows 1 and 3 gives $n_{--} + n_{+-} = m-2$; and the orthogonality of rows 2 and 3 gives $n_{--} + n_{++} = m-2$. Combining these three equations yields $2n_{--} = m-2$, so that $n \equiv 0 \pmod 4$.

Similarly, if $C_{23} = C_{32}$, say $C_{23} = C_{32} = 1$, then $n_{+-} + n_{++} = m-2$ and $n_{--} + n_{+-} = m-1$ and $n_{--} + n_{++} = m-2$ so that $2n_{--} = m-1$ and $n \equiv 2 \pmod 4$. □

Proposition 8.2.2 *Let* $C = \begin{pmatrix} 0 & \mathbf{1}^\top \\ \pm\mathbf{1} & S \end{pmatrix}$ *be a conference matrix of order* $n+1$*. Then* $S \otimes S + I \otimes J - J \otimes I$ *is the core of a conference matrix of order* $n^2 + 1$*.*

Proof. That C is a conference matrix of order $n+1$, is expressed by $SS^\top = nI - J$, $SJ = JS = 0$, $S^\top = \pm S$. □

Conference graphs

Strongly regular graphs with 'half case' parameters $(v, k, \lambda, \mu) = (4t+1, 2t, t-1, t)$ are also known as *conference graphs.* If S is the Seidel matrix of such a graph (of order v), then bordering it with a first column and top row of 1's, with 0 in the top left position, yields a symmetric conference matrix of order $n = v+1$, and conversely, starting with a symmetric conference matrix and normalizing yields the Seidel matrix S of a strongly regular graph with 'half case' parameters.

Theorem 8.2.3 (BELEVITCH [57], see also VAN LINT & SEIDEL [525])
If $(v, k, \lambda, \mu) = (4t+1, 2t, t-1, t)$ *are the parameters of a strongly regular graph, then* v *is the sum of two squares.*

For example, there is no strongly regular graph with parameters $(21, 10, 4, 5)$ because 21 is not the sum of two squares. Similarly, $v = 33$ is ruled out. Of course for all prime powers $v = 4t+1$ one has the Paley graphs (and for $v > 17$ also further examples). The smallest example of a non-prime power v was given by MATHON [544], who constructed a family of examples including $v = 45$. An example for the next smallest case, $v = 65$, was constructed by GRITSENKO [366]. The smallest open case is now $v = 85$. That is, it is unknown whether there exists a symmetric conference matrix of order 86. For a recent survey, see [37].

Switching

Given a conference matrix of order $2m+2$, Proposition 8.2.2 yields conference graphs of order $(2m+1)^2$. We can apply Proposition 1.1.4 and switch w.r.t. the union of m pairwise disjoint $(2m+1)$-cocliques and get strongly regular graphs with parameters $(v, k, \lambda, \mu) = ((2m+1)^2 + 1, m(2m+1), m^2 - 1, m^2)$. For example, we find graphs with parameters $(226, 105, 48, 49)$.

8.3 Symmetric designs

8.3.1 Generalities

A *square design*, or *symmetric design*, is a 2-(v, k, λ) design with equally many points as blocks. Thus, it has v blocks, and k blocks on each point, and $\lambda(v-1) = k(k-1)$.

A necessary condition for existence is

Theorem 8.3.1 (Bruck, Chowla & Ryser [150, 195]) *Suppose a symmetric 2-(v, k, λ) design exists. Then if v is even, $k - \lambda$ is a square. If v is odd, then the equation $X^2 = (k-\lambda)Y^2 + (-1)^{(v-1)/2}\lambda Z^2$ has a nontrivial solution.*

This theorem is the consequence of the matrix equation $A^\top A = (k-\lambda)I + \lambda J$ for the point-block incidence matrix A. The first part is easy, since $k+\lambda(v-1) = k^2$, and $(\det A)^2 = (k + \lambda(v-1))(k-\lambda)^{v-1}$ is a square. For the second part, see, e.g., [398], Theorem 10.3.1.

The adjacency matrix A of a strongly regular graph with $\mu = \lambda$ is the point-block incidence matrix of a symmetric 2-(v, k, λ) design. If $\mu = \lambda + 2$, then $A + I$ is the point-block incidence matrix of a symmetric 2-$(v, k+1, \lambda+2)$ design. Conversely, given such a design with a polarity where no (or all) points are absolute, we find a strongly regular graph again.

The Bruck-Chowla-Ryser theorem is really a result on rational matrices M satisfying $M^\top M = aI + bJ$. Now for a strongly regular graph with parameters (v, k, λ, μ) if one puts $M = 2A + (\mu-\lambda)I$, then $M^2 = ((\mu-\lambda)^2 - 4(\mu-k))I + 4\mu J$, and one finds Theorem 8.2.3 again.

8.3.2 The McFarland difference sets

Let G be an abelian group, and D a subset such that $|D \cap (D+g)| = \lambda$ for all nonzero $g \in G$. Then the design with point set G and set of blocks $\{D + g \mid g \in G\}$ is a symmetric 2-(v, k, λ) design, where $v = |G|$ and $k = |D|$. One says that D has *multiplier* -1 when it is fixed under $d \mapsto -d$. In that case the incidence matrix $A = (A_{gh})$ indexed by points g and blocks $D + h$ is symmetric. Now if $0 \notin D$, then A is the adjacency matrix of a strongly regular graph with parameters (v, k, λ, λ), and if $0 \in D$, then $A - I$ is the adjacency matrix of a strongly regular graph with parameters $(v, k-1, \lambda-2, \lambda)$.

Difference sets with multiplier -1 are rare, and McFarland conjectures that the only possible parameters are $(v, k, \lambda) = (4m^2, 2m^2 \pm m, m^2 \pm m)$ (so-called *Hadamard difference sets*) and $(v, k, \lambda) = (4000, 775, 150)$. In McFarland [554] examples with these latter parameters are constructed. From these one finds strongly regular graphs with the following parameters.

	v	k	λ	μ	comment
a	4000	774	148	150	$0 \in D$
b	4000	775	150	150	$0 \notin D$
c	4000	1935	910	960	Delsarte dual of (a)
d	4000	1984	1008	960	Delsarte dual of (b)
e	3999	1950	925	975	descendant of (c), (d)

The McFarland construction is as follows. Let V be an $(s+1)$-dimensional vector space over $\mathbb{F}_q$, let $r = \frac{q^{s+1}-1}{q-1}$ be the number of hyperplanes in V, and let K be any group of order

$r+1$. For each hyperplane H, let e_H be an arbitrary vector in V, and let k_H be some element of K, where the k_H are distinct. Put $G = V \times K$ and $D = \bigcup_H (H + e_H) \times \{k_H\}$. This is a (v, k, λ) difference set with $v = q^{s+1}(r+1)$, $k = q^s r$ and $\lambda = q^s \frac{q^s-1}{q-1}$.

(Indeed, $|D \cap (D+g)| = (r-1)q^{s-1} = \lambda$ when $g = (u, k)$ with $k \neq 0$, and $|D \cap (D+g)| = \sum_{H \ni u} q^s = \lambda$ when $g = (u, 0)$.)

If $q = 5$, $s = 2$, then $r + 1 = 32$ and we can take K to be an elementary abelian 2-group. Take $e_H = 0$ for all H. Now -1 is a multiplier, and $0 \in D$ when 0 is one of the k_H.

8.4 Latin squares

8.4.1 Generalities

A *Latin square* of order n is an $n \times n$ array, such that each of the n rows and n columns is a permutation of the same n-set. Two Latin squares A and B of the same order n are called *orthogonal* when the n^2 pairs of symbols (A_{ij}, B_{ij}) are distinct (and hence take all possible values). A set of m *MOLS* (*mutually orthogonal Latin squares*) of order n is a set of m Latin squares of order n, pairwise orthogonal.

1234	1234	1234
2143	3412	4321
3412	4321	2143
4321	2143	3412

Table 8.1: Three MOLS of order 4

A *transversal design* TD$(k; n)$ is a partial linear space with kn points and $k + n^2$ lines, with k lines (called *groups*) of size n forming a partition of the point set, and n^2 lines (called *blocks*) of size k, each meeting every group in a single point.

Lemma 8.4.1 *A set of m MOLS of order n is equivalent to a* TD$(m + 2; n)$.

Proof. Given a set of m MOLS A_h of order n $(1 \le h \le m)$, let the $m + 2$ groups be indexed by $\{1, \ldots, m, \mathrm{r}, \mathrm{c}\}$, each containing a copy of the symbol set $\{1, \ldots, n\}$. Let the n^2 blocks correspond to the n^2 positions ij. Let the block belonging to ij contain the point $(A_h)_{ij}$ in group h, and points i, j in groups r, c. One checks that this gives a 1-1 correspondence. □

Lemma 8.4.2 *A set of m MOLS of order $n > 1$ does not exist for $m \ge n$. A set of $n - 1$ MOLS of order n is equivalent to a projective plane of order n together with a designated point.*

Proof. Consider the corresponding TD$(m + 2; n)$. If P is a point outside a block B, then P is on n blocks, $m + 1$ of which meet B. Hence $m + 1 \le n$. If $n = m + 1$ then any two blocks meet, and we obtain a projective plane of order n by adding a 'point at infinity' to each group. □

Let $N(n)$ be the maximum number of mutually orthogonal Latin squares of order n for $n \ge 2$. We see that $N(n) \le n - 1$, and that $N(q) = q - 1$ for prime powers q. It was shown by Tarry [676] that $N(6) = 1$. Euler [309] conjectured that there do not exist two MOLS of order n for any $n \equiv 2 \pmod 4$, but this was disproved by Bose & Shrikhande [98], and together with Parker they proved that $N(n) > 1$ for $n > 6$ ([101]). It is known that $N(12) \geq 5$ ([466]) and $N(14) \geq 4$ ([700]). For a table with lower bounds on $N(n)$, see [1]. Here a small table with lower bounds on $N(n)$ for $n < 100$.

	0	1	2	3	4	5	6	7	8	9
0	∞	∞	1	2	3	4	1	6	7	8
10	2	10	5	12	4	4	15	16	5	18
20	4	5	3	22	7	24	4	26	5	28
30	4	30	31	5	4	6	8	36	4	5
40	7	40	5	42	5	6	4	46	10	48
50	6	5	5	52	5	6	7	7	5	58
60	5	60	5	8	63	7	5	66	5	6
70	6	70	7	72	5	7	6	6	6	78
80	9	80	8	82	6	6	6	6	7	88
90	6	7	6	6	6	6	8	96	6	8

There is some similarity with the problem of constructing sets of mutually unbiased bases (MUBs)[1]. If n factors as $\prod q_i$ where the q_i are prime powers, and $k = \min q_i + 1$, then there exists a TD$(k;n)$ and also a set of k MUBs in $\mathbb{C}^n$. The maximum number of mutually unbiased bases in $\mathbb{C}^n$ is not larger than $n+1$, just like $k \leq n+1$ for a TD$(k;n)$ ([744]). See also [667].

A *net* of degree k and order n is a partial linear space with n^2 points and kn lines, each of size n, where the lines are partitioned into k parallel classes.

Lemma 8.4.3 *A net of degree k and order n is equivalent to a* TD$(k;n)$.

Proof. The points and blocks and groups of the transversal design correspond to the lines and points and parallel classes of the net. □

Yet another equivalent structure is that of an *orthogonal array.* An OA$_\lambda(n, q, t)$ is an $n \times N$ array, where $N = \lambda q^t$, with symbols in an alphabet of size q, such that for any t rows each possible column occurs precisely λ times. One drops λ when $\lambda = 1$, and t when $t = 2$. An OA(n, q) is equivalent to a TD$(q;n)$.

8.4.2 Latin square graphs

Given a transversal design TD$(m;n)$ with $2 \leq m \leq n$, we construct a graph known as a *Latin square graph* LS$_m(n)$ by taking its blocks as vertices, where two blocks are adjacent when they meet. This graph is strongly regular with parameters $(v, k, \lambda, \mu) = (n^2, m(n-1), (m-1)(m-2)+n-2, m(m-1))$ and spectrum k^1 $(n-m)^f$ $(-m)^g$, where $f = m(n-1)$ and $g = (n+1-m)(n-1)$.

We say that a strongly regular graph has *Latin square parameters* LS$_m(n)$ when it has these parameters but is not necessarily derived from a transversal design. Such graphs are also called *pseudo Latin square graphs.* We say that a strongly regular graph has *negative Latin square parameters* NL$_m(n)$ when it has parameters LS$_{-m}(-n)$ (that is, $v = n^2$, $k = m(n+1)$, $\lambda = m(m+3) - n$, $\mu = m(m+1)$, $r = m$, $s = m - n$, $f = (n+1)(n-1-m)$, $g = m(n+1)$).

The complementary graph of a graph with parameters LS$_m(n)$ (and $m < n$) has parameters LS$_{n-m+1}(n)$.

The graph LS$_2(n)$ is the $n \times n$ grid. It is uniquely determined by its parameters for $n \neq 4$. For $2 < m < n-1$ there are many other graphs with the same parameters, for example because there are many nonisomorphic Latin squares and sets of mutually orthogonal Latin squares. But also other graphs with the same parameters exist.

[1] Two bases $\{u_i \mid 1 \leq i \leq n\}$ and $\{v_j \mid 1 \leq j \leq n\}$ of $\mathbb{C}^n$ are called *mutually unbiased* if $|u_i^* v_j| = \frac{1}{\sqrt{n}}$ for all i, j.

Cliques

Latin square graphs $\mathrm{LS}_m(n)$ have maximal cliques of size n, meeting the Hoffman bound. If $n > (m-1)^2$, then each edge lies in a unique clique of size n. For smaller n this needs not be true.

For example, in the $\mathrm{LS}_3(4)$ derived from the addition table of $\mathbb{F}_4$, each edge lies in two 4-cliques. More generally, let q be a prime power and consider in $\mathsf{PG}(3,q)$ the lines and points disjoint from a fixed line L. This is a $\mathrm{TD}(m;n)$ for $m = q+1$ and $n = q^2$ and in the corresponding graph $\mathrm{LS}_m(n)$ each edge $\{M,N\}$ lies in two n-cliques, one consisting of the lines missing L on the point $M \cap N$, the other of the lines missing L in the plane $\langle M,N\rangle$.

If $n = m$ these graphs are complete multipartite graphs $K_{n\times n}$ with n^n cliques of size n, n^{n-2} on each edge.

There do exist two MOLS of order 10, and one finds graphs with parameters $\mathrm{LS}_4(10)$ and maximal cliques of size 10. The Hall-Janko graph (cf. §10.32) also has parameters $\mathrm{LS}_4(10)$ but maximal cliques of size 4, hence is not a Latin square graph.

Switching

The block graph of a $\mathrm{TD}(m;n)$ with $n = 2m-1$ satisfies $k = 2\mu$, and we can apply Proposition 1.1.4. If there are $\frac{1}{2}(n-1)$ parallel classes (sets of n pairwise disjoint blocks), and in particular, if a $\mathrm{TD}(m+1;n)$ exists, then switching yields a strongly regular graph with parameters $(n^2+1, \frac{1}{2}n(n-1), \frac{1}{4}(n-3)(n+1), \frac{1}{4}(n-1)^2)$. In particular this applies to odd prime powers n.

8.4.3 Transversal 3-designs

A transversal design $\mathrm{TD}(k;n)$ is pairwise balanced: two points from different groups determine a unique block. When q is a power of 2, there exist triplewise balanced designs $3TD(q+2;q)$ with $q+2$ groups of size q, and q^3 blocks each meeting all groups in a single point, such that three points from different groups determine a unique block. Now there are q^3 blocks, each point is on q^2 blocks, each pair of points from different groups is on q blocks.

An equivalent object is an $(n,M,d) = (q+2,q^3,q)$ *MDS-code*, with q^3 code words of length $q+2$ and mutual distance at least q. One sees that all distances are q or $q+2$. A construction as linear code is found by labeling the positions with $\mathbb{F}_q \cup \{\sigma,\tau\}$ and the code words with triples $(x,y,z) \in \mathbb{F}_q^3$, where word (x,y,z) has entry $x+ya+za^2$ at position $a \in \mathbb{F}_q$, and entries y and z at positions σ and τ, respectively.

The block graph Γ (where blocks are adjacent when they have nonempty intersection) is strongly regular with parameters $v = q^3$, $k = \frac{1}{2}(q+2)(q^2-1)$, $\lambda = \frac{1}{4}(q^3+5q^2-2q-8)$, $\mu = \frac{1}{4}q(q+1)(q+2)$, $r = \frac{1}{2}(q-2)(q+1)$, $s = -\frac{1}{2}q-1$.

This is the Delsarte dual of the graph (with $v = q^3$, $k = (q-1)(q+2)$) obtained from a hyperoval at infinity.

The second subconstituent of Γ is strongly regular with parameters $v = \frac{1}{2}q(q-1)^2$, $k = \frac{1}{4}(q-2)(q+2)(q+1)$, $\lambda = \frac{1}{8}q^2(q+5)-q-2$, $\mu = \frac{1}{8}q(q+1)(q+2)$, $r = \frac{1}{4}(q-4)(q+1)$, $s = -\frac{1}{2}q-1$. For example, for $q = 8$ one finds $(v,k,\lambda,\mu) = (196,135,94,90)$.

See also Huang, Huang & Lin [444].

8.5 Quasi-symmetric designs

A *quasi-symmetric* design is a 2-design such that the size of the intersection of two distinct blocks takes two values. Consider a quasi-symmetric 2-(v,k,λ) design, with block intersection numbers x,y, and assume that $1 < k < v$. The number of blocks on each point is $r = \lambda(v-1)/(k-1)$ and the total number of blocks is $b = vr/k$. Let N be the point-block incidence matrix. Let A be the 0-1 matrix indexed by the blocks with (B,C)-entry 1 precisely when $|B \cap C| = x$. Then $NN^\top = rI + \lambda(J-I)$ and $N^\top N = kI + xA + y(J-I-A)$.

Now A is the adjacency matrix of a strongly regular graph. Indeed, $NN^\top$ has two different eigenvalues $r-\lambda$ and kr, so $N^\top N$ has three eigenvalues 0, $r-\lambda$ and rk, and also $A = \frac{1}{x-y}(N^\top N - (k-y)I - yJ)$ has three eigenvalues, namely $K = \frac{(r-1)k-(b-1)y}{x-y}$, $R = \frac{r-\lambda-k+y}{x-y}$ and $S = -\frac{k-y}{x-y}$ with multiplicities 1, $v-1$, and $b-v$, respectively.

We find a strongly regular graph with parameters (V,K,Λ,M) and eigenvalues R,S with multiplicities F,G where $V = b$ and K,R,S are as above (for $x > y$) so that $F = v-1$ and $G = b-v$. The values of Λ, M follow from $R+S = \Lambda - M$ and $RS = M-K$.

For example, the Steiner system $S(4,7,23)$ has $b = 253$, $r = 77$, $\lambda = 21$. It has block intersection sizes $y = 1$ and $x = 3$. The graph on the blocks, adjacent when they meet in 3 points, is strongly regular with parameters $(V,K,\Lambda,M) = (253,140,87,65)$ with spectrum $140^1\ 25^{22}\ (-3)^{230}$ (cf. §10.56).

Complement

The complementary design (found by replacing each block by its complement) is a quasi-symmetric 2-$(v, v-k, b-2r+\lambda)$ design with block intersection numbers $v-2k+x, v-2k+y$ and the same graph.

History

Quasi-symmetric designs were introduced by Goethals & Seidel [354], [355].

8.5.1 The Calderbank-Cowen inequality

The following result allows one to express the number of blocks b of a quasi-symmetric 2-design in terms of the parameters v,k,x,y.

Proposition 8.5.1 (Calderbank [167]) *Every* 1-(v,k,r) *design with* b *blocks, and two block intersection numbers* x,y*, satisfies*

$$1 - \frac{1}{b} \le \frac{k(v-k)}{v(v-1)} \left(\frac{(v-1)(2k-x-y) - k(v-k)}{(k-x)(k-y)} \right)$$

with equality if and only if the design is a 2*-design.* □

8.5.2 Neumaier's inequality

Let Γ be the strongly regular graph on the blocks of a quasi-symmetric 2-(v,k,λ) design $(X,\mathscr{B})$ with block intersection numbers x,y, where blocks are adjacent if they meet in x points. Let $r = \lambda(v-1)/(k-1)$ be the replication number (number of blocks on any point).

Proposition 8.5.2 (NEUMAIER [589]) *The sets of all blocks $S(x)$ containing a fixed point x are regular sets in Γ of size r, degree $d = \frac{(\lambda-1)(k-1)-(r-1)(y-1)}{x-y}$ and nexus $e = \frac{\lambda k - ry}{x-y}$.*

Proof. Clearly, $|S(x)| = r$. For $B \in S(x)$, with d_B neighbors in $S(x)$, count the number of pairs (y, C) with $y \neq x$ and $C \neq B$ and $x, y \in C$ and $y \in B$. This number is $(k-1)(\lambda-1)$ and also $d_B(x-1) + (r - d_B - 1)(y-1)$ so that $d = d_B$ does not depend on B and has the stated value. Similarly, for $B \notin S(x)$, with e_B neighbors in $S(x)$, we find $k\lambda = e_B x + (r - e_B)y$, so that e_B does not depend on B and has the stated value. □

Proposition 8.5.3 (NEUMAIER [589]) *The parameters of $(X, \mathscr{B})$ satisfy*

$$B(B - A) \leq AC,$$

where

$$A = (v-1)(v-2), \quad B = r(k-1)(k-2)$$

$$C = rd(x-1)(x-2) + r(r-1-d)(y-1)(y-2).$$

Equality holds if and only if $(X, \mathscr{B})$ is a 3-design.

Proof. For distinct points x, y, z, let λ_{xyz} denote the number of blocks containing these three points. Fix x and sum over all ordered pairs y, z with x, y, z distinct. One obtains $\sum 1 = A$, $\sum \lambda_{xyz} = B$, $\sum \lambda_{xyz}(\lambda_{xyz} - 1) = C$. Now $0 \leq \sum(\lambda_{xyz} - \frac{B}{A})^2 = C + B - \frac{B^2}{A}$. □

For example, there is no 2-$(24, 6, 10)$ design with $x = 2$, $y = 0$.

An equivalent inequality was given by Calderbank as a consequence of the linear programming bound in the Johnson scheme.

Proposition 8.5.4 (CALDERBANK [167]) *Let $x' = k - x$ and $y' = k - y$. Then*

$$(v-1)(v-2)x'y' - k(v-k)(v-2)(x'+y') + k(v-k)(k(v-k)-1) \geq 0,$$

with equality if and only if the design is a 3-design. □

An equivalent inequality was derived by Hobart as a consequence of inequalities for coherent configurations.

Proposition 8.5.5 (HOBART [430]) *The parameters of a quasisymmetric design (and its strongly regular intersection-x graph) satisfy*

$$\frac{v-2}{v}\left(1 + \frac{R^3}{K^2} - \frac{(R+1)^3}{(b-K-1)^2}\right) - \frac{(v-2k)^2\lambda}{k^2(k-1)(v-k)} \geq 0. \qquad \square$$

8.5.3 No triangular graph

Proposition 8.5.6 *(i) A quasi-symmetric design with* $b = v(v-1)/2$ *and* $1 < k < v-1$ *is either the trivial* 2-$(v, 2, 1)$ *design or its complementary* 2-$(v, v-2, \binom{v-2}{2})$ *design, or the unique* 4-$(23, 7, 1)$ *design.*

(ii) In particular, if also $2 < k < v-2$*, then* Γ *is not a triangular graph.*

Proof. The triangular graph $T(m)$ has multiplicities $F = m-1$ and $G = m(m-3)/2$, so that $v = m$ and $b = m(m-1)/2$, and $b = v(v-1)/2$. By [182] (1.52), if $4 \le k \le v-2$, then a quasi-symmetric 2-design with $b = v(v-1)/2$ is a 4-design, and by *ibid.* (1.54) this can happen only for 4-$(23, 7, 1)$ and for $k = v-2$. But the block graph of the former is not triangular. □

For example, there is no quasi-symmetric 2-$(27, 7, 21)$ design with $x = 3$, $y = 1$ and no 2-$(59, 27, 351)$ with $x = 15$, $y = 11$. Coster & Haemers [236] give conditions for Γ to be the complement of the triangular graph.

8.5.4 Examples

A. Steiner 2-designs

In a Steiner 2-design $S(2, m, u)$ two blocks meet in at most one point, so that we have the above situation with $x = 1$ and $y = 0$ (when $u > m^2 + m + 1$, so that both cases occur). We find a strongly regular graph with parameters $v = u(u-1)/m(m-1)$, $k = m(u-m)/(m-1)$, $\lambda = (m-1)^2 + (u-2m+1)/(m-1)$, $\mu = m^2$, $r = (u-m^2)/(m-1)$, and $s = -m$.

For example, the lines in $\mathsf{PG}(3, q)$, adjacent when they meet, form a strongly regular graph with parameters $v = (q^2+1)(q^2+q+1)$, $k = q(q+1)^2$, $\lambda = 2q^2+q-1$, $\mu = (q+1)^2$, $r = q^2-1$, $s = -q-1$.

Infinite families of known Steiner 2-designs include the following.

(a) For block size $k = 3, 4, 5$, Steiner systems $S(2, k, v)$ exist if and only if $v \equiv 1$ or k (mod $k(k-1)$). For larger block size there are only partial results (cf. [2]).

(b) For each prime power q there exist systems $S(2, q, q^n)$ (for example, the affine space $\mathsf{AG}(n, q)$), $S(2, q+1, q^d + \dots + 1)$ (for example, the projective space $\mathsf{PG}(d, q)$), and $S(2, q+1, q^3+1)$ (for example, the Hermitian unitals in $\mathsf{PG}(2, q^2)$).

(c) For q a power of 2, and $a \,|\, q$ there exist systems $S(2, a, qa-q+a)$ derived from Denniston's maximal arcs (subsets of size $qa-q+a$ of $\mathsf{PG}(2, q)$ such that the projective lines meet it in either 0 or a points).

B. Strongly resolvable 2-designs

A *resolution* of a 2-(v, k, λ) design with b blocks, r on each point, is a partition of the set of blocks into classes that are 1-designs. The number of classes of a resolution is at most $b - v + 1$. When equality holds, each class has the same size m, and there are constants x, y such that blocks in different classes meet in x points, and blocks from the same class meet in y points (Hughes & Piper [446]). Such designs are called *strongly resolvable.* One has $x = \frac{k^2}{v}$ and $y = x - \frac{r-\lambda}{m}$ and $m(b-v+1) = b$.

For example, the planes in $\mathsf{AG}(3, q)$ form a strongly resolvable design with $v = q^3$, $k = q^2$, $\lambda = q+1$, $r = q^2+q+1$, $b = q(q^2+q+1)$, $m = q$, $x = q$, $y = 0$.

The corresponding strongly regular graphs are imprimitive (complete multipartite, or union of cliques).

C. Residuals of biplanes

Let $(X, \mathscr{B})$ be a symmetric 2-(v, k, λ) design, and fix $B_0 \in \mathscr{B}$. The *derived design* of $(X, \mathscr{B})$ at B_0 is the 2-$(k, \lambda, \lambda - 1)$ design $(B_0, \{B \cap B_0 \mid B \in \mathscr{B},\ B \neq B_0\})$. The *residual design* of $(X, \mathscr{B})$ at B_0 is the 2-$(v - k, k - \lambda, \lambda)$-design $(X \setminus B_0, \{B \setminus B_0 \mid B \in \mathscr{B},\ B \neq B_0\})$.[2] For example, the residual of $\mathsf{PG}(2, n)$ is $\mathsf{AG}(2, n)$.

A *biplane* is a symmetric 2-(v, k, λ) design with $\lambda = 2$. A biplane has $v = 1 + \binom{k}{2}$. Biplanes are known for $k = 2, 3, 4, 5, 6, 9, 11, 13$ ([12], [19], [450], [483]).

The residual of a biplane is a 2-$(\binom{k-1}{2}, k-2, 2)$ design with block intersection numbers 1 and 2, hence is quasi-symmetric. By Hall & Connor [399], any such design can be extended to a biplane.

D. Quasi-symmetric designs from 5-designs

In §6.2.1 we made quasi-symmetric designs with parameters 2-$(21, 6, 4)$, 2-$(22, 6, 5)$ (with $x = 2$, $y = 0$), and 2-$(21, 7, 12)$, 2-$(22, 7, 16)$, 2-$(23, 7, 21)$ (with $x = 3$, $y = 1$) from the Steiner system $S(5, 8, 24)$ (with intersection numbers 0, 2, 4).

Tonchev [703] observed that one can also start from a 5-(48,12,8) design (with intersection numbers 0, 2, 4, 6) and shorten three times to obtain a quasi-symmetric 2-$(45, 9, 8)$ design with intersection numbers 1, 3.

E. Codimension 2 subspaces of projective spaces

Let V be a vector space of dimension n over $\mathbb{F}_q$. Any two subspaces of dimension $n - 2$ meet in either an $(n - 4)$- or an $(n - 3)$-space.

For example, the planes in $\mathsf{PG}(4, 2)$ give a quasi-symmetric 2-(31,7,7) design with $x = 3$, $y = 1$.

F. Designs with the symmetric difference property

Kantor [477] says that a design has the *symmetric difference property* when the symmetric difference $B \,\Delta\, C \,\Delta\, D$ of any three blocks is either a block or the complement of a block. He shows that a symmetric design with the symmetric difference property has parameters 2-$(2^{2m},\ 2^{2m-1} + \epsilon 2^{m-1},\ 2^{2m-2} + \epsilon 2^{m-1})$, where the complement of a design with $\epsilon = 1$ has $\epsilon = -1$.

An example of such designs is given by the tensor product of m copies of the Hadamard matrix $J_4 - 2I_4$ of order 4, if one interprets this matrix as the point-block incidence matrix with ϵ $(-\epsilon)$ denoting incidence (nonincidence). Let V be a $2m$-dimensional vector space over $\mathbb{F}_2$, and Q a nondegenerate quadratic form with maximal (minimal) Witt index for $\epsilon = 1$ $(\epsilon = -1)$. Let B be the set of singular vectors of Q. This same design can also be constructed by taking V as the point set, and the translates $B + v$ $(v \in V)$ as blocks. Its group of automorphisms is $2^{2m}.\mathsf{Sp}(2m, 2)$.

The derived and residuals of these designs are quasi-symmetric (Cameron). For example, from 2-$(64, 28, 12)$ one obtains quasi-symmetric 2-$(28, 12, 11)$ and 2-$(36, 16, 12)$ designs. Quasi-symmetry follows from the symmetric difference

[2] These are not to be confused with the derived/residual of a t-design, which are $(t - 1)$-designs; in t-design terminology, the designs here are the dual of the derived/residual of the dual.

property: given three blocks B, C, D where $|B \cap C \cap D| = a$ one finds that $|B \,\Delta\, C \,\Delta\, D| = 12 + 4a$ so that $a \in \{4, 6\}$.

8.5.5 Classification

NEUMAIER [589] defines the block graph of a quasi-symmetric design with intersection numbers x, y where $x > y$, as the graph with the blocks as vertices, adjacent when they meet in x points. He shows:

Proposition 8.5.7 *(i) A quasi-symmetric design with disconnected block graph is a multiple (union of identical copies) of a symmetric design.*

(ii) A quasi-symmetric design with complete multipartite block graph is a strongly resolvable design.

(iii) A quasi-symmetric design with intersection numbers 0, 1 *is a Steiner* 2-*design.*

(iv) A quasi-symmetric design with intersection numbers 1, 2 *is the residual of a biplane, or the* 2-(5, 3, 3) *design.* □

8.5.6 Table

We give a small table with exceptional parameter sets, i.e., parameter sets of quasi-symmetric designs satisfying Neumaier's inequality and Proposition 8.5.3, and not in one of the classes of Proposition 8.5.7. Since the complement of a quasi-symmetric design is quasi-symmetric again, we can restrict ourselves to the cases with $k \leq v/2$. The table covers the parameters with $v \leq 100$. Column ex(istence): = graph does not exist, − design does not exist, + design exists, ! design is unique, 5 there are 5 nonisomorphic such designs.

v	k	λ	y	x	V	K	Λ	M	R	S	ex	ref
19	7	7	1	3	57	42	31	30	4	−3	=	[732]
19	9	16	3	5	76	45	28	24	7	−3	=	[89]
20	8	14	2	4	95	54	33	27	9	−3	=	[21]
20	10	18	4	6	76	35	18	14	7	−3	=	[20]
21	6	4	0	2	56	45	36	36	3	−3	!	Ex. D
21	7	12	1	3	120	77	52	44	11	−3	!	Ex. D
21	8	14	2	4	105	52	29	22	10	−3	−	[165]
21	9	12	3	5	70	27	12	9	6	−3	−	[165]
22	6	5	0	2	77	60	47	45	5	−3	!	Ex. D
22	7	16	1	3	176	105	68	54	17	−3	!	Ex. D
22	8	12	2	4	99	42	21	15	9	−3	−	[165]
23	7	21	1	3	253	140	87	65	25	−3	!	Ex. D
24	8	7	2	4	69	20	7	5	5	−3	−	[121]
28	7	16	1	3	288	105	52	30	25	−3	−	[702]
28	12	11	4	6	63	32	16	16	4	−4	+	Ex. F
29	7	12	1	3	232	77	36	20	19	−3	−	[166]
31	7	7	1	3	155	42	17	9	11	−3	5	Ex. E, [703]
33	9	6	1	3	88	60	41	40	5	−4	−	[165]
33	15	35	6	9	176	45	18	9	12	−3	?	
35	7	3	1	3	85	14	3	2	4	−3	−	[165]
35	14	13	5	8	85	14	3	2	4	−3	?	
36	16	12	6	8	63	30	13	15	3	−5	+	Ex. F
37	9	8	1	3	148	84	50	44	10	−4	−	[415]
39	12	22	3	6	247	54	21	9	15	−3	?	
41	9	9	1	3	205	96	50	40	14	−4	?	
41	17	34	5	8	205	136	93	84	13	−4	−	[166]

continued...

v	k	λ	y	x	V	K	Λ	M	R	S	ex	ref
41	20	57	8	11	246	140	85	72	17	−4	?	
42	18	51	6	9	287	160	96	80	20	−4	?	
42	21	60	9	12	246	119	64	51	17	−4	?	
43	16	40	4	7	301	192	128	112	20	−4	?	
43	18	51	6	9	301	150	83	66	21	−4	−	[166]
45	9	8	1	3	220	84	38	28	14	−4	!	Ex. D, [703], [414]
45	15	42	3	6	396	260	178	156	26	−4	?	
45	18	34	6	9	220	84	38	28	14	−4	?	
45	21	70	9	13	330	63	24	9	18	−3	?	
46	16	8	4	6	69	48	32	36	2	−6	?	
46	16	72	4	7	621	320	184	144	44	−4	?	
49	9	6	1	3	196	60	23	16	11	−4	+	[445]
49	13	13	1	4	196	156	125	120	9	−4	?	
49	16	45	4	7	441	176	85	60	29	−4	?	
51	15	7	3	5	85	54	33	36	3	−6	−	[165]
51	21	14	6	9	85	70	57	60	2	−5	−	[166]
52	16	20	4	7	221	64	24	16	12	−4	−	[166]
55	15	7	3	5	99	48	22	24	4	−6	?	
55	15	63	3	6	891	320	148	96	56	−4	?	
55	16	40	4	8	495	78	29	9	23	−3	?	
56	12	9	0	3	210	176	148	144	8	−4	−	[578]
56	15	42	3	6	616	205	90	57	37	−4	−	[166]
56	16	6	4	6	77	16	0	4	2	−6	+	[704], [577]
56	16	18	4	8	231	30	9	3	9	−3	+	[501]
56	20	19	5	8	154	105	72	70	7	−5	?	
56	21	24	6	9	176	105	64	60	9	−5	−	[166]
57	9	3	1	3	133	24	5	4	5	−4	?	
57	12	11	0	3	266	220	183	176	11	−4	−	[578]
57	15	30	3	6	456	140	58	36	26	−4	?	
57	21	10	7	9	76	21	2	7	2	−7	=	[378]
57	21	25	6	9	190	105	60	55	10	−5	?	
57	24	23	9	12	133	44	15	14	6	−5	−	[166]
57	27	117	12	17	532	81	30	9	24	−3	−	[166]
60	15	14	3	6	236	55	18	11	11	−4	?	
60	30	58	14	18	236	55	18	11	11	−4	−	[168]
61	21	21	6	9	183	70	29	25	9	−5	?	
61	25	160	9	13	976	300	128	76	56	−4	?	
63	15	35	3	7	651	90	33	9	27	−3	+	Ex. E
63	18	17	3	6	217	150	105	100	10	−5	?	
63	24	92	8	12	651	182	73	42	35	−4	?	
64	24	46	8	12	336	80	28	16	16	−4	+	[80], [472]
65	20	19	5	8	208	75	30	25	10	−5	?	
66	30	29	12	15	143	72	36	36	6	−6	+	[105]
69	18	30	3	6	460	255	150	130	25	−5	?	
69	33	176	15	21	782	99	36	9	30	−3	−	[165]
70	10	6	0	2	322	225	160	150	15	−5	−	[168]
70	30	58	10	14	322	225	160	150	15	−5	?	
71	14	39	2	5	1065	266	103	54	53	−4	?	
71	31	93	11	15	497	310	201	180	26	−5	?	
71	35	136	15	19	568	315	186	160	31	−5	?	
72	18	34	3	6	568	279	150	124	31	−5	?	
72	32	124	12	16	639	350	205	175	35	−5	?	
72	36	140	16	20	568	279	150	124	31	−5	?	
73	10	15	1	4	876	105	38	9	32	−3	?	
73	28	126	10	16	876	105	38	9	32	−3	?	
73	32	124	12	16	657	328	179	148	36	−5	?	
75	27	117	9	15	925	108	39	9	33	−3	?	
76	16	12	1	4	285	220	171	165	11	−5	−	[166]
76	26	52	6	10	456	325	236	220	21	−5	?	
76	30	116	10	14	760	345	176	140	41	−5	?	

continued...

v	k	λ	y	x	V	K	Λ	M	R	S	ex	ref
76	36	21	16	18	95	40	12	20	2	−10	=	[21]
76	36	42	16	20	190	45	12	10	7	−5	?	
76	36	105	16	21	475	96	32	16	20	−4	−	[166]
77	33	24	12	15	133	88	57	60	4	−7	−	[30], [78]
78	26	100	6	10	924	611	418	376	47	−5	?	
78	28	216	8	12	1716	875	490	400	95	−5	?	
78	33	64	13	18	364	66	20	10	14	−4	−	[166]
78	36	30	15	18	143	70	33	35	5	−7	+	[105]
79	19	57	4	9	1027	114	41	9	35	−3	?	
81	30	290	10	15	2160	476	178	84	98	−4	?	
81	39	247	18	25	1080	117	42	9	36	−3	?	
84	28	54	8	12	498	161	64	46	23	−5	?	
85	15	4	1	3	136	105	80	84	3	−7	?	
85	15	6	0	3	204	175	150	150	5	−5	?	
85	35	34	10	15	204	175	150	150	5	−5	?	
85	40	52	16	20	238	162	111	108	9	−6	?	
85	40	130	15	20	595	450	345	325	25	−5	?	
87	24	92	6	12	1247	126	45	9	39	−3	?	
88	22	14	2	6	232	198	169	168	6	−5	?	
88	28	63	8	13	638	112	36	16	24	−4	−	[166]
88	33	32	8	13	232	198	169	168	6	−5	−	[166]
88	40	65	16	20	319	168	92	84	14	−6	?	
91	21	18	3	6	351	210	129	120	15	−6	?	
91	26	160	6	10	2016	715	314	220	99	−5	?	
91	28	18	7	10	195	98	49	49	7	−7	−	[166]
91	35	51	11	15	351	210	129	120	15	−6	?	
91	36	56	12	16	364	198	112	102	16	−6	?	
91	39	19	15	17	105	78	55	66	1	−12	−	[165]
91	40	52	16	20	273	102	41	36	11	−6	?	
92	26	100	6	10	1288	429	180	124	61	−5	?	
92	27	108	7	12	1288	234	80	34	50	−4	−	[166]
93	18	51	3	8	1426	135	48	9	42	−3	?	
93	30	145	9	16	1426	135	48	9	42	−3	?	
93	45	330	21	29	1426	135	48	9	42	−3	?	
93	45	825	20	25	3565	1260	555	385	175	−5	?	
96	36	42	12	16	304	108	42	36	12	−6	?	
96	40	78	16	24	456	35	10	2	11	−3	=	§8.18
99	15	5	1	3	231	140	85	84	8	−7	−	[165]
99	36	20	12	15	154	48	12	16	4	−8	?	
100	12	5	0	2	375	264	188	180	14	−6	?	
100	36	105	12	18	825	128	40	16	28	−4	−	[166]

Table 8.2: Parameters of sporadic quasi-symmetric designs

8.5.7 Parameter conditions from coding theory

We give some of the necessary conditions used to rule out certain parameter sets for quasi-symmetric designs. Notation is as above.

Lemma 8.5.8 *(i)* $k = y \pmod{x-y}$ *and* $r = \lambda \pmod{x-y}$.

(ii) If a set Z of size w meets all blocks in an even number of points, then $w(w-1)\lambda - wr = 0 \pmod 8$.

(iii) If a set Z of size w meets all blocks in an odd number of points, then $w(w-1)\lambda + wr - b = 0 \pmod 8$.

Proof. (i) This follows directly since $R - S = \frac{r-\lambda}{x-y}$ and $S = -\frac{k-y}{x-y}$ are integral.

(ii), (iii): Let Z meet n_i blocks in i points. Then $\sum n_i = b$, and $\sum i n_i = wr$, and $\sum \binom{i}{2} n_i = \binom{w}{2}\lambda$. Now (ii) follows from $8 \mid \sum i(i-2)n_i = w(w-1)\lambda - wr$ and (iii) from $8 \mid \sum (i-1)(i+1)n_i = w(w-1)\lambda + wr - b$. □

A binary code is called *doubly even* when all code words have a weight divisible by 4. A doubly even binary code is self-orthogonal. We need the following well-known result.

Proposition 8.5.9 ([343], [538]) *Let C be a doubly even binary code.*
(i) If C has parameters $[v, v/2]$, then $v = 0 \pmod 8$.
(ii) If C has parameters $[v, (v-1)/2]$, then $v = \pm 1 \pmod 8$. □

Proposition 8.5.10 (Calderbank [165]) *Suppose $r \neq \lambda \pmod 4$.*
(i) If $x = y = 0 \pmod 2$, then $k = 0 \pmod 4$ and $v = \pm 1 \pmod 8$.
(ii) If $x = y = 1 \pmod 2$, then $k = v \pmod 4$ and $v = \pm 1 \pmod 8$.

Proof. (i) Let C be the binary code spanned by the characteristic vectors of the blocks. Then C is self-orthogonal since k is even by part (i) of the lemma. Let C' be a maximal self-orthogonal code containing C. Apply Lemma 8.5.8 (ii) with Z the support of a code word in C'. If $w = 2 \pmod 4$, then $r = \lambda \pmod 4$ which was excluded. Hence C' is doubly even, and $k = 0 \pmod 4$. If C' is self-dual then its length v is divisible by 8, contradicting $r(k-1) = \lambda(v-1)$ and $r \neq \lambda \pmod 4$. Hence C' has dimension $(v-1)/2$ and $v = \pm 1 \pmod 8$.

(ii) By part (i) of the lemma, $k = 1 \pmod 2$. If v is even, then by $r(k-1) = \lambda(v-1)$ and $r = \lambda \pmod 2$ it follows that $\lambda = 0 \pmod 4$ and $r = 2 \pmod 4$. Let Z be the complement of a block. Then Z meets all blocks evenly, and by part (ii) of the lemma $|Z| = 0 \pmod 4$. Hence $k = v \pmod 4$ and v is odd. Let C_1 be the binary code spanned by the blocks, extended by a parity check. It is self-orthogonal, and contained in a self-dual $[v+1, (v+1)/2]$ code. Shorten that latter code again to obtain a self-orthogonal $[v, (v-1)/2]$ code. Again by part (ii) of the lemma, this code is doubly even, which shows that $v = \pm 1 \pmod 8$. □

For example, there is no quasi-symmetric 2-$(21, 8, 14)$ design with intersection numbers $2, 4$, and no quasi-symmetric 2-$(21, 9, 12)$ design with intersection numbers $3, 5$.

Proposition 8.5.11 (Calderbank & Frankl [168]) *Suppose $k = 2 \pmod 4$ and $x = y = 0 \pmod 2$. Then $w(w-1)\lambda + wr - b = 0 \pmod 8$ has a solution w.*

Proof. Let C be the binary code spanned by the characteristic vectors of the blocks. Then C is self-orthogonal, but not doubly even. Let K be the doubly even kernel of C. Then K is generated by the sums of two blocks, and has codimension 1 in C. Let $z \in K^\perp \setminus C^\perp$. Then z (viewed as a set of points) meets each block in an odd number of points. Now apply Lemma 8.5.8 (iii). □

For example, there is no 2-$(70, 10, 6)$ design with intersection numbers $0, 2$. Here $b = 2 \pmod 8$ and $r = \lambda = 6 \pmod 8$ but $6w(w-1) + 6w + 6 = 6(w^2+1) \neq 0 \pmod 8$.

In Calderbank [166] conditions are given for the case where x, y are congruent modulo an odd prime p. In Bagchi [30] and in Blokhuis & Calderbank [78] conditions are given obtained by use of the Smith normal form.

8.5.8 Haemers cocliques

Let a *Haemers coclique* C in a strongly regular graph Γ be a coclique that has equality both in the Hoffman and in the Cvetković bound. We repeat Proposition 1.1.8, adding some detail.

Proposition 8.5.12 (HAEMERS [376], Theorem 2.1.7; see also [132], 9.4.1 (iii)) *Let Γ be a strongly regular graph with point set X and eigenvalues k, r, s with multiplicities $1, f, g$ (where $k > r > s$), and let C be a coclique in Γ with $|C| = 1 + \frac{v-k-1}{r+1} = g$. Then the graph Γ' induced on $X \setminus C$ is strongly regular with eigenvalues $k' = k + s$, $r' = r$, $s' = r + s$ and multiplicities 1, $f - g + 1$, $g - 1$, respectively.*

The restriction $1 + \frac{v-k-1}{r+1} = g$ enables one to express the parameters in two variables, say r, m, where $m = -s$. We find

$$k = \tfrac{m(m+r)}{r+1}, \quad \mu = \tfrac{m(m-r^2)}{r+1}, \quad \lambda = \tfrac{(m-1)(m-r^2-r)}{r+1}, \quad g = \tfrac{m^2+rm-r^2-r}{m-r^2}.$$

For the graph Γ' we find

$$k' = \tfrac{m(m-1)}{r+1}, \quad \mu' = \tfrac{(m-r^2)(m-r-1)}{r+1}, \quad \lambda' = \tfrac{(m-r-2)(m-r-r^2)}{r+1}.$$

In this situation one finds a quasisymmetric design $(C, X \setminus C)$ where a block $x \in X \setminus C$ is incident with a point $c \in C$ when $c \sim x$. The number of points is $|C| = g$, the block size is m, and the intersection numbers are $\lambda - \lambda' = m - r^2 - r$ and $\mu - \mu' = m - r^2$. In particular, the coding theory restrictions for quasisymmetric designs apply.

These same parameter values were derived by SHRIKHANDE [647] under slightly different hypotheses.

If C_1, C_2 are two Haemers cocliques in Γ, then $|C_1 \cap C_2| = \frac{r(m-1)}{m-r^2}$. Indeed, one finds a symmetric design $(C_1 \setminus C_2, C_2 \setminus C_1)$ with block size m and block intersection number $m - r^2$ and using '$\lambda(v-1) = r(k-1)$' in this situation yields $|C_1 \setminus C_2| = \frac{m^2-r^2}{m-r^2}$, and $|C_1 \cap C_2|$ follows.

The Hoffman bound for cocliques in Γ' is $\frac{m^2-r^2}{m-r^2}$ which equals $|C_1 \setminus C_2|$, so any point of $X \setminus C_2$ outside $C_1 \setminus C_2$ is adjacent to $m - r$ vertices of $C_1 \setminus C_2$, and hence to r vertices of $C_1 \cap C_2$.

If C_1, C_2, C_3 are three Haemers cocliques in Γ, then $|C_1 \cap C_2 \cap C_3| = \frac{r^2-r}{m-r^2}$. Indeed, this follows from $m \cdot |C_1 \cap C_2 \setminus C_3| = r \cdot |C_3 \setminus (C_1 \cup C_2)|$.

ADM et al. [5] discuss this situation, and observe (the above, and also) that Γ has at most $g + 1$ Haemers cocliques. Indeed, the characteristic functions of these cocliques are linearly independent (their Gram matrix is nonsingular) and live in $W + \langle \mathbf{1} \rangle$, where W is the s-eigenspace (cf. the proof of Proposition 1.1.3), which has dimension $g + 1$.

Krein graphs without triangles

Consider strongly regular graphs Σ without triangles and with $q^2_{22} = 0$. All parameters can be expressed in terms of a single variable, say r. Let us use capitals for the parameters of Σ. We find $V = r^2(r+3)^2$, $K = r^3 + 3r^2 + r$, $\Lambda = 0$, $M = r^2 + r$, $R = r$, $S = -r^2 - 2r$, $F = (r^2 + 2r - 1)(r^2 + 3r + 1)$, $G = r(r^2 + 3r + 1)$. (That is, we have Smith graphs with parameters $\mathrm{NL}_r(r^2 + 3r)$.) Since $q^2_{22} = 0$, Theorem 1.3.11 implies that each second subconstituent $\Gamma = \Sigma_2(x)$ of Σ is strongly regular, and one finds that Γ has parameters $v = (r^2 + 2r - 1)(r^2 + 3r + 1)$, $k = r^3 + 2r^2$, $\lambda = 0$, $\mu = r^2$, $r = r$, $s = -r^2 - r$,

$f = (r^2+r-1)(r^2+3r+1)$, $g = (r+1)(r^2+2r-1) = K-1$. We see that Γ has parameters as above (with $m = r^2+r$), and the sets $\Sigma(y) \setminus \{x\}$ for $y \sim x$ form a system of $g+1$ Haemers cocliques in Γ, so that these are the only ones. For $r = 1$ the graph Σ is the complement of the Clebsch graph, for $r = 2$ the Higman-Sims graph, and no such graph exists for $r = 3$. Nothing is known for $r > 3$.

Unitary two-graphs

Taylor [677] constructed unitary two-graphs (cf. §8.10.1) that after suitably switching yield a strongly regular graph with parameters $v = q^3+1$, $k = \frac{1}{2}q(q^2+1)$, $r = \frac{1}{2}(q-1)$, $s = -\frac{1}{2}(q^2+1)$. A graph with these parameters contains at most one Haemers coclique when $q > 3$ since the intersection of two would have nonintegral size $\frac{(q-1)^2}{q+1}$. For $q = 3$ these are the parameters of $\overline{T(8)}$ which does have $g+1 = 8$ Haemers cocliques.

8.6 Partial geometries

A *partial geometry* $pg(K, R, T)$ is a partial linear space $(X, \mathscr{L})$ such that each line has K points, each point is on R lines, and given a point x outside a line L, there are precisely T lines on x meeting L. (In the literature one also meets the notation $pg(s, t, \alpha)$, where $K = s+1$, $R = t+1$, $T = \alpha$.) We shall assume that $K, R > 1$ and $T > 0$.

The dual of a $pg(K, R, T)$ is a $pg(R, K, T)$.

The collinearity graph Γ of a $pg(K, R, T)$ is strongly regular (or complete) with parameters

$$\begin{aligned}
v &= K + K(K-1)(R-1)/T, \\
k &= R(K-1), \\
\lambda &= (R-1)(T-1) + K - 2, \\
\mu &= RT, \\
r &= K - T - 1, \\
s &= -R, \\
f &= \frac{K(K-1)R(R-1)}{T(K+R-T-1)}, \\
g &= \frac{(K-1)(K-T)(T+(K-1)(R-1))}{T(K+R-T-1)}.
\end{aligned}$$

The lines form maximal cliques in Γ meeting the Hoffman bound: $K = 1 + k/(-s)$. Conversely, if a strongly regular graph possesses a collection $\mathscr{C}$ of cliques meeting the Hoffman bound such that each edge is in a unique such clique, then $(X, \mathscr{C})$ is a partial geometry.

Clearly $1 \le T \le \min(K, R)$.

If $T = 1$, the partial geometry is a generalized quadrangle $\mathsf{GQ}(s, t)$, where $K = s+1$, $R = t+1$.

If $T = K$, the partial geometry is a 2-$(v, K, 1)$ design, that is, a Steiner system $S(2, K, v)$, and Γ is a clique.

If $T = K-1$, the partial geometry is a transversal design $\mathrm{TD}(K; R)$, and Γ is complete K-partite on KR vertices.

If $T = R$, the partial geometry is a dual design, and the collinearity graph Γ is the block intersection graph of the design, see §8.5.4A.

If $T = R - 1$, the partial geometry is a dual transversal design (i.e., a net), and Γ is a Latin square graph, see §8.4.2.

A strongly regular graph with the same parameters as the collinearity graph of a $pg(K, R, T)$ is called *pseudo-geometric*. Theorem 8.6.3 below gives a sufficient condition for a pseudo-geometric graph to be geometric. A very simple criterion is the following.

Proposition 8.6.1 *Let Γ be a pseudo-geometric graph with the parameters of a $pg(K, R, T)$. If $\mathscr{C}$ is a collection of K-cliques of Γ such that each edge of Γ is in precisely one member of $\mathscr{C}$, then $(V\Gamma, \mathscr{C})$ is a $pg(K, R, T)$.*

Proof. The Hoffman bound for cliques is K, so by Proposition 1.1.7 (ii) each vertex outside any $C \in \mathscr{C}$ is collinear with T vertices inside. $\square$

Proposition 8.6.2 ([178], Th. 7.6) *For a partial geometry $pg(K, R, T)$ one has $(R-1)(K-2T) \le (K-2)(K-T)^2$ and $(K-1)(R-2T) \le (R-2)(R-T)^2$.*

Proof. Apply the second Krein inequality to $pg(K, R, T)$ and its dual. $\square$

We already made the same observation (in §2.2.10) in the special case of generalized quadrangles of order (s, t) (the case $K = s + 1$, $R = t + 1$, $T = 1$) where $t \le s^2$ or $s = 1$, and $s \le t^2$ or $t = 1$. One may check that the first Krein inequality does not yield nontrivial information here.

History

Partial geometries were introduced by Bose [92].

8.6.1 Examples

We give some examples of partial geometries $pg(K, R, T)$ with $1 < T < \min(K-1, R-1)$.

(i) Thas [682] observed that if K is a maximal n-arc in $\mathsf{PG}(2, q)$ (i.e., a subset such that each line meets it in either 0 or n points), then $|K| = 1+(q+1)(n-1)$, and the complement of K is a $pg(q+1-n, q+1-\frac{q}{n}, q+1-n-\frac{q}{n})$ if we take the n-secants as lines. Maximal arcs are known whenever q is a power of 2, and $n|q$ (Denniston [281]).

(ii) De Clerck, Dye & Thas [268] constructed partial geometries $pg(2^{2n-1}, 2^{2n-1}+1, 2^{2n-2})$. Consider a $4n$-dimensional vector space V with nondegenerate hyperbolic quadric Q. Let Σ be a spread (partition of Q into maximal totally singular subspaces). Take as points the nonsingular 1-spaces (points), and as lines the $2n$-spaces that meet Q in an element of Σ, with natural incidence.

Different constructions for (the dual of) the $n = 2$ example of this series had earlier been given by Cohen [203], and by Haemers & Van Lint [382]. For the isomorphism of these three constructions, see [479], [701]. For nonisomorphic partial geometries with the same parameters, see [552]. Some of these were generalized to infinite families in [266].

(iii) Van Lint & Schrijver [524] construct a $pg(6, 6, 2)$. Consider the ternary code $C = \langle \mathbf{1} \rangle$ of length 6. Partition the 3^5 cosets into three sets A_i

of size 3^4, where $A_i = \{u + C \mid \sum u_h = i\}$ for $i \in \mathbb{F}_3$. Let A_i be the set of points, and A_j the set of lines, for arbitrary distinct i, j, where incidence is having Hamming distance 1. This yields $pg(6, 6, 2)$, in fact a system of three linked such designs ([181]).

(iv) Haemers [377] constructs a $pg(5, 18, 2)$ with group A_7 by taking as points the 175 edges of the Hoffman-Singleton graph Γ, adjacent when they are disjoint and together in a pentagon, and as lines a selection of the Petersen graphs in Γ. See §10.19.

(v) Consider a 6-dimensional vector space V over $\mathbb{F}_3$, and let $H = \mathsf{P}V$ be its hyperplane at infinity. Let $\mathscr{L}$ be a set of 21 pairwise disjoint lines in H with the property that every hyperplane of H meets their union in either 21 or 30 points. Such a set was constructed by Mathon. Let V be the set of points, and take all translates of the 2-spaces $L \in \mathscr{L}$ as lines. This yields a $pg(9, 21, 2)$. (See [267], [39].)

(vi) Thas [684] constructs a $pg(27, 28, 18)$. From a spread in a hyperbolic quadric in $\mathsf{PG}(4h+3, 3)$ one obtains a $pg(3^{2h+1}, 3^{2h+1}+1, 2\cdot 3^{2h})$. Such a spread is known only for ($h = 0$ and) $h = 1$.

(vii) Mathon [549] and Kuijken [505] construct $pg(q, \frac{1}{2}(q^2+1), \frac{1}{2}(q-1))$ whenever q is a power of 3. The collinearity graph is the descendant (on q^3 vertices) of Taylor's unitary 2-graph, cf. §8.10.1.

8.6.2 Enumeration

There are precisely 2 nonisomorphic $pg(5, 7, 3)$ (Mathon [546]).

8.6.3 Nonexistence

There are some sporadic nonexistence results.

$s+1$	$t+1$	α	nonexistence reference
4	5	2	De Clerck [265]
5	28	2	Östergård & Soicher [598]
6	9	4	Lam, Thiel, Swiercz & McKay [507]

8.6.4 The claw bound

Let Γ be a strongly regular graph with the usual parameters. We derive a bound on r given μ and s by showing that if r is sufficiently large then Γ is the collinearity graph of a partial geometry, and then using standard inequalities for its dual.

It turns out to be convenient to use the variables $m = -s$ and $n = r - s$.

Theorem 8.6.3 *Let Γ be a primitive strongly regular graph with integral eigenvalues $r = n - m$ and $s = -m$. Let $f(m, \mu) = \frac{1}{2}m(m-1)(\mu+1) + m - 1$. Then*

(i) (Bruck [148]) *If $\mu = m(m-1)$ and $n > f(m, \mu)$ then Γ is the collinearity graph of a partial geometry $pg(K, R, T)$ with $T = R-1$, that is, is a Latin square graph $\mathrm{LS}_m(n)$.*

(ii) (Bose [92]) *If* $\mu = m^2$ *and* $n > f(m,\mu)$ *then* Γ *is the collinearity graph of a partial geometry* $pg(K,R,T)$ *with* $T = R$*, that is, the block graph of a* 2-$(mn+m-n,m,1)$ *design.*

(iii) ('Claw bound', Neumaier [587]) *If* $\mu \neq m(m-1)$ *and* $\mu \neq m^2$ *then* $n \leq f(m,\mu)$.

In other words: If $r+1 > \frac{1}{2}s(s+1)(\mu+1)$ then $\mu = s(s+1)$ or $\mu = s^2$.

For example, for $m=3$ it follows that a graph with the parameters of a Latin square graph $\mathrm{LS}_3(n)$ is actually such a graph for $n > 23$, and that a graph with the parameters of the block graph of a Steiner triple system $S(2,3,u)$ is actually such a graph for $n > 32$, that is, for $u = 2n+3 > 67$. In [56] examples are given of strongly regular graphs with parameters $(70,27,12,9)$ and $(100,27,10,6)$ that are not the block graph of an $S(2,3,21)$ or a Latin square graph $\mathrm{LS}_3(10)$.

This theorem is proved below (as Theorem 8.6.15).

Strongly regular graphs with given smallest eigenvalue

A direct consequence of the claw bound and the μ-bound is

Theorem 8.6.4 (Sims, cf. [623], p. 99) *The strongly regular graphs with integral smallest eigenvalue* $s = -m$*, where* $m \geq 2$*, are*

(i) complete multipartite graphs with classes of size m*,*
(ii) Latin square graphs $\mathrm{LS}_m(n)$*,*
(iii) block graphs of Steiner systems 2-$(mn+m-n,m,1)$*,*
(iv) finitely many further graphs. □

Completing sets of MOLS

As we saw earlier, necessary for a set of $m-2$ MOLS of order n to exist is $m \leq n+1$, and if equality holds one has a projective plane of order n. The *deficiency* of a set of MOLS is $\delta = n-m+1$, the number of MOLS missing to have a projective plane. The complementary graph of a Latin square graph $\mathrm{LS}_m(n)$ has parameters $\mathrm{LS}_\delta(n)$. The above result by Bruck says that if $n > f(\delta)$ (where f is a fixed polynomial of degree 4), then this complementary graph is itself a Latin square graph, and one can combine the two to get a full set of MOLS, and hence a projective plane. Metsch [562] improved Bruck's bound, and has a polynomial f of degree 3.

Pseudo-generalized quadrangles

The collinearity graph of a generalized quadrangle $\mathsf{GQ}(s,t)$ is strongly regular with parameters $(v,k,\lambda,\mu) = (s+1)(st+1), s(t+1), s-1, t+1)$. Let us call a graph *pseudo-GQ* if it has these same parameters. For example, the Cameron graph is a pseudo-$\mathsf{GQ}(10,2)$. For such graphs, the Krein condition yields $t \leq s^2$. The claw bound implies $s \leq \frac{1}{2}t(t+1)(t+2)$. Guo, Koolen, Markowsky & Park [372] improve this to $s \leq t\lfloor \frac{8}{3}t+1 \rfloor$.

This rules out, e.g., $(v,k,\lambda,\mu) = (12825, 280, 55, 5)$, a pseudo-$\mathsf{GQ}(56,4)$.

8.6.5 Claws and cliques

The results announced above are proved using the Bruck-Bose-Laskar claw-and-clique method ([148, 92, 94]). Let an *s-claw* be an induced $K_{1,s}$ subgraph. Suppose we can show that each j-claw can be extended to a $(j+1)$-claw in at least M ways for $j < m$, but that no $(m+1)$-claw exists. It follows that the M points that can be added to an $(m-1)$-claw are mutually adjacent. In this way one finds the large cliques that are going to be the lines of a partial geometry.

Let Γ be a connected strongly regular graph with parameters (v, k, λ, μ) and integral eigenvalues k, r, s with $s < 0$. All parameters can be expressed in terms of the variables μ, m, and n, where $m = -s$ and $n = r - s$. We have $\lambda = \mu + n - 2m$ and $k = \mu + m(n - m)$.

An *s-claw* (a, S) is a vertex a together with an independent set S of neighbors of a, where $s = |S|$. (Note that s is no longer used to denote the negative eigenvalue $-m$ of Γ.)

A *grand clique* is a maximal clique C with $|C| > 1 + \frac{1}{2}(\lambda + \mu)$ $(=\frac{1}{2}n+\mu+1-m)$. (The precise definition of 'grand clique' varies in the literature. We follow Neumaier [588].)

Lemma 8.6.5 *Each edge of Γ lies in at most one grand clique.*

Proof. If the distinct maximal cliques C_1 and C_2 have an edge uv in common, then $C_1 \cup C_2 \leq 2 + \lambda$ and $C_1 \cap C_2 \leq \mu$, so that $|C_1| + |C_2| \leq 2 + \lambda + \mu$, and C_1, C_2 cannot both be grand cliques. □

Lemma 8.6.6 *Any clique C contains at most $\frac{k}{m} + 1 = n + 1 - m + \frac{\mu}{m}$ vertices, with equality if and only if every vertex outside C is adjacent to the same number of vertices of C, and then that number is $\frac{\mu}{m}$.*

(This follows by quadratic counting or by eigenvalue interlacing. It is the Delsarte-Hoffman bound, cf. §1.1.14.)

Lemma 8.6.7 *(i) If Γ contains a grand clique C, then $n > 2(m-1)\frac{\mu}{m}$.*

(ii) In the collinearity graph of a partial geometry with $n > 2(m-1)\frac{\mu}{m}$, the grand cliques are exactly the lines.

Proof. (i) $\frac{1}{2}n + \mu + 1 - m < |C| \leq n + 1 - m + \frac{\mu}{m}$.

(ii) In a partial geometry lines have size $n + 1 - m + \frac{\mu}{m}$ hence are grand cliques by the proof of (i). Conversely, if C is a grand clique, it has at least two points, so meets some line in at least an edge, but that line is a grand clique, and the edge is in at most one grand clique, so the line coincides with C. □

Lemma 8.6.8 *(a) If Γ has an $(m+1)$-claw, then $n \leq \frac{1}{2}m(m-1)(\mu+1)+m-1$.*

(b) Let $1 \leq s \leq m-1$. Then every s-claw is contained in at least M $(s+1)$-claws, where $M = n - 1 - (m-2)(\mu + 1 - m)$.

(c) If Γ has a maximal s-claw with $s \leq m$, then $m \leq \mu$.

(d) If (a, S) is a maximal m-claw, then there are at least $m(n-2)-(m-2)\mu$ other m-claws (a, S') such that $|S \cap S'| = m - 1$.

Proof. Let (a, S) be an s-claw, and let T be the set of neighbors of a not in S. For $x \in T$, let a_x be the number of neighbors of x in S. Then

$$\sum_{x\in T} 1 = |T| = k - s = \mu + m(n-m) - s,$$
$$\sum_{x\in T} a_x = s\lambda = s(\mu + n - 2m),$$
$$\sum_{x\in T} a_x(a_x - 1) \le s(s-1)(\mu - 1).$$

Now the four parts of the lemma follow: (a) Take $s = m + 1$. The desired conclusion follows from $0 \le \sum_{x\in T}(a_x-1)(a_x-2) \le m(m-1)(\mu+1)+2m-2-2n$.

(b) Take $s \le m-1$. The number of $x \in T$ nonadjacent to all vertices in S is at least $\sum_{x\in T}(1-a_x) = \mu+m(n-m)-s(\mu+n-2m+1) \ge n-1-(m-2)(\mu+1-m)$.

(c) Take $s \le m$. Now $(m-1)(m-\mu) \le \sum_{x\in T}(1-a_x) \le 0$.

(d) Take $s = m$. Since $a_x > 0$ for all $x \in T$, the number of $x \in T$ adjacent to precisely one vertex in S is at least $\sum_{x\in T}(2-a_x) = m(n-2)-(m-2)\mu$. □

Lemma 8.6.9 *If Γ does not have $(m+1)$-claws, and $n > 2(m-1)(\mu+1-m)$, then each edge is in exactly one grand clique.*

Proof. Since $1 \le m \le \mu$, so that $2(m-1)(\mu+1-m) \ge 1+(m-2)(\mu+1-m)$, each edge xy of Γ can be extended to an m-claw by Lemma 8.6.8 (b). Given the m-claw (x, S) on the edge xy, the $(m-1)$-claw $(x, S\backslash y)$ can be extended to an m-claw in at least M ways and we find a clique of size $M+1 = n-(m-2)(\mu+1-m)$ on the edge xy. By the hypothesis, this is contained in a grand clique. □

Lemma 8.6.10 *If Γ does not have $(m+1)$-claws, and $n > 2(m-1)(\mu+1-m)$, then each vertex lies in exactly m grand cliques.*

Proof. Let (a, S) be an m-claw, and for $y \in S$, let C_y be the grand clique containing the clique $\{a\} \cup \{z \mid z \sim a,\ (a, (S \setminus y) \cup \{z\})$ is a m-claw$\}$. The C_y have pairwise intersection $\{a\}$ and by Lemma 8.6.8 (d) cover at least $m(n-1)-(m-2)\mu$ vertices. The vertex a has $k = \mu + m(n-m)$ neighbors, so at most $k - (m(n-1)-(m-2)\mu) = (m-1)(\mu-m)$ are not in any C_y. If C is another grand clique containing a, then $\frac{1}{2}n + \mu - m + 1 < |C| \le (m-1)(\mu-m)+1$, a contradiction. □

Lemma 8.6.11 *Let Σ be a set of cliques such that each point is in exactly m members of Σ, and each edge is in some member of Σ. Then the vertices and members of Σ are the points and lines of a partial geometry $pg(K, R, T)$ with parameters $R = m$, $K = \frac{\mu}{m} + n + 1 - m$, $T = \frac{\mu}{m}$.*

Proof. By Lemma 8.6.6, K is an upper bound for the size of a clique. Since $k = m(K-1)$, all members of Σ have size K, and the statement follows from the 'equality' part of Lemma 8.6.6. □

Lemma 8.6.12 *If Γ does not have $(m+1)$-claws, and $n > 2(m-1)(\mu+1-m)$, then Γ is the collinearity graph of a partial geometry $pg(K, R, T)$ with parameters as above.* □

Proposition 8.6.13 *A strongly regular graph is the collinearity graph of a generalized quadrangle if and only if $\mu = m$ and there are no $(m+1)$-claws.*

Proof. A generalized quadrangle is a partial geometry $pg(K, R, T)$ with $T = 1$, and the stated conditions are satisfied. Conversely, let Γ be a strongly regular graph with $\mu = m$ and without $(m+1)$-claws. Then $k = \mu + m(n-m) = \mu(\lambda+1)$. Since there are no $(m+1)$-claws, and the neighbors of a given point form a regular graph of valency λ, it follows that $\Gamma(x) \simeq mK_{\lambda+1}$ for each vertex x. Now apply Lemma 8.6.11. □

Lemma 8.6.14 *If Γ does not have $(m+1)$-claws, then either*
(a) $\mu = m^2$, or
(b) $\mu = m(m-1)$, or
(c) $\mu = m$, $n \le m(m-1)$, or
(d) $n \le 2(m-1)(\mu+1-m)$.

Proof. If (d) does not hold, then Γ is the collinearity graph of a $pg(K, R, T)$ by Lemma 8.6.12. We have $m = R$ and $\mu = RT$, and $n > 2(m-1)(\mu+1-m)$ can be rewritten $K - T - 1 + R > 2(R-1)(RT+1-R)$.

If $T = 1$, then $\mu = m$, and $n \le m(m-1)$ is the inequality $(K-1) \le (R-1)^2$ that follows from the Krein conditions (cf. §2.2.10) if $R-1 > 1$. For $R = m = 2$, this case is part of (b).

If $T = R - 1$, then $\mu = m(m-1)$.

If $T = R$, then $\mu = m^2$.

Now suppose that $1 < T < R - 1$.

If $R \ge 3T$, then the Krein condition $(K-1)(R-2T) \le (R-2)(R-T)^2$ (Lemma 8.6.2) for the line graph yields the contradiction

$$(R-2)(R-T)^2 \ge 2R(R-1)(T-1)(R-2T) \ge R(R-1)(T-1)(R-T).$$

Finally, if $R < 3T$, then the absolute bound for the line graph yields a contradiction. Indeed, this line graph has $v = R + \frac{R(R-1)(K-1)}{T} > \frac{2R^2(R-1)^2(T-1)}{T}$ and $g = \frac{(R-1)(R-T)(T+(R-1)(K-1))}{T(K+R-T-1)} < \frac{(R-1)^2(R-T)}{T}$. Now $v \le \frac{1}{2}g(g+3)$ implies $4R^2T(T-1) \le (R-T)((R-1)^2(R-T)+3T)$. Since $R < 3T$, also $6RT(T-1) \le (R-1)^2(R-T) + 3T$. For $R \le 3T - 1$ this yields a contradiction. □

We prove Theorem 8.6.3, restated here.

Theorem 8.6.15 *Let Γ be a strongly regular graph with integral smallest eigenvalue $-m$, where $m \ge 2$. Let $f(m, \mu) = \frac{1}{2}m(m-1)(\mu+1) + m - 1$, and suppose that $n > f(m, \mu)$.*

(i) If $\mu = m(m-1)$, then Γ is the collinearity graph of a partial geometry $pg(K, R, T)$ with $T = R - 1$, that is, is the line graph of a transversal 2-design with $\lambda = 1$.

(ii) If $\mu = m^2$, then Γ is the collinearity graph of a partial geometry $pg(K, R, T)$ with $T = R$, that is, is the line graph of a 2-design with $\lambda = 1$.

(iii) Otherwise $n \le 2(m-1)(\mu+1-m)$.

Proof. If Γ contains an $(m+1)$-claw, then $n \le f(m, \mu)$ by Lemma 8.6.8 (a), contrary to our assumption. If we are not in case (iii), then Γ is the collinearity graph of a $pg(K, R, T)$ by Lemma 8.6.12. By Lemma 8.6.14 we may assume $\mu = m$, $n \le m(m-1)$, but that contradicts $n > f(m, \mu)$. □

The final case (iii) is eliminated by the following proposition.

Proposition 8.6.16 *No strongly regular graph* Γ *has parameters satisfying* $\frac{1}{2}m(m-1)(\mu+1) + m - 1 < n \le 2(m-1)(\mu+1-m)$.

Proof. The inequality immediately implies $\frac{1}{2}m < 2$, i.e., $m = 2$ or $m = 3$. A somewhat messy computation using the absolute bound and divisibility conditions eliminates $m = 2, 3$. □

8.7 Semipartial geometries

Debroey & Thas [272] introduced the concept of *semipartial geometry*, generalizing that of partial geometry. A *semipartial geometry* $spg(s+1, t+1, \alpha, \mu)$ is a partial linear space $(X, \mathscr{L})$ such that (i) each line has $s+1$ points, (ii) each point is on $t+1$ lines, (iii) given a point x outside a line L, there are either 0 or α lines on x meeting L, and (iv) any two noncollinear points x, y are both collinear with μ points.

Because of (iv), the collinearity graph of a semipartial geometry is strongly regular (with valency $k = s(t+1)$, and $\lambda = s - 1 + (\alpha - 1)t$, and $\mu = \mu$). A semipartial geometry with $\alpha = 1$ is called a *partial quadrangle* (Cameron [171]).

De Clerck & Thas [270] further generalized the concept of semipartial geometries, and defined $(0, \alpha)$-*geometries* as connected partial linear spaces such that all lines have $s+1$ points, and given a point x outside a line L, there are either 0 or α lines on x meeting L. One loses the strong regularity of the collinearity graph, but gains good inductive properties.

8.7.1 Examples of partial quadrangles

Below we give some examples of partial quadrangles (i.e., of semipartial geometries with $\alpha = 1$). Let a $pq(s+1, t+1, \mu)$ be a $spg(s+1, t+1, 1, \mu)$. The collinearity graph is strongly regular with parameters $(v, k, \lambda, \mu) = (1 + s(t+1) + s^2t(t+1)/\mu, s(t+1), s-1, \mu)$.

(i) Any strongly regular graph with $\lambda = 0$ gives a $pq(2, k, \mu)$ of which the lines are the edges of the graph.

(ii) Any generalized quadrangle $\mathsf{GQ}(s, t)$ is a $pq(s+1, t+1, t+1)$.

(iii) Let $(X, \mathscr{L})$ be a $\mathsf{GQ}(s, s^2)$. Deleting $x^\perp$ for some fixed point x yields a $pq(s, s^2+1, s^2-s)$ with s^4 points.

(iv) (This is case $a = 1$ of example (viii) in §8.7.2.)

Let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$, H a hyperplane in $\mathsf{P}V$, and S a *cap* in H, that is, a subset such that each line meets it in 0, 1, or 2 points. Let X be the set of points of $\mathsf{P}V$ not in H, and let $\mathscr{L}$ be the collection of lines in $\mathsf{P}V$ and not in H that meet S. If each point of $H \setminus S$ is on the same number h of 2-secants of S, then $(X, \mathscr{L})$ is a $pq(q, |S|, 2h)$ with q^n points. It is a generalized quadrangle precisely if $2h = |S|$, that is, if S allows no tangents (and then S is necessarily a hyperoval in a plane).

q	n	$\lvert S\rvert$	v	k	λ	μ	collinearity graph
2^e	3	2^e+2	2^{3e}				$\mathsf{GQ}(2^e-1, 2^e+1)$
q	4	q^2+1	q^4				e.g. $VO_4^-(q)$, $VSz(q)$, see §3.3.1
3	5	11	243	22	1	2	Berlekamp-Van Lint-Seidel graph
3	6	56	729	112	1	20	Games graph
4	6	78	4096	234	2	14	Hill graph

We have seen (§7.1.1) that the construction 'join two vectors when the line they determine hits a fixed set S at infinity' yields a strongly regular graph if and only if S is a two-character set. It follows that S here is a two-character set. A two-character set yields a partial quadrangle only if it is a cap. See also Theorem 7.1.1.

The strongly regular graph defined by S as a two-character set, or equivalently, the collinearity graph of the partial quadrangle defined by S as a cap, is rank 3 if the stabilizer of S in the automorphism group of H acts transitively on both S and $H \setminus S$. In the above examples, this is the case for the first line if $e = 2$ (see §10.24), and for the second line if S is either the quadric $\mathsf{O}_4^-(q)$ (the graph is $VO_4^-(q)$) or the Suzuki-Tits ovoid (see §2.5.5; the graph is $VSz(q)$, see §3.3.1).

(v) Cossidente & Penttila [233] show that when q is odd, there exists a hemisystem in the $\mathsf{U}_4(q)$ generalized quadrangle, that is, a hemisystem of points in the dual $\mathsf{GQ}(q, q^2)$. By Proposition 2.7.9 this point set induces a $pq(\frac{1}{2}(q+1), q^2+1, \frac{1}{2}(q-1)^2)$. Further hemisystems in the $\mathsf{U}_4(q)$ generalized quadrangle were constructed by Bamberg et al. [41] (for $q \le 11$) and by Bamberg et al. [44].

(vi) Bamberg et al. [40] generalize the previous example and construct hemisystems (with the same parameters) in flock generalized quadrangles $\mathsf{GQ}(q^2, q)$.

8.7.2 Examples of semipartial geometries

Below we give some further examples of semipartial geometries. Many are due to Debroey & Thas [272]. For a survey, see De Clerck & Van Maldeghem [271].

(i) Any partial geometry $pg(K, R, T)$ is a semipartial geometry $spg(K, R, T, RT)$.

(ii) For a Moore graph (strongly regular with $\lambda = 0$, $\mu = 1$), one can take the point neighborhoods as lines. In this way a Moore graph Γ with parameters $(k^2+1, k, 0, 1)$ yields an $spg(k, k, k-1, (k-1)^2)$ of which the collinearity graph is $\overline{\Gamma}$.

(iii) Let $\binom{X}{i}$ be the collection of all i-subsets of an m-set X. Then $(\binom{X}{2}, \binom{X}{3})$ is an $spg(3, m-2, 2, 4)$ of which the collinearity graph is $T(m)$.

(iv) Let $\left[{V \atop i}\right]$ be the collection of all i-subspaces of an m-dimensional vector space V. Then $(\left[{V \atop 2}\right], \left[{V \atop 3}\right])$ is an $spg(\left[{3 \atop 1}\right], \left[{m-2 \atop 1}\right], q+1, (q+1)^2)$ of which the collinearity graph is the Grassmann graph $J_q(m, 2)$.

(v) Let V be a $2n$-dimensional vector space over $\mathbb{F}_q$ provided with a nondegenerate symplectic form. Let X be the collection of projective points and $\mathscr{L}$ the collection of hyperbolic lines. Then $(X, \mathscr{L})$ is an $spg(q+1, q^{2n-2}, q, (q-1)q^{2n-2})$ of which the collinearity graph is the complement of the symplectic graph $\mathsf{Sp}(2n, q)$.

(vi) Let V be a $2n$-dimensional vector space over $\mathbb{F}_2$ provided with a nondegenerate quadratic form of type $\varepsilon = \pm 1$. Let X be the collection of nonsingular projective points and $\mathscr{L}$ the collection of elliptic lines. Then $(X, \mathscr{L})$ is an $spg(3,\ 2^{2n-3}{-}\varepsilon 2^{n-2},\ 2,\ 2^{2n-3}{-}\varepsilon 2^{n-1})$ of which the collinearity graph is the complement of the graph from §3.1.2 (Wilbrink [733]).

(vii) Let V be an $(n+2)$-dimensional vector space over $\mathbb{F}_q$, and W an n-dimensional subspace. Let X be the collection of 2-spaces (lines) of V missing W and $\mathscr{L}$ the collection of 3-spaces (planes) of V meeting W in a single 1-space (point). Then $(X, \mathscr{L})$ is an $spg(q^2, \left[{n \atop 1}\right], q, q(q+1))$ of which the collinearity graph is the bilinear forms graph $H_q(2,n)$ from §3.4.1.

(viii) Let V be an n-dimensional vector space over $\mathbb{F}_q$, H its hyperplane at infinity (a $\mathsf{PG}(n-1,q)$), and S a subset of H such that every line in H meets S in either 0, 1 or $a+1$ points, for some fixed a, and such that every point of H not in S is on the same number h of $(a+1)$-secants. Let $\mathscr{L}$ be the collection of lines in V of which the direction is element of S. Then $(V, \mathscr{L})$ is an $spg(q, |S|, a, a(a+1)h)$ with q^n points.

We have seen (§7.1.1) that the construction 'join two vectors when the line they determine hits a fixed set S at infinity' yields a strongly regular graph if and only if S is a two-character set. It follows that S here is a two-character set (with an additional condition on the sizes of line intersections).

For example, if $n = 3$, then S could be a Baer subplane of H, where $|S| = q + \sqrt{q} + 1$ and $a = \sqrt{q}$, or a unital, where $S = q\sqrt{q} + 1$ and $a = \sqrt{q}$. In the first case the collinearity graph is the bilinear forms graph $H_{\sqrt{q}}(2,3)$. In the second case it is the case $m = 3$, $\varepsilon = -1$ and $q \to \sqrt{q}$ of the graph from §3.3.1.

(ix) Let V be a 6-dimensional vector space over $\mathbb{F}_q$ provided with a nondegenerate elliptic quadric Q. Let p be a nonsingular point. Let X be the set of hyperbolic lines (2-spaces) on p, and let $\mathscr{L}$ be the set of planes (3-spaces) on p that meet Q in two intersecting lines. Then $(X, \mathscr{L})$ is an $spg(q, q^2{+}1, 2, 2q(q{-}1))$ (Metz [567]). The collinearity graph is the graph $NO_5^-(q)$, see §3.1.4.

8.8 Zara graphs

A *Zara graph* (with parameters m, e) is a finite graph Γ such that (Z1) every maximal clique has cardinality m, and (Z2) for each maximal clique M each vertex outside is adjacent to precisely e vertices of M. These graphs were first studied by Zara [747]. A structure theory was developed by Blokhuis, Kloks & Wilbrink [81].

Examples of Zara graphs are the polar spaces, the graphs $\overline{L_2(m)}$ and $\overline{T(2n)}$, the folded Johnson graph $\overline{J}(8,4)$ and its complement, and the McLaughlin graph. Also the graph on the nonsingular points of a vector space of dimension $2n$ over $\mathbb{F}_2$ provided with a hyperbolic quadratic form, adjacent when orthogonal. (Here $m = 2^{n-1}$ and $e = 2^{n-2}$.) Also the graph on a vector space of dimension $2n$ over $\mathbb{F}_q$ provided with a hyperbolic quadratic form Q, with $x \sim y$ when $Q(y-x) = 0$. (Here $m = q^n$, $e = q^{n-1}$.) Also the graph on the norm 1 vectors in a vector space of dimension 6 over $\mathbb{F}_3$ provided with a nondegenerate symmetric bilinear form, adjacent when orthogonal. (Here $m = 6$, $e = 2$.)

Easy ways to get new Zara graphs out of old ones:

(1) If Γ_i ($1 \le i \le t$) are Zara graphs with parameters m_i, e_i, and there is an a such that $e_i = m_i - a$ for all i, then their join $\bigvee \Gamma_i$ (with vertex set the disjoint union of the vertex sets $\mathsf{V}\Gamma_i$, and edges xy when xy is an edge in one of the Γ_i or when $x \in \mathsf{V}\Gamma_i$, $y \in \mathsf{V}\Gamma_j$ for $i \ne j$) is again a Zara graph with parameters $m = \sum m_i$ and $e = m - a$.

(2) If Γ is a Zara graph with parameters m, e, then the t-clique extension is a Zara graph with parameters tm, te.

(3) If Γ is a Zara graph with parameters m, e, and C a clique in Γ of size c, then $C^\perp \setminus C$ is a Zara graph with parameters $m - c$, $e - c$.

Let $x^\perp = \{x\} \cup \Gamma(x)$ and $x^* = \{y \mid x^\perp = y^\perp\}$. The graph Γ is called *reduced* if it is coconnected (i.e., if $\overline{\Gamma}$ is connected) and all equivalence classes x^* are reduced to single vertices. Blokhuis, Kloks & Wilbrink [81] show that reduced Zara graphs are strongly regular. Also that reduced graphs satisfying (Z2) but not (Z1) in the definition of Zara graphs are $m \times m'$ grids. Also that if S is a singular subspace of a coconnected Zara graph (that is, a set of the form $S = C^{\perp\perp}$ for some clique C, or, equivalently, an intersection of maximal cliques), then the Zara graph $S^\perp \setminus S$ is coconnected.

For characterizations and related material, see [73], [83], [617], [747], [748].

8.9 Terwilliger graphs

Terwilliger [681] developed a structure theory for a certain class of graphs called Terwilliger graphs. For regular connected Terwilliger graphs, the reduced graphs are strongly regular. There are only few examples.

A *Terwilliger graph* is a non-complete graph Γ such that for any two vertices x, y at distance two the subgraph $\Gamma(x) \cap \Gamma(y)$ is complete of the same size μ.

If Γ is a connected Terwilliger graph for a given μ, then its local graphs $\Gamma(x)$ are Terwilliger graphs for $\mu' = \mu - 1$.

For arbitrary Γ, and $x \in \mathsf{V}\Gamma$, let $x^* = \{y \mid x^\perp = y^\perp\}$. The graph Γ is called *reduced* when all x^* are single vertices. We shall write Γ^* for the *reduced graph* of Γ, that has as vertices the classes x^*, where x^* is adjacent to y^* when $x \sim y$. (Now Γ^* is reduced, and Γ is a clique extension of Γ^*.) The *radical* of Γ is $\{x \mid x^\perp = \mathsf{V}\Gamma\}$.

Proposition 8.9.1 *Let Γ be a regular connected Terwilliger graph. Suppose that $|\Gamma_3(x) \cap \Gamma(y)| = |\Gamma(x) \cap \Gamma_3(y)|$ whenever $d(x, y) = 2$. Then, for all $p \in \mathsf{V}\Gamma$, the graph $\Delta := p^\perp \setminus p^*$ is a coconnected regular Terwilliger graph with $\mu_\Delta = \mu - |p^*|$. If $\mu_\Delta = 0$, then Δ is a union of cliques. Otherwise, it has diameter* 2.

Proof. Γ is regular and connected and not a clique, so $p^\perp$ is not a clique, and hence Δ is nonempty and not a clique. It follows that Δ is a Terwilliger graph with $\mu_\Delta = \mu - |p^*|$. Let Δ_0 be a cocomponent of Δ containing a pair x, y of nonadjacent vertices. Then $\Delta \setminus \Delta_0$ is contained in $x^\perp \cap y^\perp$, hence is a clique, hence is contained in the radical of Δ, hence is empty. It follows that Δ is coconnected. We show that Δ is regular. It suffices to show that nonadjacent vertices x, y of Δ have the same valency in Δ. For each subset E of $x^\perp \cap y^\perp$, and for $\{s, t\} = \{x, y\}$ define $E^s_t := \{z \in \Gamma(s) \cap \Gamma_2(t) \mid \{x, y, z\}^\perp = E\}$. Then $|\Gamma(s)| = |\Gamma(s) \cap \Gamma_3(t)| + \mu + \sum_E |E^s_t|$ and $|\Delta(s)| = \mu_\Delta + \sum_{E \ni p} |E^s_t|$. Each vertex u of E^s_t has $|E|$ neighbors in E, and $\mu - |E|$ neighbors in E^t_s, so

$|E_t^s|(\mu - |E|) = |E_s^t|(\mu - |E|)$ and for $|E| < \mu$ it follows that $|E_y^x| = |E_x^y|$. Since $|\Gamma(x)| = |\Gamma(y)|$, it follows that $|E_y^x| = |E_x^y|$ also holds for $E = x^\perp \cap y^\perp$, and hence $|\Delta(x)| = |\Delta(y)|$. □

Proposition 8.9.2 *Let Γ be a regular connected Terwilliger graph of diameter 2. Then all vertex equivalence classes x^* ($x \in \mathsf{V}\Gamma$) have the same cardinality, and Γ^* is a strongly regular Terwilliger graph.*

Proof. For $p \in \mathsf{V}\Gamma$, let k_p be the valency of $p^\perp \setminus p^*$. If $p \sim q$ and $p^* \neq q^*$, then $k_p = |\{p,q\}^\perp| - |p^*| - 1$, so that $\kappa := k_p + |p^*|$ is independent of the choice of p. Counting edges between $p^\perp \setminus p^*$ and $\Gamma_2(p)$ we find $|p^\perp \setminus p^*|\,(k-\kappa) = (v - |p^\perp|)\mu$, so that $(k+1-|p^*|)(k-\kappa+\mu) = (v-|p^*|)\mu$ since $|p^\perp| = k+1$. This equation determines $|p^*|$ uniquely, because $k > \kappa$. □

A graph Γ is called *edge-regular* with parameters (v,k,λ) when it has v vertices, is regular of valency k, and each edge is in λ triangles. A graph Γ is called *amply regular* with parameters (v,k,λ,μ) when it is edge-regular with parameters (v,k,λ), and any two vertices at distance 2 are joined by μ paths of length 2. Every strongly regular graph and every distance-regular graph is amply regular. Let a *singular line* of a graph Γ be a set of the form $\{x,y\}^{\perp\perp}$, where $x \sim y$. Singular lines are complete subgraphs. If Γ is edge-regular then two singular lines have at most one point in common (cf. [123], 1.14.1).

Proposition 8.9.3 *Let Γ be a reduced amply regular Terwilliger graph with parameters (v,k,λ,μ), where $\mu > 1$. Then, for any $p \in \mathsf{V}\Gamma$ the reduced graph $\Gamma(p)^*$ is strongly regular with parameters $v^* = k/s$, $k^* = (\lambda - s + 1)/s$, $\mu^* = (\mu-1)/s$, $\lambda^* = ((\lambda-s+1)(\lambda-2s+1) - (\mu-1)(k-\lambda-1))/(s(\lambda-s+1))$, where s is the size of the equivalence classes of $\Gamma(p)$. Here s is independent of the choice of p. The singular lines of Γ have size $s+1$, and every vertex is in k/s singular lines. In particular, $\mu = s+1$ or $\mu \geq s^2+s+1$. Also $s \leq \lambda^*+1$.*

Proof. We can apply the previous proposition since $\Gamma(p)$ has diameter 2. Let $s = s_p$ be the common cardinality of the vertex equivalence classes in the graph $\Gamma(p)$. For distinct p,q,r, the vertex r belongs to q^* in $\Gamma(p)$ when p,q,r are contained in the singular line $\{p,q\}^{\perp\perp}$. But then r belongs to p^* in $\Gamma(q)$. It follows that s_p is independent of p. The formulae for v^*, k^*, λ^*, μ^* follow by simple counting. Starting from Γ, and repeatedly taking local graphs, we eventually arrive at a graph with $\mu = 1$. A graph with $\mu^* = 1$ has lines of size λ^*+2, so $s \leq \lambda^*+1$. □

With $\mu = 1$ the known examples are the pentagon, the Petersen graph, and the Hoffman-Singleton graph. The smallest open parameter set is $(400, 21, 2, 1)$. (See also [93].) Collins [210] gives parameter conditions for strongly regular Terwilliger graphs with $\mu = 2$, and in particular shows that an example must have $v > 5.8 \times 10^{58}$.

8.10 Regular two-graphs

A *two-graph* $\Omega = (X, \Delta)$ is a set X provided with a collection Δ of triples called *coherent*, such that each 4-subset of X contains an even number of coherent triples. The relation between a two-graph and a switching class of graphs was given in §1.1.12. The two-graph is called *regular* of degree a when every pair from X is in precisely a coherent triples. Now each descendant is strongly regular with $a = k = 2\mu$ and $v + 1 = |X| = 3k - 2\lambda$. If these descendants have spectrum $k^1\ r^f\ s^g$, and the switching class of Ω contains a strongly regular graph Γ, then Γ has spectrum either $(k-r)^1\ r^{f+1}\ s^g$ or $(k-s)^1\ r^f\ s^{g+1}$.

Conversely, the switching class of a strongly regular graph with parameters (v, k, λ, μ) belongs to a regular two-graph if and only if $v = |X| = 2(2k - \lambda - \mu)$. If this is the case, then it has degree $a = 2(k - \mu)$, and the descendants are strongly regular with parameters $(v - 1, 2(k - \mu), k + \lambda - 2\mu, k - \mu)$.

The *spectrum* of a two-graph is the spectrum of the Seidel matrix S of any (and then all) graph(s) in its switching class. A two-graph with $v > 1$ is regular if and only if it has precisely two distinct eigenvalues. If the eigenvalues of a regular two-graph are ρ_1, ρ_2, then $v = 1 - \rho_1\rho_2$ and $a = -\frac{1}{2}(\rho_1 + 1)(\rho_2 + 1)$. (Both are immediate from $(S - \rho_1 I)(S - \rho_2 I) = 0$.)

8.10.1 Examples

Trivial two-graphs

The two-graph (X, Δ) with $\Delta = \emptyset$ is represented by the edgeless graph with adjacency matrix $A = 0$, Seidel matrix $S = J - I$ and spectrum $(v-1)^1\ (-1)^{v-1}$. Here $a = 0$.

The two-graph (X, Δ) with $\Delta = \binom{X}{3}$, the set of all triples in X, is represented by the complete graph K_n with adjacency matrix $A = J - I$, Seidel matrix $S = -(J - I)$ and spectrum $1^{v-1}\ (1 - v)^1$. Here $a = v - 2$.

Unitary two-graphs

Let V be a 3-dimensional vector space over $F = \mathbb{F}_{q^2}$, where q is odd, provided with a Hermitian form h. Let U be the corresponding unital. Then $|U| = q^3+1$. One obtains a regular two-graph with $a = \frac{1}{2}(q-1)(q^2+1)$ on the set U by taking the triples $\{\langle x\rangle, \langle y\rangle, \langle z\rangle\}$ in U for which $h(x, y)h(y, z)h(z, x)$ is a nonsquare in F if $q \equiv 1 \pmod 4$, and is a square if $q \equiv 3 \pmod 4$. The spectrum of this two-graph is q^2 with multiplicity q^2-q+1, and $-q$ with multiplicity $q(q^2-q+1)$.

Proof. The condition on the quadratic character of $h(x, y)h(y, z)h(z, x)$ does not depend on the choice of the vectors x, y, z in their spans, and does not depend on the order of x, y, z. The product of these triple products for the four 3-subsets of a 4-set is a square, so the triple product is a nonsquare for an even number of triples. This shows that we have a two-graph.

Now let us compute a. Since $\mathsf{U}_3(q)$ acts 2-transitively on U, we may fix two points $\langle x\rangle, \langle y\rangle$ and count the number of coherent triples containing them. The trace $\operatorname{tr} s = s + \overline{s}$ of $s \in F^*$ vanishes when $s^{q-1} = -1$. Such an s is a square precisely when $(q + 1)/2$ is even. Thus, the condition on $h(x, y)h(y, z)h(z, x)$ is that it has the same quadratic character as s where $\operatorname{tr} s = 0$, $s \neq 0$. Take the Hermitian form h defined by $h(x, y) = \overline{x_1}y_3 + \overline{x_2}y_2 + \overline{x_3}y_1$. Take $x = (1, 0, 0)$ and $y = (0, 0, 1)$. Let $z = (s, t, 1)$. It is isotropic when $s + \overline{s} + t\,\overline{t} = 0$, and $h(x, y)h(y, z)h(z, x) = s$. If $t = 0$, $s \neq 0$, i.e., if $z \in \langle x, y\rangle$, $z \neq y$, then all $q - 1$ choices for s yield a coherent triple. If $t \neq 0$ then $\operatorname{tr} s \neq 0$ gives $\frac{1}{2}(q^2 - 1) - (q - 1)$ choices for s with the desired quadratic character, and $q+1$ choices for t given s, so that $a = (q-1)+\frac{1}{2}(q-1)^2(q+1)$, as desired. □

This two-graph is due to TAYLOR [677].

The descendants Ω_w^* of this two-graph are strongly regular with parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ s^g$, where

$$\begin{aligned} v &= q^3, & r &= \frac{1}{2}(q-1),\\ k &= \frac{1}{2}(q-1)(q^2+1), & s &= -\frac{1}{2}(q^2+1),\\ \lambda &= \frac{1}{4}(q-1)^3-1, & f &= (q-1)(q^2+1)\\ \mu &= \frac{1}{4}(q-1)(q^2+1), & g &= q(q-1). \end{aligned}$$

This graph is pseudo-geometric with the parameters of $pg(K, R, T)$ where $K = q$, $R = \frac{1}{2}(q^2+1)$ and $T = \frac{1}{2}(q-1)$. Spence [668] showed that is is geometric for $q = 3$ (one gets $\mathsf{GQ}(2,4)$), but not for $q = 5, 7$. Mathon [549] and Kuijken [505] showed that it is geometric whenever q is a power of 3.

We saw that collinear triples of the unital are members of the two-graph, so that after switching the set $L \setminus \{w\}$ has become a clique, for each line L passing through w. Thus, the graph Ω_w^* has a partition into cliques of size q (achieving the Hoffman bound) and any point outside such a clique has exactly $\frac{1}{2}(q-1)$ neighbors in it. Consequently, if we take the union of any $\frac{1}{2}(q^2+1)$ of these cliques, we get a regular subgraph of degree $\frac{1}{4}(q-1)(q^2+3)$, and adding the point w again and switching yields a strongly regular graph with parameters

$$\begin{aligned} v &= q^3+1, & r &= \frac{1}{2}(q-1),\\ k &= \frac{1}{2}q(q^2+1), & s &= -\frac{1}{2}(q^2+1),\\ \lambda &= \frac{1}{4}(q-1)(q^2+3), & f &= (q-1)(q^2+1)\\ \mu &= \frac{1}{4}(q+1)(q^2+1), & g &= q^2-q+1. \end{aligned}$$

The other possible valency for strongly regular graphs in the switching class of this regular two-graph is $\frac{1}{2}(q-1)q^2$. For $q = 5$ there is such a graph (with parameters $(v, k, \lambda, \mu) = (126, 50, 13, 24)$ and spectrum $50^1\ 2^{105}\ (-13)^{20}$). For $q \equiv 3 \pmod 8$ there is no such graph ([380]).

The regular two-graph on 276 points

Consider the graph Γ on the $23 + 253 = 276$ symbols and blocks of the Steiner system $S(4, 7, 23)$, where the symbols form a coclique, a symbol is adjacent to the 77 blocks containing it, and two blocks are adjacent when they meet in precisely 1 point. This graph Γ belongs to the switching class of a regular two-graph Ω on 276 points. The degree of Ω is 112. Its spectrum is $55^{23}\ (-5)^{253}$. The automorphism group of Ω is Co_3, acting 2-transitively. Uniqueness was proved by Goethals & Seidel [356].

The descendants Ω_w^* of Ω are McLaughlin graphs with parameters $(v, k, \lambda, \mu) = (275, 112, 30, 56)$ and spectrum $112^1\ 2^{252}\ (-28)^{22}$. (See §10.61.)

The switching class of Ω contains (many, see [593]) strongly regular graphs with parameters $(v, k, \lambda, \mu) = (276, 140, 58, 84)$ and spectrum $140^1\ 2^{252}\ (-28)^{23}$. The parameter set $(v, k, \lambda, \mu) = (276, 110, 28, 54)$ with spectrum $110^1\ 2^{253}\ (-28)^{22}$ is ruled out by the absolute bound (and also by the Krein conditions).

The regular two-graph on 176 points

There is a regular two-graph on 176 vertices, with spectrum 35^{22} $(-5)^{154}$ and $a = 72$. Related strongly regular graphs have parameters (i) $(v, k, \lambda, \mu) = (175, 72, 20, 36)$ or (ii) $(176, 70, 18, 34)$ or (iii) $(176, 90, 38, 54)$. Examples of each are known. ((i) Graph on the edges of the Hoffman-Singleton graph (§10.19), (ii) M_{22} graph on 176 vertices (§10.51), (iii) Graph constructed by Haemers from (i)+K_1 by switching with respect to the union of 18 pairwise disjoint 5-cliques.) It is unknown whether the two-graph is unique. (But graph (ii) is unique.)

The regular two-graph on 126 points

There is a regular two-graph on 126 vertices, with spectrum 25^{21} $(-5)^{105}$ and $a = 52$. Related strongly regular graphs have parameters (i) $(v, k, \lambda, \mu) = (125, 52, 15, 26)$ or (ii) $(126, 50, 13, 24)$ or (iii) $(126, 65, 28, 39)$. Examples of each are known. ((i), (iii): see above onder Unitary two-graphs, (ii) Goethals graph, §10.42.) It is unknown whether the two-graph is unique. (But graph (ii) is unique [222].)

8.10.2 Enumeration

Small regular two-graphs have been classified. The table below gives the numbers of nonisomorphic nontrivial regular two-graphs with eigenvalue -3 or -5 or with $v \leq 50$.

v	6	10	14	16	18	26	28	30
ρ_1, ρ_2	$\pm\sqrt{5}$	± 3	$\pm\sqrt{13}$	$-3, 5$	$\pm\sqrt{17}$	± 5	$-3, 9$	$\pm\sqrt{29}$
#	1	1	1	1	1	4	1	6

v	36	38	42	46	50	126	176	276
ρ_1, ρ_2	$-5, 7$	$\pm\sqrt{37}$	$\pm\sqrt{41}$	$\pm\sqrt{45}$	± 7	$-5, 25$	$-5, 35$	$-5, 55$
#	227	≥ 191	≥ 18	≥ 97	≥ 54	≥ 1	≥ 1	1

For $v < 30$, see Bussemaker et al. [161]. The case $v = 30$ was settled by Spence [669] (and independently by Bussemaker). The regular two-graphs on 36 vertices were enumerated by McKay & Spence [556]. Nonexistence of nontrivial regular two-graphs on 76 or 96 vertices was shown by Azarija & Marc [20, 21].

8.10.3 Completely regular two-graphs

Let (X, Δ) be a two-graph. A subset C of X is called a *clique* when each triple from C is coherent. If C is a clique, and $x \notin C$, then x determines a partition $\{C_x, C'_x\}$ of C into two possibly empty parts such that a triple xyz with $y, z \in C$ is coherent precisely when y and z belong to the same part of the partition.

Proposition 8.10.1 (Taylor [677]) *Let C be a nonempty clique of the regular two-graph Ω with eigenvalues ρ_1, ρ_2, where $\rho_2 < 0$ and ρ_2 has multiplicity m. Then*

(i) $|C| \leq 1 - \rho_2$, with equality if and only if $|C_x| = |C'_x|$ for each $x \notin C$, and

(ii) $|C| \leq m$.

Proof. See [677], Propositions 5.2 and 5.3, or [132], Proposition 10.3.4. □

In a regular two-graph each pair is in $a_2 = a$ coherent triples, that is, in a_2 3-cliques, and each coherent triple is in a_3 4-cliques, where a_3 is the number of common neighbors of two adjacent vertices in any descendant, so that $a_3 = -\frac{1}{4}(\rho_1+3)(\rho_2+3)+1$.

A *completely regular two-graph* is a two-graph in which there are constants a_i such that each i-clique with $i \le -\rho_2$ is contained in precisely a_i $(i+1)$-cliques, where $a_i > 0$. Neumaier [589] introduced this concept and gave parameter restrictions strong enough to leave only a finite list of feasible parameters. There are five examples, and two open cases. See Table 8.3 below.

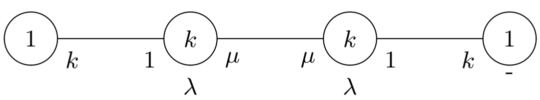
#	ρ_1	ρ_2	v	a_2	a_3	a_4	a_5	a_6	a_7	existence
1	3	−3	10	4	1					unique
2	5	−3	16	6	1					unique
3	9	−3	28	10	1					unique
4	7	−5	36	16	6	2	1			unique [83]
5	19	−5	96	40	12	2	1			none [83]
6	25	−5	126	52	15	2	1			none [589]
7	55	−5	276	112	30	2	1			unique [356]
8	21	−7	148	66	25	8	3	2	1	none [589]
9	41	−7	288	126	45	12	3	2	1	none [73]
10	161	−7	1128	486	165	36	3	2	1	?
11	71	−9	640	288	112	36	10	4	3	none [83]
12	351	−9	3160	1408	532	156	30	4	3	?
13	253	−11	2784	1270	513	176	49	12	5	none [589]

Table 8.3: Parameters of completely regular two-graphs

Blokhuis & Wilbrink [83] observed that a descendant of a completely regular two-graph is a Zara graph with $m = -\rho_2$ and $e = (m-1)/2$. In the cases with $v = 10, 16, 28$ that Zara graph is a generalized quadrangle $\mathsf{GQ}(2,t)$ with $t = 1, 2, 4$. In the case with $v = 36$ that Zara graph is locally 4×4 and hence the folded Johnson graph $\overline{J}(8,4)$ ([74]).

8.10.4 Covers and quotients

Taylor graphs



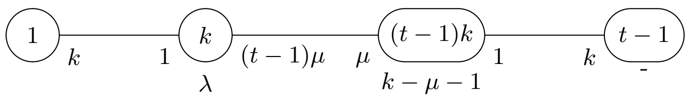
$v = 2(k+1)$

A distance-regular antipodal double cover of a complete graph K_n is called a *Taylor graph*. Such graphs have intersection array $\{k, \mu, 1; 1, \mu, k\}$ and are equivalent to regular 2-graphs on $n = k+1$ vertices. (Taylor & Levingston [679]).

Let X be an n-set, and fix $\infty \in X$. The regular 2-graph (X, Δ) corresponds to the graph Γ with vertex set $\{x^+, x^- \mid x \in X\}$ and (for $x \ne y$) edges $x^\varepsilon y^\eta$ where $\varepsilon = \eta$ if $\infty \in \{x, y\}$ or $\{\infty, x, y\} \in \Delta$, and $\varepsilon = -\eta$ otherwise. Note that the isomorphism type of Γ does not depend on the choice of ∞. Conversely, given Γ we find Δ as the image of the set of triangles in Γ under the map $x^\varepsilon \mapsto x$. For a detailed discussion, see [123], §1.5.

The local graphs of Γ are the descendants of (X, Δ), and hence strongly regular.

Krein covers



$v = t(k+1)$

More generally one can look at distance-regular antipodal t-covers of K_n. Here $v = tn$, $n = k+1$, $k-1-\lambda = (t-1)\mu$. These graphs have intersection array $\{k, (t-1)\mu, 1;\ 1, \mu, k\}$ and spectrum $k^1\ \theta^f\ (-1)^k\ (-k/\theta)^g$ where θ and $-k/\theta$ are the solutions of $\theta^2 + (\mu-\lambda)\theta - k = 0$ with $\theta > 0$, and $f = (t-1)k(k+1)/(k+\theta^2)$, $g = \theta^2(t-1)(k+1)/(k+\theta^2)$. The Krein condition $q_{33}^3 \geq 0$ gives the inequality $k \leq \theta^3$ when $t > 2$.

Godsil [345] shows that when equality holds, the local graphs are strongly regular, with parameters $(v_0, k_0, \lambda_0, \mu_0)$ and spectrum $k_0^1\ r_0^{f_0}\ s_0^{g_0}$, where

$$
\begin{aligned}
v_0 &= k = \theta^3, & r_0 &= \theta - \frac{\theta+1}{t},\\
k_0 &= \lambda = (\theta-1)\left(\frac{(\theta+1)^2}{t} - \theta\right), & s_0 &= \frac{-\lambda}{\theta-1} = \theta - \frac{(\theta+1)^2}{t},\\
\lambda_0 &= \frac{(\theta+1)^3}{t^2} - \frac{3(\theta+1)^2}{t} + 3\theta, & f_0 &= (\theta-1)((\theta+1)^2 - t\theta),\\
\mu_0 &= \frac{(\theta+1)^3}{t^2} - \frac{(\theta+1)(2\theta+1)}{t} + \theta, & g_0 &= (t-1)\theta(\theta-1).
\end{aligned}
$$

It follows that in this situation $t \mid (\theta+1)$, with equality when the local graph is a union of cliques.

Godsil also observed that given a distance-regular antipodal t-cover Γ of K_n and a group G preserving the fibers, acting fixpoint-freely, the quotient graph Γ/G is a distance-regular antipodal (t/g)-cover of K_n, where $g = |G|$, with the same eigenvalues as Γ. If Γ satisfied $k = \theta^3$, then so does Γ/G.

A case where this happens is that of a generalized quadrangle $\mathsf{GQ}(q, q^2)$ with spread. The collinearity graph Γ of that generalized quadrangle minus the lines of the spread is a distance-regular antipodal $(q+1)$-cover of K_n, where $n = q^3+1$, with eigenvalues $k = q^3$, $\theta = q$, -1, and $-q^2$ ([123], Theorem 12.5.2). There is a cyclic group G of order $q+1$ acting on the fibers. It follows that for each $t \mid (q+1)$ the local graph of Γ/G is strongly regular with the above parameters (with $\theta = q$). For example, with $(q,t) = (5,3)$ or $(8,3)$ we find strongly regular graphs with parameters $(v,k,\lambda,\mu) = (125, 28, 3, 7)$ or $(512, 133, 24, 38)$. For $t = 2$ we find the descendants of Taylor's 2-graph again.

8.11 Pseudocyclic association schemes

A d-class association scheme $(X, \mathscr{R})$ is called *pseudocyclic* when all nontrivial multiplicities are equal, i.e., when $m_1 = \cdots = m_d$.

Proposition 8.11.1 (Mathon [543]; Hollmann [439], p. 84; cf. [123], 2.2.7) *A d-class association scheme is pseudocyclic if and only if for some constant m we have $n_i = m$ $(1 \le i \le d)$ and $\sum_i p_{ii}^h = m - 1$ $(1 \le h \le d)$.* □

Given two association schemes $(X, \mathscr{R})$ and $(X', \mathscr{R}')$, one can take the direct product (tensor product) in the obvious way. It has point set $X \times X'$ and relations $R_{ij} = \{(xx', yy') \mid (x,y) \in R_i \text{ and } (x', y') \in R'_j\}$ with eigenvalues $\mu\nu$ where μ (resp. ν) runs through the eigenvalues of R_i (resp. R'_j).

Proposition 8.11.2 (cf. Fujisaki [331]) *Let $(X, \mathscr{R})$ be a pseudocyclic association scheme with d classes on $n = dm + 1$ points. In the direct product of $(X, \mathscr{R})$ with itself the three relations $R = \bigcup_{j=1}^d R_{jj}$ and $R' = R \cup \bigcup_{j=1}^d R_{j0}$ and $R'' = R' \cup \bigcup_{j=1}^d R_{0j}$ define strongly regular graphs with Latin square parameters $\mathrm{LS}_t(n)$, with $t = m, m+1, m+2$, respectively.*

Proof. The eigenvalues of R are $\sum_{j=1}^d P_{ij} P_{i'j}$ for $0 \le i, i' \le d$. Using Proposition 1.3.2 we see that this equals dm^2 if $i = i' = 0$, $n - m$ if $i = i' \ne 0$, and $-m$ if $i \ne i'$. The eigenvalues of R' are $\sum_{j=1}^d (P_{ij} P_{i'j} + P_{ij})$ which equals $dm(m+1)$ if $i = i' = 0$, $n - m - 1$ if $i = 0, i' \ne 0$, and $-m - 1$ if $i \ne 0$. The eigenvalues of R'' are $\sum_{j=1}^d (P_{ij} P_{i'j} + P_{ij} + P_{i'j})$ which equals $dm(m+2)$ if $i = i' = 0$, $n - m - 2$ if $i = 0, i' \ne 0$ or $i' = 0, i \ne 0$, and $-m - 2$ if $i, i' \ne 0$. □

Examples of pseudocyclic d-class schemes are the cyclotomic schemes on $\mathbb{F}_q$. (Let K be a subgroup of $\mathbb{F}_q^*$ of index d with $-1 \in K$. Let a be a primitive element. Let $(x,y) \in R_i$ when $y - x \in a^{i-1}K$ $(1 \le i \le d)$.)

Not many examples are known for non-primepower $|X|$. Mathon [543] and Hollmann [440] found the two pseudocyclic 3-class association schemes on 28 points. Hollmann [439] constructed a 3-class example on 496 points. These generalize to examples with $(d, m) = (\frac{1}{2}q - 1, q + 1)$ for $q = 2^e$, and $(d, m) = ((\frac{1}{2}q - 1)/e, e(q + 1))$ for $q = 2^e$, e prime. See [123] p. 390 and [441].

8.12 Tensor products of skew schemes

In §1.3 we defined symmetric association schemes. More generally one can look at association schemes that are not necessarily symmetric. One drops the condition that the relations R_i are symmetric, and requires instead that for each i the converse of the relation R_i is also one of the relations R_j.

In the special case of a 2-class association scheme that is not symmetric, there are three relations: identity and R and the converse of R, so that the Bose-Mesner algebra is generated by three matrices $I, A, A^\top$ with $I + A + A^\top = J$. The relation R describes a tournament (a directed complete graph). If the number of points is v, then $AJ = JA = kJ$ where $v = 2k + 1$, and the algebra is automatically commutative. One finds $AA^\top = (m-1)J + mI$ and $k = 2m - 1$ and $v = 4m - 1$. Let $S = A - A^\top$. Then $S = -S^\top$, and the matrix $C = \left(\begin{smallmatrix} 0 & \mathbf{1}^\top \\ \mathbf{1} & S \end{smallmatrix}\right)$ of order $v + 1$ is a conference matrix. (Equivalently, $H = \left(\begin{smallmatrix} 1 & \mathbf{1}^\top \\ -\mathbf{1} & S+I \end{smallmatrix}\right)$ is a skew Hadamard matrix.) Conversely, each conference matrix (or skew Hadamard matrix) of order $4m$ yields a skew 2-class association scheme (cf. Theorem 8.2.1).

The *tensor product* of two association schemes $(X, \mathscr{R})$ and $(Y, \mathscr{S})$, symmetric or not, is the association scheme $(X \times Y, \mathscr{T})$ where the points (x, y) and (x', y')

are in relation $T_{(i,j)}$ when $(x,x') \in R_i$ and $(y,y') \in S_j$. The *symmetrization* of a not necessarily symmetric association scheme $(X,\mathscr{R})$, is the symmetric association scheme $(X,\mathscr{R}')$ where $\mathscr{R}' = \{R \cup R^\top \mid R \in \mathscr{R}\}$.

Theorem 8.12.1 (PASECHNIK [600]) *Let H, H' be skew-symmetric Hadamard matrices of order $4m$. The symmetrization of the tensor product of the two corresponding association schemes is an amorphic 4-class association scheme on $(4m-1)^2$ points, with valencies $n_0 = 1$, $n_1 = n_2 = 4m-2$, $n_3 = n_4 = 8m^2 - 8m + 2$. The relations R_3 and R_4 define strongly regular graphs with Latin square parameters $\mathrm{LS}_{2m-1}(4m-1)$.*

For example, one finds graphs with parameters $(v,k,\lambda,\mu) = (225, 98, 43, 42)$.

The graphs here satisfy the 4-vertex condition (cf. §8.16.1) if and only if $m = 1$, hence are not rank 3 for $m > 1$.

8.13 Cospectral graphs

Seidel switching gives classes of graphs with the same Seidel spectrum. One also has switching-type constructions that preserve the ordinary spectrum. In many cases these can be used to show that a strongly regular graph is not determined uniquely by its parameters.

8.13.1 Godsil-McKay switching

Let Γ be a graph with vertex set X, and let $\{C_1, \ldots, C_t, D\}$ be a partition of X such that $\{C_1, \ldots, C_t\}$ is an equitable partition of $X \setminus D$ (that is, any two vertices in C_i have the same number of neighbors in C_j for all i, j), and for every $x \in D$ and every $i \in \{1, \ldots, t\}$ the vertex x has either 0, $\frac{1}{2}|C_i|$ or $|C_i|$ neighbors in C_i. Construct a new graph Γ' by interchanging adjacency and nonadjacency between $x \in D$ and the vertices in C_i whenever x has $\frac{1}{2}|C_i|$ neighbors in C_i. Then Γ and Γ' are cospectral (GODSIL & MCKAY [348]).

For discussion and examples, see [132], §§1.8.3, 14.2.3. For example, GM-switching (for $t = 1$) with respect to a diagonal turns the 4×4 grid into the Shrikhande graph. MUNEMASA [576] shows that the Van Dam-Koolen graphs arise by GM-switching from Grassmann graphs. See also [3], [447], [52].

8.13.2 Wang-Qiu-Hu switching

Let Γ be a graph with vertex set X, and let $\{C_1, C_2, D\}$ be a partition of X, where the subgraphs induced on C_1, C_2, and $C_1 \cup C_2$ are regular, and C_1 and C_2 have the same size and degree. Suppose that each $x \in D$ either has the same number of neighbors in C_1 and C_2, or satisfies $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$. Construct a new graph Γ' by interchanging adjacency and nonadjacency between $x \in D$ and $C_1 \cup C_2$ when $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$. Then Γ and Γ' are cospectral (WANG, QIU & HU [721]).

One may check that GM-switching with $t = 1$, $|C_1| = 4$ is equivalent to WQH-switching with $|C_1| = 2$. IHRINGER & MUNEMASA [452] construct new strongly regular graphs by applying WQH-switching to polar graphs.

8.14 Equiangular sets of lines

Let x_i $(1 \le i \le n)$ be unit vectors in $\mathbb{R}^d$ or $\mathbb{C}^d$. The set of lines $\{\langle x_i \rangle \mid 1 \le i \le n\}$ is called *equiangular* when there is a constant α such that for any two distinct i, j one has $|x_i^* x_j| = \alpha$. (Here $x^* = \overline{x^\top}$ denotes the conjugate transpose of x.) In $\mathbb{R}^d$ this says that the cosine of the angle between any two of these lines is α.

The size of a set of equiangular lines is bounded as a function of d.

Proposition 8.14.1 (See [132], §10.6.2.)

(i) ('Absolute bound') *A set of equiangular lines in* $\mathbb{R}^d$ *has size at most* $\frac{1}{2}d(d+1)$.

(ii) A set of equiangular lines in $\mathbb{C}^d$ *has size at most* d^2.

(iii) ('Special bound') *If* $\{x_i \mid 1 \le i \le n\}$ *is a set of unit vectors such that* $|x_i^* x_j| \le \alpha$ *for any two distinct indices* i, j, *and* $\alpha^2 d < 1$, *then* $n \le \frac{d(1-\alpha^2)}{1-\alpha^2 d}$. □

Part (i) is due to M. Gerzon[3] (see [514]).

Part (ii) is due to Delsarte, Goethals & Seidel [279]. Complex systems of lines with equality in (ii) are known as *SICPOVMs* ('symmetric informationally complete positive operator-valued measures'). Examples are known for $1 \le d \le 21$ and many further values of d. It is conjectured (Zauner [749], p. 61) that they exist for all d. An example for $d = 3$ are the 9 vectors in $\mathbb{C}^3$ given by the cyclic shifts of $\frac{1}{\sqrt{2}}(0, 1, -a)$ where $a^3 = 1$. A nice example for $d = 8$ was given by Hoggar [437, 438].

Complex systems of lines with equality in (iii) are known as *equiangular tight frames* (ETFs). Equivalently, an equiangular tight frame is a $d \times n$ matrix F of which the columns are equiangular unit vectors, and the rows are mutually orthogonal, all with the same length a, so that $FF^* = aI$. Now $a = \frac{n}{d}$. (See also [132], §10.6.2.)

In the real case, equality in (iii) leads to strongly regular graphs. If $G = F^\top F = (x_i^\top x_j)$ is the Gram matrix of the set of vectors, so that $G = I + \alpha S$ for a matrix S that has zero diagonal and off-diagonal entries ± 1, then S is the Seidel adjacency matrix of a graph in the switching class of a regular 2-graph with eigenvalues $\frac{n-d}{\alpha d}$ and $\frac{-1}{\alpha}$ with multiplicities d and $n - d$, respectively. This graph will be strongly regular precisely when $\mathbf{1}$ is eigenvector of G, so that $G\mathbf{1} = 0$ or $G\mathbf{1} = \frac{n}{d}\mathbf{1}$. The former happens if and only if $F\mathbf{1} = 0$. The latter if and only if $\mathbf{1}^\top$ lies in the row space of F.

Indeed, if $F^\top y = \mathbf{1}$, then $G\mathbf{1} = F^\top F F^\top y = \frac{n}{d}\mathbf{1}$.

This leads to a number of constructions.

(i) (Goethals & Seidel [355]) If there exists a Steiner system $S(2, k, v)$ (with $b = \frac{v(v-1)}{k(k-1)}$ blocks, and $r = \frac{v-1}{k-1}$ blocks on each point), and a Hadamard matrix H of order $r + 1$, then there exists an equiangular tight frame F with $d = b$ and $n = v(r + 1)$ and $\alpha = \frac{1}{r}$, obtained by substituting rows of H for the 1's in the block-point incidence matrix of the design (and dividing by $\sqrt{r}$), taking r distinct rows for the r 1's in a single column.

[3] Michael Gerzon was an audio pioneer from Oxford. He was interested in the question of equiangular lines in connection with the problem of sending many signals through a small number of channels with minimal crosstalk. [PJC]

If H is normalized to have top row $\mathbf{1}^\top$, and the remaining rows are used in the substitution process, then $F\mathbf{1} = 0$, and we find a strongly regular graph with $V = v(r+1)$ vertices, and eigenvalues $K = \frac{(v+1)(r+1)}{2} - 1$, $R = \frac{r-1}{2}$, and $S = -\frac{v+k}{2}$.

(ii) (Fickus et al. [324]) If there exists a Steiner system $S(2,k,v)$ with a parallel class, and a Hadamard matrix H of order $r+1$, then there exists a strongly regular graph with $V = v(r+1)$ vertices, and eigenvalues $K = \frac{(v-k+1)(r+1)}{2} - 1$, $R = \frac{r-1}{2}$, and $S = -\frac{v+k}{2}$.

Here we use the parallel class, combined with an all-1 row of H to see that $\mathbf{1}^\top$ lies in the row space of F.

(iii) See also [325].

The asymptotic behavior of $N_\alpha(d)$, the maximum number of vectors in $\mathbb{R}^d$ with pairwise inner products $\pm\alpha$, for fixed α and large d was determined in [465].

8.15 Spherical designs

A finite nonempty subset X of the unit sphere Ω in the Euclidean space $\mathbb{R}^m$ is called a *spherical t-design* if for each polynomial $F = F(x_1, \ldots, x_m)$ of degree at most t the average of F over Ω equals the average over the set X, i.e.,

$$\frac{1}{|X|} \sum_{x \in X} F(x) = \frac{1}{\operatorname{vol} \Omega} \int_\Omega F(x) dx.$$

Spherical designs were introduced by Delsarte, Goethals & Seidel [278]. Many of the results stated below can be found here.

A polynomial F is called *harmonic* when $\sum_i \frac{\partial^2 F}{\partial x_i^2} = 0$. Let $\operatorname{Hom}(k)$ be the space of homogeneous polynomials of degree k, and $\operatorname{Harm}(k)$ be the subspace of harmonic polynomials. Then $\dim \operatorname{Hom}(k) = \binom{k+m-1}{m-1}$ and $\dim \operatorname{Harm}(k) = \binom{k+m-1}{m-1} - \binom{k+m-3}{m-1}$.

Proposition 8.15.1 ([278], [716]) *For a finite nonempty subset X of the unit sphere in $\mathbb{R}^m$, the following are equivalent.*

(i) X is a spherical t-design,

(ii) $\sum_{x \in X} F(x) = 0$ for all $F \in \operatorname{Harm}(k)$ and any k with $1 \le k \le t$,

(iii) for any $y \in \mathbb{R}^m$,

$$\frac{1}{|X|} \sum_{x \in X} \langle x, y \rangle^k = \begin{cases} \frac{1 \cdot 3 \cdots (k-1)}{m(m+2) \cdots (m+k-2)} \langle y, y \rangle^{k/2} & \text{if } k \text{ is even, } 0 \le k \le t, \\ 0 & \text{if } k \text{ is odd, } 0 \le k \le t. \end{cases}$$

For example, from the (distance-regular) collinearity graph of the dual polar space $\mathsf{U}_6(2)$ one gets (by taking the columns of the idempotent E_3) a set of 891 vectors in $\mathbb{R}^{22}$ with inner products 1, $-\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{8}$ with frequencies 1, 42, 336, 512. Using (iii) one sees that this is a spherical 5-design.

For a survey, see Bannai & Bannai [46].

8.15.1 Tight spherical designs

The set X is said to have *degree s* when the inner product between two distinct elements of X takes precisely s values. Put $n = |X|$. If X has degree s, we have the upper bound

$$n \le \binom{m+s-1}{s} + \binom{m+s-2}{s-1}, \text{ or } n \le 2\binom{m+s-2}{s-1},$$

with the sharper inequality if X is antipodal. If X is a spherical t-design, we have the lower bound

$$n \ge \binom{m+e-1}{e} + \binom{m+e-2}{e-1}, \text{ or } n \ge 2\binom{m+e-1}{e}$$

for $t = 2e$ and $t = 2e+1$, respectively. In case of equality, the spherical t-design is called *tight*. For example, the set of $2\binom{28}{5}$ shortest vectors of the Leech lattice in $\mathbb{R}^{24}$ is an antipodal spherical 11-design of degree 6 (with inner products $-1, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$) and has equality in both upper and lower bound.

A spherical $2e$-design is tight if and only if it has degree e. A spherical $(2e+1)$-design is tight if and only if it has degree $e+1$ and is antipodal.

t	m	N	inner products	comment
1	m	2	-1	pair of vectors $\pm e$
2	m	$m+1$	$-\frac{1}{m}$	simplex
3	m	$2m$	$-1, 0$	cross polytope (vectors $\pm e_i$, $i = 1, \ldots, m$)
4	6	27	$-\frac{1}{2}, \frac{1}{4}$	Schläfli graph
4	22	275	$-\frac{1}{4}, \frac{1}{6}$	McLaughlin graph
5	3	12	$-1, \pm\frac{1}{\sqrt{5}}$	icosahedron
5	7	56	$-1, \pm\frac{1}{3}$	28 equiangular lines
5	23	552	$-1, \pm\frac{1}{5}$	276 equiangular lines
7	8	240	$-1, \pm\frac{1}{2}, 0$	roots of E_8
7	23	4600	$-1, \pm\frac{1}{3}, 0$	2300 equiangular lines (invariant under $2 \times \mathsf{Co}_2$)
11	24	196560	$-1, \pm\frac{1}{2}, \pm\frac{1}{4}, 0$	shortest vectors in the Leech lattice
$N-1$	2	N	$\cos\frac{2\pi i}{N}, 1 \le i \le \frac{1}{2}N$	regular N-gon

Table 8.4: Tight spherical t-designs of size N in $\mathbb{R}^m$

For $t \ne 4, 5, 7$ all examples of tight spherical t-designs are known. For $m = 2$, the tight spherical t-designs are the regular $(t+1)$-gons. No tight spherical t-designs exist in $\mathbb{R}^m$ with $m \ge 3$ for $t = 2e \ge 6$ or $t = 2e+1 \ge 9$, except in case $m = 24$, $t = 11$ (Bannai & Damerell [47], [48]). Uniqueness of the examples with $(t, m) = (5, 7), (7, 8), (7, 23), (11, 24)$ was shown in Bannai & Sloane [50].

There is a 1-1 correspondence between tight spherical 4-designs and tight spherical 5-designs: any example X of the latter (of size N in $\mathbb{R}^m$) has degree 3 and inner products $-1, -a, a$, and shifting and scaling the set $\{x \in X \mid (x, x_0) = a\}$ for some fixed $x_0 \in X$ yields a tight spherical 4-design (of size $\frac{1}{2}N - 1$ in $\mathbb{R}^{m-1}$). This procedure can be reversed. Tight spherical 4-designs are obtained from strongly regular graphs with equality in the absolute bound, see Proposition 8.15.2 below.

Any tight spherical 5-design in $\mathbb{R}^m$ with $m > 3$ lives in dimension $m = (2h+1)^2 - 2$ for some integer h. Examples are known for $h = 1, 2$ and there are none for $h = 3, 4$. Any tight spherical 7-design in $\mathbb{R}^m$ has $m = 3h^2 - 4$ for some integer h. Examples are known for $h = 2, 3$ and there are none for $h = 4, 5$. These and further nonexistence results are due to Bannai, Munemasa & Venkov [49] and Nebe & Venkov [586].

If X is a spherical t-design of degree s, and $t \ge 2s - 2$, then X, with the inner products as relations, is an s-class association scheme (Delsarte, Goethals & Seidel [278], Thm. 7.4).

8.15.2 Spherical designs from association schemes

Given a d-class association scheme $(X, \mathscr{R})$ with primitive idempotent E of rank m (with $EJ = 0$), one can represent the point $x \in X$ by the vector $\bar{x} \in \mathbb{R}^m$ given by column x of E. Now $\langle \bar{x}, \bar{y} \rangle = e_x^\top E^\top E e_y = E_{xy}$. It follows that $\overline{X}$ has degree at most d. (See also §1.3.5.)

If $(X, \mathscr{R})$ is primitive (no union of relations is a nontrivial equivalence relation), then the map $x \mapsto \bar{x}$ is injective.

In particular, if $(X, \mathscr{R})$ is a primitive strongly regular graph on v vertices, then $\overline{X}$ has degree (at most) 2, and it follows that $v \le \frac{1}{2}m(m+3)$ if m is the multiplicity of an eigenvalue other than k. This is the *absolute bound*, see §1.3.7. For the McLaughlin graph we have equality: $v = 275$, $g = 22$.

Since $\mathrm{tr}E = m$, the scaled vectors $c\bar{x}$, where $c = \sqrt{|X|/m}$, lie on the unit sphere. Since $\sum_{x \in X} \bar{x} = 0$, and $\sum_{x \in X} \langle \bar{x}, y \rangle^2 = y^\top E y = \langle y, y \rangle$ for arbitrary $y \in \mathbb{R}^m$, we always get a spherical 2-design.

Proposition 8.15.2 *Let Γ be a primitive strongly regular graph, and let $\overline{X}$ be the spherical design formed by the columns of a primitive idempotent E of rank $m > 1$. Then*
(i) $\overline{X}$ is always a spherical 2-design,
(ii) $\overline{X}$ is a spherical 3-design if and only if $q_{ii}^i = 0$, where $E = E_i$,
(iii) $\overline{X}$ is a spherical 4-design if and only if $v = \frac{1}{2}m(m+3)$,
(iv) $\overline{X}$ is never a spherical 5-design.

We see that the absolute bound holds with equality if and only if $\overline{X}$ is a tight spherical 4-design. See also [346], [46], [132] (Chapter 10).

8.15.3 Bounds on the number of K_4's

Bondarenko et al. [89], [88] derive lower bounds for the number of K_4 subgraphs of a strongly regular graph by looking not only at the images $\bar{x}$ of the vertices but also at the images $\bar{x} + \bar{y}$ of the edges xy. As a corollary they show nonexistence for strongly regular graphs with parameters $(v, k, \lambda, \mu) = (76, 30, 8, 14)$, $(460, 153, 32, 60)$, $(5929, 1482, 275, 402)$, $(6205, 858, 47, 130)$.

8.16 Higher regularity conditions

8.16.1 The t-vertex condition

Hestenes & Higman [419] introduced the t-vertex condition. A graph Γ is said to satisfy the *t-vertex condition*, when for all triples (T, x_0, y_0) of a t-vertex graph T with two distinct distinguished vertices x_0, y_0, and all pairs of distinct vertices x, y of Γ, where $x \sim y$ if and only if $x_0 \sim y_0$, the number $n(x, y)$ of isomorphic copies of T in Γ, where the isomorphism maps x_0 to x and y_0 to y, does not depend on the choice of the pair x, y.

Clearly, a rank 3 graph satisfies the t-vertex condition for all t. If the graph Γ satisfies the t-vertex condition, where Γ has v vertices and $3 \le t \le v$, then Γ also satisfies the $(t-1)$-vertex condition. A graph Γ satisfies the 3-vertex condition if and only if it is strongly regular (or complete or empty).

Details on the parameters of graphs satisfying the 4-vertex condition (partly due to the no longer accessible [657]) are given in [419]. In particular, one has the simplified criterion for the 4-vertex condition:

Proposition 8.16.1 (Sims [657]) *A strongly regular graph* Γ *with parameters* (v, k, λ, μ) *satisfies the 4-vertex condition, with parameters* (α, β)*, if and only if the number of edges in* $\Gamma(x) \cap \Gamma(y)$ *is* α *(resp.* β*) whenever the vertices* x, y *are adjacent (resp. nonadjacent). In this case,* $k(\binom{\lambda}{2} - \alpha) = \beta(v - k - 1)$.

It immediately follows that the collinearity graph of a generalized quadrangle satisfies the 4-vertex condition (with $\alpha = \binom{\lambda}{2}$ and $\beta = 0$).

Reichard [625] shows that the collinearity graph of a generalized quadrangle satisfies the 5-vertex condition (but not necessarily the 6-vertex condition) and that the collinearity graph of a generalized quadrangle $\mathsf{GQ}(s, s^2)$ satisfies the 7-vertex condition (but not necessarily the 8-vertex condition).

One conjectures that graphs that satisfy the t-vertex condition for sufficiently large t must be rank 3.

Higman [421] and Kaski et al. [485] show that the block graph of a Steiner triple system satisfies the 4-vertex condition precisely for $\mathsf{PG}(n, 2)$ and $\mathsf{AG}(2, 3)$.

For two infinite series of graphs satisfying the 5-vertex condition, see [459], [134], [460], [624].

Below a table with parameters of small rank 4 graphs satisfying the 4-vertex condition.

v	k	λ	μ	α	β	group	ref
144	55	22	20	87	90	$\mathsf{M}_{12}.2$	§10.46
280	36	8	4	1	4	$\mathsf{HJ}.2$	§10.32
300	104	28	40	78	160	$\mathsf{PGO}_5(5)$	§3.1.4, $NO_5^-(5)$
325	144	68	60	1153	900	$\mathsf{PGO}_5(5)$	§3.1.4, $NO_5^+(5)$
512	196	60	84	420	840	$2^9.\Gamma\mathsf{L}_3(8)$	§8.4.3
729	112	1	20	0	0	$3^6.2.\mathsf{L}_3(4).2$	§10.75
1120	729	468	486	69498	74358	$\mathsf{PSp}_6(3).2$	§3.2.4
1849	462	131	110	2980	1845	$43^2:(42 \times \mathsf{D}_{22})$	§7.4.5, $e = 4$

Brouwer, Ihringer & Kantor (2021, unpublished) showed for several infinite series of strongly regular graphs that they satisfy the 4-vertex condition, and also provided a prolific construction of such graphs (with the parameters of the symplectic polar graphs).

8.16.2 t-Isoregularity

A graph is said to be *t-tuple regular* (Cameron & Van Lint [182], pp. 112–113) when for any set S of vertices with $|S| \le t$ the size of $S^\perp$ (the set of all vertices adjacent to each vertex in S) depends on the isomorphism type of S only. Elsewhere, this same concept is called *t-isoregularity*.

A graph Γ is t-isoregular if and only if its complement $\overline{\Gamma}$ is.

For $t = 1$ we find the regular graphs. For $t = 2$ we find the graphs that are strongly regular or complete or edgeless. For $t = 3$ we find the graphs that are strongly regular with strongly regular subconstituents, or complete, or edgeless. For $t = 4$ we find the graphs aK_m and their complements $K_{a\times m}$, 3×3, and the graphs with equality in the absolute bound. For $t = 5$ we find the graphs aK_m and their complements $K_{a\times m}$, the pentagon, and 3×3. (See Buczak [153], Cameron [172] (Note added in proof), and [182] (8.21).) This result is independently due to Ya. Yu. Gol'fand. See also [624], [625].

8.17 Asymptotics

8.17.1 Graph isomorphism

The problem of testing whether two graphs are isomorphic is of both theoretical and practical importance. On the practical side McKay's `nauty` works well (and improvements exist). See [555]. On the theoretical side it is unknown whether a polynomial-time algorithm exists. Babai has claimed a quasipolynomial-time algorithm, retracted the claim after Helfgott pointed out a flaw, and repaired his proof again, a few days later. Babai & Helfgott currently claim an algorithm that runs in time $\exp(O(\log v)^3)$ for graphs on v vertices. This result is so far unpublished. See [25], [26], [416]. For graphs of bounded valency, and for graphs with bounded eigenvalue multiplicity, graph isomorphism can be decided in polynomial time (Luks [527], Babai, Grigor'ev & Mount [27]).

For the graph isomorphism problem, the most difficult cases are graphs that are very similar without being isomorphic, and strongly regular graphs with the same parameters are good test cases. They have the same spectrum, and vertices or pairs of adjacent or nonadjacent vertices cannot be distinguished by counting neighbors or common neighbors. There is literature about isomorphism testing in this special case. See [22], [671], [23], [24].

Since quantum computation may be more powerful than classical computation, people have been searching for efficient quantum algorithms for the isomorphism problem. One type of attempt is getting an invariant from quantum walks. For example, [306] defines an invariant (the spectrum of a certain matrix of order vk in case of a regular graph of valency k on v vertices) and conjectures that it distinguishes nonisomorphic strongly regular graphs. Godsil, Guo & Myklebust [347] give a counterexample, and show that it does not distinguish two strongly regular graphs with parameters (756, 130, 4, 26), the collinearity graphs of two different generalized quadrangles $\mathsf{GQ}(5, 25)$.

8.17.2 Pseudo-randomness

Pyber [622] proves that large connected strongly regular graphs other than the complete multipartite graphs or block graphs of Steiner 2-designs or Latin square graphs have a big eigenvalue gap, that is, that $\max(|r|, |s|)$ is much smaller than k. It follows that these graphs are highly pseudo-random. And, for example, are Hamiltonian.

A graph Γ is called *pseudo-random* when it sufficiently resembles a random graph, say, a graph on v vertices where edges are chosen independently with some probability p.

A first precise definition was given by Thomason [698, 699], who introduced the concept of *jumbled graph*. A graph is (p, α)-jumbled when for every h-subset H of its vertex set the number $e(H)$ of edges contained in H satisfies $|e(H) - p\binom{h}{2}| \leq \alpha h$.

Chung, Graham & Wilson [196] consider weak pseudo-randomness for a series of graphs with increasing number of vertices v, while p is fixed, and show the equivalence of many properties, one of which is $e(H) = p\binom{h}{2} + o(v^2)$ for each subset H.

Let a (v, k, M)-graph be a regular graph of valency k on v vertices, with eigenvalues $k = \theta_1 \geq \cdots \geq \theta_v$ where $|\theta_i| \leq M$ for $i > 1$. If M is much smaller than k, such graphs have good randomness properties. For example, if S, T are two subsets of the vertex set of sizes s, t, respectively, and $e(S, T)$ ordered edges xy have $x \in S$, $y \in T$, then $|e(S,T) - \frac{kst}{v}|^2 \leq M^2 st(1 - \frac{s}{v})(1 - \frac{t}{v})$. (See [132], 4.3.2.)

A survey of pseudo-random graphs is given by Krivelevich & Sudakov [504].

Let Γ be a primitive strongly regular graph with parameters (v, k, λ, μ) and spectrum $k^1\ r^f\ s^g$, where $r > 0 > s$. Let $m := -s$, and $M := \max(r, m)$.

Lemma 8.17.1 $M < k^{1/2} v^{1/4}$.

Proof. By Proposition 1.3.14, $v \le \frac{1}{2}f(f+3)$ and $v \le \frac{1}{2}g(g+3)$. It follows that $f > \sqrt{v}$ and $g > \sqrt{v}$ for $v > 5$. Since $k^2 + fr^2 + gs^2 = \mathrm{tr}A^2 = kv$, it follows that $M < k^{1/2}v^{1/4}$. □

Proposition 8.17.2 $|\lambda - \mu| < v^{3/4}$.

Proof. By the lemma, $|\lambda - \mu| = |r + s| < M < k^{1/2}v^{1/4} < v^{3/4}$. □

Proposition 8.17.3 $\frac{m}{k} < 2v^{-1/6}$.

Proof. By the lemma, $M < k^{1/2}v^{1/4}$. This certainly suffices in case $k > \frac{1}{4}v$. So suppose $k \le \frac{1}{4}v$. We have $m = -s = r + \mu - \lambda \le r + \mu$. Also $v = 1 + k + \frac{k(k-1-\lambda)}{\mu} \le k^2 + 1$. If $r \ge \mu$, then $m \le 2r$. Since $rm = -rs = k - \mu \le k$, it follows that $m \le \sqrt{2k}$, and the conclusion follows from $k \ge \sqrt{v-1}$. So suppose $r < \mu$. Then $m \le r + \mu < 2\mu = \frac{2k(k-1-\lambda)}{v-k-1} < \frac{2k^2}{v-k}$ so that $\frac{m}{k} < \frac{2k}{v-k}$. Also $m^4 \le M^4 < vk^2$, so that $(\frac{m}{k})^4 < \frac{v}{k^2}$. Hence $(\frac{m}{k})^6 < \frac{v}{k^2}(\frac{2k}{v-k})^2 < \frac{64}{v}$. □

Proposition 8.17.4 *If Γ is not a Latin square graph or the block graph of a Steiner 2-design, then $\frac{r}{k} < v^{-1/10}$.*

Proof. In the half case, $v = 4t + 1$, $k = 2t$, and $r = \frac{1}{2}(-1 + \sqrt{v})$, and the conclusion holds. So, we may assume that r, s are integral, and $s \le -2$. If $k > v^{7/10}$, the conclusion follows from the lemma. Since $k = rm + \mu$, we have $k > rm$, and if $m > v^{1/10}$, the conclusion follows. By the Claw Bound (Theorem 8.6.3), we have $r \le \frac{1}{2}s(s+1)(\mu+1) - 1$. If $m \ge \mu$, then $\sqrt{v-1} \le k = rm + \mu \le \frac{1}{2}m^2(m-1)(\mu+1) - m + \mu < \frac{1}{2}m^4$ so that $m > v^{1/8}$, and we are done. Remains the case $k \le v^{7/10}$, $m \le v^{1/10}$, $m < \mu$. Then $v \ge 2^{10}$ and $v \ge 8k$. Now we have $k = rm + \mu \le \frac{1}{2}m^2(m-1)(\mu+1) - m + \mu \le \frac{1}{2}(m^3 + 2)\mu$. Since $\mu = \frac{k(k-\lambda-1)}{v-k-1} < \frac{k^2}{v-k}$, this yields $k \le \frac{1}{2}(m^3+2)\frac{k^2}{v-k}$, so that $2(v-k) \le (m^3+2)k$ and $2(\frac{7}{8}v) \le \frac{m^3+2}{m^3}(km^3) \le \frac{10}{8}v$, a contradiction. □

Theorem 8.17.5 *Let Γ be a primitive strongly regular graph. If Γ is not a Latin square graph or the block graph of a Steiner 2-design, then $M/k < 2v^{-1/10}$.* □

Let us call the graphs of the theorem, just here, *general*, and mention two applications.

Brouwer [118] showed that the toughness t of a connected non-complete regular graph satisfies $t > k/M - 2$. So general strongly regular graphs are very tough.

Krivelevich & Sudakov [503] showed that a (v, k, M)-graph is Hamiltonian if v is sufficiently large and $M/k \le \frac{(\log\log v)^2}{1000 \log v (\log\log\log v)}$. So large general strongly regular graphs are Hamiltonian.

The above estimates and results were due to Pyber. Other bounds are due to Spielman [671] and Babai & Wilmes [28]. For example, the latter show that

$$\lambda + 1 < \max\{4\sqrt{2v},\ \frac{6}{\sqrt{13}-1}\sqrt{k(\mu-1)}\}$$

for edge-regular graphs where v, k, λ have the usual meaning, and μ is an upper bound for the number of common neighbors of two vertices at distance 2.

8.18 Conditions in case $\mu = 1$ or $\mu = 2$

In a strongly regular graph with $\mu = 1$, the local graphs are unions of cliques, so that the graph is the collinearity graph of a partial linear space with lines of size $\lambda + 1$. The number of lines on each point is $k/(\lambda+1)$. The total number of lines is $vk/(\lambda+1)(\lambda+2)$. In particular, these numbers are integers.

For example, there is no strongly regular graph with parameters $(v, k, \lambda, \mu) = (209, 16, 3, 1)$ or $(726, 29, 4, 1)$.

If $\mu = 2$, then in the local graph two vertices at distance 2 have a unique common neighbor, so that the local graph is the collinearity graph of a partial linear space. This yields a lower bound on k.

Theorem 8.18.1 (Brouwer & Neumaier [139]) *A connected partial linear space with girth at least 5 and more than one line (lines possibly of varying size) in which every point has λ neighbors, contains $k \geq \frac{1}{2}\lambda(\lambda+3)$ points.*

This can be applied to the connected components of the local graph. If $(\lambda + 1) \nmid k$ then not every component can be a single line, and there must be a big component. For example, there are no strongly regular graphs with parameters $(v, k, \lambda, \mu) = (456, 35, 10, 2)$ or $(736, 42, 8, 2)$. The first of these also fails the claw bound. Slightly more information is available, which rules out the parameter set $(1944, 67, 10, 2)$.

Bagchi [31] slightly strengthened these results and showed for $\mu = 1$ that $k \geq (\lambda+1)(\lambda+2)$, eliminating, e.g., the parameter sets $(1666, 45, 8, 1)$ and $(2745, 56, 7, 1)$, and for $\mu = 2$ that the graph is either a grid graph or satisfies $k \geq \frac{1}{2}\lambda(\lambda+3)$.

8.19 Coloring

We sketch what is known about the chromatic number $\chi(\Gamma)$ of strongly regular graphs Γ. Eigenvalue methods provide lower bounds. For more detail, see [132], §3.6, and [323]. Explicit constructions provide upper bounds.

Proposition 8.19.1 (Hoffman [434]) *If the graph Γ is not edgeless, and has largest eigenvalue $\theta_{\max}$ and smallest eigenvalue $\theta_{\min}$, then $\chi(\Gamma) \geq 1 - \theta_{\max}/\theta_{\min}$.*

When equality holds, the coloring is called a *Hoffman coloring.* Since clearly $\chi(\Gamma) \geq |V\Gamma|/\alpha(\Gamma)$, where $\alpha(\Gamma)$ is the independence number of Γ, a Hoffman coloring of a regular graph is a partition of its vertex set into cocliques reaching the Hoffman bound. Haemers & Tonchev [385] investigate strongly regular graphs with a Hoffman coloring, and give a table with the examples on at most 100 vertices. Their smallest open case was settled in [163].

Proposition 8.19.2 (Haemers [376], 2.2.2) *If the graph Γ on v vertices has eigenvalues $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_v$, and $\theta_2 > 0$, and θ_v has multiplicity m, then $\chi(\Gamma) \geq \min(m, 1 - \theta_v/\theta_2)$.*

Corollary 8.19.3 *If the strongly regular graph Γ with distinct eigenvalues $k > r > s$ is not the pentagon and not complete multipartite, then $\chi(\Gamma) \geq 1 - s/r$.*

HAEMERS [376] determined all primitive strongly regular graphs with chromatic number at most 4. There are three examples with $\chi(\Gamma) = 3$:

v	k	λ	μ	graph
5	2	0	1	pentagon
9	4	1	2	$L_2(3)$
10	3	0	1	Petersen graph

and 18 examples with $\chi(\Gamma) = 4$:

v	k	λ	μ	graph
15	6	1	3	$\overline{T(6)}$
16	5	0	2	complement of the Clebsch graph
16	6	2	2	$L_2(4)$
16	6	2	2	Shrikhande graph
16	9	4	6	$\overline{L_2(4)}$
50	7	0	1	Hoffman-Singleton graph
56	10	0	2	Gewirtz graph
64	18	2	6	11 incidence graphs of triples of linked designs.

Here the graphs with parameters (64,18,2,6) are derived from the systems of three linked 2-(16,6,2) designs.

Linked designs A *system of linked designs* (CAMERON [170]) is a particular type of coherent configuration (or of Buekenhout-Tits geometry). One has sets of objects $X_0, \ldots, X_{r-1}$ and incidence relations between X_i and X_j for $i \neq j$, such that (i) each pair (X_i, X_j) with $i \neq j$ determines a square (a.k.a symmetric, or projective) 2-design, and (ii) for any three sets X_i, X_j, X_k the number of $x \in X_i$ incident with both $y \in X_j$ and $z \in X_k$ depends only on whether y and z are incident. Such a system is called a system of $r-1$ linked designs (where one arbitrarily chooses one set X_0 as the point set, and views the remaining X_i as the sets of blocks for $r-1$ designs).

Cameron describes systems of linked designs derived from $\mathsf{Sp}(2m, q)$, where $q = 2^n$, and systems (due to Goethals) derived from Kerdock codes, and the construction of systems of linked 2-(16,6,2) designs from the Steiner system $S(5, 8, 24)$. MATHON [545] analyzes the case of linked 2-(16,6,2) designs and finds that there are 3 pairs, 12 triples, and unique 4-, 5-, 6- and 7-sets of such designs. The 12 nonisomorphic triples lead to 11 nonisomorphic 4-colorable strongly regular graphs with parameters (64,18,2,6). See also [385], [250] (§5.4), [495], [612].

FIALA & HAEMERS [323] show that a strongly regular Γ with $\chi(\Gamma) = 5$ has one of 43 parameter sets, and completely settle 34 of these 43 cases.

Edge coloring

The edge-chromatic number of strongly regular graphs is studied in CIOABĂ, GUO & HAEMERS [200]. By Vizing's theorem the edge chromatic number of a graph is either the maximum degree, or one more, and the corresponding graphs are called of *Vizing class* 1 and 2, respectively. A regular graph of valency k is of Vizing class 1 when it has an edge coloring with k colors, that is, when it has a 1-factorization. Such a graph necessarily has an even number of vertices. These authors conjecture that every connected strongly regular graph with an even number of vertices is of Vizing class 1, except for the Petersen graph. This is true if the valency is at most 18, and for several infinite families of graphs.

8.20 Graphs that are locally strongly regular

Let Δ be a fixed graph. A graph Γ is called *locally* Δ when the induced subgraph on each vertex neighborhood $\Gamma(x)$ is isomorphic to Δ. Let $\mathscr{D}$ be a class of graphs. A graph Γ is called *locally* $\mathscr{D}$ when each vertex neighborhood $\Gamma(x)$ is isomorphic to a member of $\mathscr{D}$. We met this concept earlier, and saw that a Fischer graph is locally Fischer, and looked, e.g., at locally cotriangular graphs.

Weetman [722] showed that if Δ is regular of degree > 1 and has girth at least 6 then there exist infinite graphs Γ that are locally Δ. The typical example is the triangulation of the plane that is locally a hexagon.

Conversely, Weetman [723] shows that in many cases if Δ is strongly regular, the diameter of Γ is bounded.

Theorem 8.20.1 (Weetman [723]) *Let Δ be strongly regular with parameters (v, k, λ, μ). If (i) $v \le 2k + 1$, or (ii) $\mu > \lambda$, or (iii) Δ is the collinearity graph of a partial geometry, then any locally Δ graph has diameter at most $k + 1$.*

Weetman conjectures that any graph that is locally strongly regular (with $\mu > 0$) is finite. It is true that any graph that is locally Δ, where Δ is strongly regular on at most 195 vertices (with $\mu > 0$), is finite ([117]).

8.21 Dropping regularity

A frequently rediscovered result says what happens if we drop the regularity condition from the definition of strongly regular graph.

Proposition 8.21.1 (Bose & Dowling [93]) *Let Γ be a graph, not complete, not edgeless, such that any two adjacent (resp. nonadjacent) vertices have λ (resp. μ) common neighbors. Then either Γ is strongly regular, or $\mu = 0$ and Γ is the disjoint union of complete subgraphs of sizes 1 or $\lambda + 2$, or $\mu = 1$ and Γ is the union of complete subgraphs of size $\lambda + 2$ with a single common vertex.*

The particular case $\lambda = \mu = 1$ of this proposition in known as the *friendship problem*. See also [412].

8.22 Directed strongly regular graphs

A *directed strongly regular graph* (dsrg) is a (0,1)-matrix A with zero diagonal such that the linear span of I, J and A is closed under matrix multiplication. This concept was defined by Duval [298], and most of the theory is due to him. The matrix A is the adjacency matrix of a directed graph without loops, so that xy is an edge when $A_{xy} = 1$.

One defines (integral) parameters (v, k, t, λ, μ) by: v is the number of vertices, k is the constant indegree and outdegree (that is, $AJ = JA = kJ$), and $A^2 = tI + \lambda A + \mu(J - I - A)$. If we regard an undirected edge as the combination of two oppositely directed edges, then t is the number of undirected edges on each vertex.

These dsrg's come in complementary pairs: together with A also $J - I - A$ satisfies the definition. If the first one has parameters (v, k, t, λ, μ) then its complement has parameters $(v, v-k-1, v-2k-1+t, v-2k-2+\mu, v-2k+\lambda)$.

For a given set of parameters, dsrg's also come in pairs: together with A also its transpose $A^\top$ satisfies the definition. (One arises from the other by reversing all arrows.) The corresponding dsrg's may or may not be isomorphic.

If the graph is undirected ($A = A^\top$), then we have a strongly regular graph (and $t = k$). Let us assume that the graph is not undirected, that is, that A is not symmetric. Then $t < k < v - 1$.

Spectrum

Since the case $A = J - I$ was excluded, the algebra spanned by I, A and J is 3-dimensional. It follows that A has precisely 3 distinct eigenvalues, say k, r, s, with multiplicities 1, f and g, respectively.

The eigenvalues r, s different from k are roots of $x^2 + (\mu - \lambda)x + \mu - t = 0$ so are algebraic integers. We distinguish two cases, depending on whether $f = g$.

Proposition 8.22.1 *A directed strongly regular graph with $f \neq g$ has integral eigenvalues k, r, s with $r \geq 0$ and $s < 0$, and satisfies $\mu \leq t$ and $t \neq 0$.*

Proof. If $f \neq g$, we can solve r, s from $r + s = \lambda - \mu$ and $fr + gs = -k$ to find that r and s are rational numbers, and therefore integers.

At least one of r, s is negative since $\operatorname{tr} A = 0$. But $J - I - A$ has eigenvalues $v - 1 - k$, $-1 - s$, $-1 - r$, and also has a negative eigenvalue, so we may assume that $r \geq 0$ and $s < 0$. In particular, $r \neq s$, some linear combination of A, I and J is a projection, and A is diagonalizable. Moreover, $rs \leq 0$, so $\mu \leq t$, and hence $t \neq 0$. □

Proposition 8.22.2 *Directed strongly regular graphs with $t = 0$ are equivalent to Hadamard matrices of order 4μ that have 1's on the diagonal and are skew-symmetric off-diagonal.*

Proof. Suppose H is a Hadamard matrix as described. By suitably multiplying rows and columns by -1, we may assume that H has an all-1 top row. Let B be the matrix obtained by deleting the first row and column from H. From $HH^\top = 4\mu I$, we find $BJ = J$ and $B + B^\top = 2I$ and $BB^\top = 4\mu I - J$. Now let $A = \frac{1}{2}(J - B)$. Then A is a (0,1)-matrix with zero diagonal satisfying $A + A^\top = J - I$ and $AJ = JA = \frac{v-1}{2}J$ and $A^2 + A = \mu(J - I)$, so that this is a directed strongly regular graph with parameters $(v, k, t, \lambda, \mu) = (4\mu - 1, 2\mu - 1, 0, \mu - 1, \mu)$.

Conversely, let a directed strongly regular graphs satisfy $t = 0$. By the above, $f = g = (v - 1)/2$. From $(\lambda - \mu)(v - 1)/2 = -k$ it follows that $k = (v - 1)/2$ and $\mu = \lambda + 1$. From $k^2 = t + \lambda k + \mu(v - 1 - k)$ we see $k = 2\mu - 1$. The graph is a tournament: $A^\top = J - I - A$, and bordering the $(1, -1)$ matrix $J - 2A$ first with a first column of all -1's and then with a top row of all 1's we find a Hadamard matrix H as desired. □

In the below we exclude the case $t = 0$.

The 2-dimensional case

An important subclass is that where already the linear span of A and J is closed under matrix multiplication. This happens when $t = \mu$, and then $A^2 = (\lambda - \mu)A + \mu J$ so that the eigenvalues of A are k, $\lambda - \mu$, and 0. Conversely, when A has eigenvalue 0, we are in this case. Put $d = \mu - \lambda$. Then the multiplicities of the eigenvalues k, $-d$, 0 are 1, k/d and $v - 1 - k/d$, respectively.

The 1-dimensional case

If A is a 0-1 matrix, then $B = J - 2A$ is a ± 1 matrix, and it is possible that the 1-space generated by B is closed under matrix multiplication. This happens when $t = \mu$ and $\frac{1}{2}v = 2k - \lambda - \mu$, and then $k - \mu \in \{\lambda, \mu\}$. Conversely, if B is a ± 1 matrix with constant row sums and 1's on the diagonal such that B^2 is a multiple of B, then $A = (J - B)/2$ is the adjacency matrix of a dsrg in this subcase. Note that the set of such ± 1 matrices B is closed under taking tensor products.

Combinatorial parameter conditions

We already saw that $0 < \mu \leq t$. (If $\mu = 0$ then $A = J - I$, which was excluded.) Also, that $0 \leq \lambda < t < k < v$. (Indeed, $\lambda < t$, since $\lambda + 1 - t = (r+1)(s+1) \leq 0$.)

Duval gave one more condition. We have $-2(k - t - 1) \leq \mu - \lambda \leq 2(k - t)$. (Indeed, consider a directed edge from x to y. Paths of length 2 from x to y contribute to λ, paths in the opposite direction to μ. Thus, the difference between μ and λ is counted by the at most $2(k - t)$ paths that cannot be reversed. The other inequality follows similarly, or by applying the first to the complementary graph.)

No Abelian Cayley graphs

From the spectrum we can draw one more useful conclusion (Klin et al. [493]). Let us write $A = A_s + A_a$ where A_s is the symmetric 0-1 matrix (with row sums t) describing adjacency via an undirected edge, and A_a is the 0-1 matrix describing the remaining, directed, edges (with row sums $k - t$). Since A_a has a nonzero real eigenvalue, namely $k - t$, and its square has trace zero, A_a must also have non-real eigenvalues. On the other hand, both A and A_s only have real eigenvalues. It follows that A and A_s cannot be diagonalized simultaneously, so that A_s and A_a do not commute. But then these matrices do not describe differences in the same Abelian group. Thus, a directed strongly regular graph cannot be a Cayley graph of an Abelian group.

Examples

There are many constructions, and we just give a few random examples.

(i) If there exists a dsrg A with parameters (v, k, t, λ, μ), and $t = \mu$, then for any $m \geq 1$ there is also a dsrg with parameters $(mv, mk, mt, m\lambda, m\mu)$, obtained by taking $A \otimes J$, where J is of order m ([298]).

(ii) Let $m \geq 1$ be an integer. Then there exists a dsrg with parameters $(v, k, t, \lambda, \mu) = (4m + 2, 2m, m, m - 1, m)$ found by taking the vertices x_i and y_i ($0 \leq i \leq 2m$, indices mod $2m + 1$), and directed edges $x_i \to x_{i+j}, y_{i+j}$, $y_i \to x_{i-j}, y_{i-j}$ where $1 \leq j \leq m$ ([493]).

(iii) Let μ, k be positive integers such that $\mu | (k-1)$. Then there exists a dsrg with parameters $(v, k, t, \lambda, \mu) = ((k^2 - 1)/\mu, k, \mu + 1, \mu, \mu)$ found by taking as vertices the integers mod v and letting $x \to y$ be an edge when $x + ky = 1, 2, ..., k$ (mod v) ([470]).

(iv) If there exists a strongly regular graph with parameters (v, k, λ, μ) where $\mu = \lambda + 1$, then there is a dsrg with parameters $(v', k', t', \lambda', \mu') = (vk, (v - k - 1)k, (v - k - 1)(k - \mu), (v - k - 2)(k - \mu), (v - k - 1)(k - \mu))$. Construction: take the edges of the srg, and let $xy \to uv$ when u is at distance 2 from y. For example, the Petersen graph produces a dsrg$(30, 18, 12, 10, 12)$.

Further constructions abound. Surveys can be found elsewhere.

Chapter 9

p-Ranks

Let M be an integral matrix. The *p-rank* of M, denoted $\mathrm{rk}_p M$, is the rank of M over the field $\mathbb{F}_p$.

Designs or graphs with the same parameters can sometimes be distinguished by considering the p-rank of associated matrices. For example, there are three nonisomorphic 2-(16,6,2) designs, with point-block incidence matrices of 2-rank 6, 7 and 8, respectively.

Tight bounds on the occurrence of certain configurations are sometimes obtained by computing a rank in some suitable field, since p-ranks of integral matrices may be smaller than their ranks over $\mathbb{R}$. For example, the Blokhuis-Moorhouse theorem (Theorem 2.6.2) gives good bounds on the size of partial ovoids in an orthogonal polar space.

9.1 Points and hyperplanes of a projective space

The following result was found independently by Goethals & Delsarte [352] and by MacWilliams & Mann [533]. A nicer proof was given by Smith [662].

Theorem 9.1.1 *Let A be the 0-1 incidence matrix of points and hyperplanes of $\mathsf{PG}(d,q)$, where $q = p^e$. Then* $\mathrm{rk}_p A = \binom{d+p-1}{d}^e + 1$.

We already encountered the special case of $\mathsf{PG}(2,4)$ in Theorem 6.2.2.

More generally, Hamada [402, 403] determined the p-rank of the incidence matrix of points and i-subspaces in $\mathsf{PG}(d,q)$.

See also Assmus & Key [17] and [145], §4.

9.2 Graphs

On the 2-rank

Let A be a symmetric integral matrix with zero diagonal. Then $\mathrm{rk}_2 A$ is even.

The diagonal of a symmetric (0,1)-matrix A (written as row vector) is element of its $\mathbb{F}_2$-rowspan $\langle A\rangle_2$.

Adding a multiple of J

Let M be an integral matrix of order v with row sums k. Given a field F, let $\mathrm{rk}_F(M)$ be the rank of M over F, that is the dimension of the row space $\langle M\rangle_F$. Consider $\mathrm{rk}_F(M+bJ)$ for integral b. Since J has rank 1, all matrices $M+bJ$ differ in rank by at most 1, so either all have the same rank r, or two ranks r, $r+1$ occur, and in the latter case rank $r+1$ occurs whenever $\mathbf{1} \in \langle M+bJ\rangle_F$.

If $\mathbf{1} \notin \langle M\rangle$ and $\mathbf{1} \in \langle M+bJ\rangle$ for some $b \neq 0$, then $\mathbf{1} \in \langle M+bJ\rangle$ for all $b \neq 0$. Thus, either $\mathrm{rk}_F(M+bJ)$ is independent of b, or there is precisely one value of b (in F) for which this rank is lower.

Now let $F = \mathbb{F}_p$. The matrix $M+bJ$ has row sums $k+bv$.

If $p \nmid v$, then $\mathbf{1} \in \langle M+bJ\rangle_p$ when $k+bv \not\equiv 0 \pmod p$. On the other hand, if $k+bv \equiv 0 \pmod p$, then all rows have zero row sum (mod p) while $\mathbf{1}$ has not, so that $\mathbf{1} \notin \langle M+bJ\rangle_p$. Thus, we are in the second case, where the smaller p-rank occurs for $b = -k/v$ only.

If $p \mid v$ and $p \nmid k$, then all row sums are nonzero (mod p) for all b, and we are in the former case: the rank is independent of b, and $\langle M+bJ\rangle_p$ always contains $\mathbf{1}$.

Finally, if $p|v$ and also $p|k$, then further inspection is required.

9.3 Strongly regular graphs

For strongly regular graphs the interesting primes p are those with $p \mid (r-s)$. All other ranks are already determined by the parameters.

Let Γ be a strongly regular graph with adjacency matrix A, and assume that A has integral eigenvalues k, r, s with multiplicities $1, f, g$, respectively. We investigate the p-rank of a linear combination of A, I and J.

The following proposition shows that only the case $p|(r-s)$ is interesting. More detail is given in [126]. See also [132], Ch. 13.

Proposition 9.3.1 *Let $M = A + bJ + cI$ where b, c are integers. Then M has eigenvalues $\theta_0 = k + bv + c$, $\theta_1 = r + c$, $\theta_2 = s + c$, with multiplicities $m_0 = 1$, $m_1 = f$, $m_2 = g$, respectively.*

(i) If none of the θ_i vanishes (mod p), then $\mathrm{rk}_p M = v$.

(ii) If precisely one θ_i vanishes (mod p), then M has p-rank $v - m_i$.

Put $e := \mu + b^2 v + 2bk + b(\mu - \lambda)$.

(iii) If $\theta_0 \equiv \theta_1 \equiv 0 \pmod p$, $\theta_2 \not\equiv 0 \pmod p$, then $\mathrm{rk}_p M = g$ if and only if $p|e$, and $\mathrm{rk}_p M = g+1$ otherwise.

(iii)′ If $\theta_0 \equiv \theta_2 \equiv 0 \pmod p$, $\theta_1 \not\equiv 0 \pmod p$, then $\mathrm{rk}_p M = f$ if and only if $p|e$, and $\mathrm{rk}_p M = f+1$ otherwise.

(iv) In particular, if $k \equiv r \equiv 0 \pmod p$ and $s \not\equiv 0 \pmod p$, then $\mathrm{rk}_p A = g$. And if $k \equiv s \equiv 0 \pmod p$ and $r \not\equiv 0 \pmod p$, then $\mathrm{rk}_p A = f$.

(v) If $\theta_1 \equiv \theta_2 \equiv 0 \pmod p$, then $\mathrm{rk}_p M \le \min(f+1, g+1)$.

Proof. See [132], §13.7. □

Idempotents

If $p \mid (r-s)$ then Proposition 9.3.1 only says that $\mathrm{rk}_p M \le \min(f+1, g+1)$. Looking at the idempotents sometimes improves this bound by 1: We have

$E_1 = \frac{1}{r-s}(A - sI - \frac{k-s}{v}J)$ and $E_2 = \frac{1}{s-r}(A - rI - \frac{k-r}{v}J)$. Thus, if $k - s$ and v are divisible by the same power of p (so that $\frac{k-s}{v}$ can be interpreted in $\mathbb{F}_p$), then $\mathrm{rk}_p(A - sI - \frac{k-s}{v}J) \le \mathrm{rk}\, E_1 = f$, and, similarly, if $k - r$ and v are divisible by the same power of p then $\mathrm{rk}_p(A - rI - \frac{k-r}{v}J) \le \mathrm{rk}\, E_2 = g$.

For $M = A + bJ + cI$ and $p|(r + c)$, $p|(s + c)$ we have $ME_1 = JE_1 = 0$ (over $\mathbb{F}_p$) so that $\mathrm{rk}_p\langle M, \mathbf{1}\rangle \le g + 1$, and hence $\mathrm{rk}_p M \le g$ (and similarly $\mathrm{rk}_p M \le f$) in case $\mathbf{1} \notin \langle M\rangle$.

The half case

If r, s are nonintegral, we are in the half case, with $(v, k, \lambda, \mu) = (4t+1, 2t, t-1, t)$ and $r, s = (-1 \pm \sqrt{v})/2$. The analog of Proposition 9.3.1 for this case is

Proposition 9.3.2 *Let $M = A + cI$ where c is an integer. Then M has eigenvalues $\theta_0 = k + c$, $\theta_1, \theta_2 = \frac{1}{2}(-1 \pm \sqrt{v}) + c$, with multiplicities $m_0 = 1$ and $m_1 = m_2 = (v - 1)/2$, so that $\theta_1\theta_2 = c^2 - c - \mu$. Let p be a prime.*

(i) If $p \nmid \theta_0\theta_1\theta_2$, then $\mathrm{rk}_p M = v$.

(ii) If $p \nmid \theta_1\theta_2$ but $p|\theta_0$, then $\mathrm{rk}_p M = v - 1$.

Now suppose that $p|(c^2 - c - \mu)$. If $p = 2$, this does not happen when μ is odd, and happens for all c when μ is even. If $p > 2$, then $p|(c^2 - c - \mu)$ is equivalent to $(2c - 1)^2 \equiv v \pmod{p}$, and there are 0, 1 or 2 solutions for $c \pmod{p}$, depending on whether v is a nonsquare, zero or a square $\pmod{p}$.

(iii) If $\mu \equiv 0 \pmod{p}$, then $p \mid c(c-1)$, and $\mathrm{rk}_p A = (v-1)/2$ and $\mathrm{rk}_p(A+I) = (v + 1)/2$.

(iv) If v is a nonzero square $\pmod{p}$ and $v \not\equiv 1 \pmod{p}$, then $\mathrm{rk}_p M = (v + 1)/2$ for the two values of c satisfying $(2c - 1)^2 \equiv v \pmod{p}$.

Proof. See [126], §4. □

This proposition covers all cases except that where $p \mid v$ where p is odd. In that case we only know $\mathrm{rk}_p M \le (v+1)/2$, with equality in case $p \,||\, v$ (Proposition 9.3.5). For Paley and Peisert graphs the precise values are given in Propositions 9.3.3 and 9.3.4.

Table

In the table below we give for a few strongly regular graphs for each prime p dividing $r - s$ the p-rank of $A - sI$ and the unique b_0 such that $\mathrm{rk}_p(A - sI - b_0J) = \mathrm{rk}_p(A - sI - bJ) - 1$ for all $b \not\equiv b_0 \pmod{p}$, or '-' in case $\mathrm{rk}_p(A - sI - bJ)$ is independent of b.

(When $p \nmid v$ we are in the former case, and $b_0 = (k - s)/v$ follows from the parameters. When $p \mid v$ and $p \nmid \mu$, we are in the latter case.)

Since $\overline{A} = J - I - A$ for the complementary graph, the table line for the complement would have the same minimal p-rank, and $\overline{b_0} = 1 - b_0$.

This table extends that in [126].

Name	ref	v	k	λ	μ	r^f	s^g	p	$\mathrm{rk}_p(A - sI)$	b_0
3×3, Paley(9)	§1.1.8	9	4	1	2	1^4	$(-2)^4$	3	4	-
$T(5)$	§1.1.7	10	6	3	4	1^4	$(-2)^5$	3	5	2
$T(6)$	§1.1.7	15	8	4	4	2^5	$(-2)^9$	2	4	0
Folded 5-cube	§10.7	16	5	0	2	1^{10}	$(-3)^5$	2	6	-
4×4	§1.1.8	16	6	2	2	2^6	$(-2)^9$	2	6	-

continued...

Name	ref	v	k	λ	μ	r^f	s^g	p	$\mathrm{rk}_p(A-sI)$	b_0
$T(7)$	§1.1.7	21	10	5	4	3^6	$(-2)^{14}$	5	7	2
5×5	§1.1.8	25	8	3	2	3^8	$(-2)^{16}$	5	8	-
Paley(25)	§1.1.9	25	12	5	6	2^{12}	$(-3)^{12}$	5	9	-
Schläfli	§10.10	27	16	10	8	4^6	$(-2)^{20}$	2	6	0
								3	7	-
$T(8)$	§1.1.7	28	12	6	4	4^7	$(-2)^{20}$	2	6	0
								3	8	2
3 Chang graphs	§10.11	28	12	6	4	4^7	$(-2)^{20}$	2	8	-
								3	8	2
$\overline{J}(8,4)$	§10.13	35	16	6	8	2^{20}	$(-4)^{14}$	2	6	0
								3	14	1
6×6	§1.1.8	36	10	4	2	4^{10}	$(-2)^{25}$	2	10	-
								3	10	-
$\mathsf{G}_2(2)$	§10.14	36	14	4	6	2^{21}	$(-4)^{14}$	2	8	-
								3	14	-
$T(9)$	§1.1.7	36	14	7	4	5^8	$(-2)^{27}$	7	9	2
$NO_6^-(2)$	§10.15	36	15	6	6	3^{15}	$(-3)^{20}$	2	7	1
								3	15	1
$\mathsf{Sp}_4(3)$	§10.16	40	12	2	4	2^{24}	$(-4)^{15}$	2	16	-
								3	11	1
$\mathsf{O}_5(3)$	§10.16	40	12	2	4	2^{24}	$(-4)^{15}$	2	10	-
								3	15	1
$\mathsf{U}_4(2)$	§10.17	45	12	3	3	3^{20}	$(-3)^{24}$	2	15	1
								3	15	2
$T(10)$	§1.1.7	45	16	8	4	6^9	$(-2)^{35}$	2	8	0
Paley(49)	§1.1.9	49	24	11	12	3^{24}	$(-4)^{24}$	7	16	-
Hoffman-Singleton	§10.19	50	7	0	1	2^{28}	$(-3)^{21}$	5	21	-
Gewirtz	§10.20	56	10	0	2	2^{35}	$(-4)^{20}$	2	20	-
								3	20	1
$\mathsf{Sp}_6(2)$	§10.21	63	30	13	15	3^{35}	$(-5)^{27}$	2	7	1
$\mathsf{GQ}(3,5)$	§10.24	64	18	2	6	2^{45}	$(-6)^{18}$	2	14	-
$2^6:\mathsf{O}_6^-(2)$	§10.25	64	27	10	12	3^{36}	$(-5)^{27}$	2	8	-
Halved folded 8-cube	§10.26	64	28	12	12	4^{28}	$(-4)^{35}$	2	8	-
M_{22}	§10.27	77	16	0	4	2^{55}	$(-6)^{21}$	2	20	0
Brouwer-Haemers	§10.28	81	20	1	6	2^{60}	$(-7)^{20}$	3	19	-
Paley(81)	§1.1.9	81	40	19	20	4^{40}	$(-5)^{40}$	3	16	-
Higman-Sims	§10.31	100	22	0	6	2^{77}	$(-8)^{22}$	2	22	-
								5	23	-
Hall-Janko	§10.32	100	36	14	12	6^{36}	$(-4)^{63}$	2	36	0
								5	23	-
Flags of $\mathsf{PG}(2,4)$	§10.33	105	32	4	12	2^{84}	$(-10)^{20}$	2	18	0
								3	20	2
$\mathsf{GQ}(3,9)$	§10.34	112	30	2	10	2^{90}	$(-10)^{21}$	2	22	-
								3	20	1
$NO_6^+(3)$	§10.35	117	36	15	9	9^{26}	$(-3)^{90}$	2	27	1
								3	21	-
001... in $S(5,8,24)$	§10.37	120	42	8	18	2^{99}	$(-12)^{20}$	2	20	-
								7	20	5
$\overline{NO_8^+(2)}$	§10.39	120	56	28	24	8^{35}	$(-4)^{84}$	2	8	0
								3	36	2
S_{10}	§10.40	126	25	8	4	7^{35}	$(-3)^{90}$	2	27	1
								5	36	3
$NO_6^-(3)$	§10.41	126	45	12	18	3^{90}	$(-9)^{35}$	2	27	1
								3	36	-
Goethals	§10.42	126	50	13	24	2^{105}	$(-13)^{20}$	3	21	-
								5	20	3
$\mathsf{O}_8^+(2)$	§10.43	135	70	37	35	7^{50}	$(-5)^{84}$	2	9	1
								3	50	-
Faradžev-Klin-Muzychuk	§10.45	144	39	6	12	3^{104}	$(-9)^{39}$	2	40	-
								3	32	-
$\mathsf{Sp}_4(5)$		156	30	4	6	4^{90}	$(-6)^{65}$	2	66	-
								5	36	1
2nd sub McL	§10.48	162	56	10	24	2^{140}	$(-16)^{21}$	2	20	0
								3	21	-

continued...

Name	ref	v	k	λ	μ	r^f	s^g	p	$\mathrm{rk}_p(A-sI)$	b_0
Edges of Ho-Si		175	72	20	36	2^{153}	$(-18)^{21}$	2	20	0
								5	21	-
01... in $S(5,8,24)$	§10.51	176	70	18	34	2^{154}	$(-18)^{21}$	2	22	-
								5	22	3
A switched version of the previous graph		176	90	38	54	2^{153}	$(-18)^{22}$	2	22	-
								5	22	3
Cameron	§10.54	231	30	9	3	9^{55}	$(-3)^{175}$	2	55	1
								3	56	1
Berlekamp-Van Lint-Seidel	§10.55	243	22	1	2	4^{132}	$(-5)^{110}$	3	67	-
Delsarte		243	110	37	60	2^{220}	$(-25)^{22}$	3	22	-
$S(4,7,23)$	§10.56	253	112	36	60	2^{230}	$(-26)^{22}$	2	22	0
								7	23	5
$VO_8^-(2)$	§10.59	256	119	54	56	7^{136}	$(-9)^{119}$	2	10	-
$VO_8^+(2)$	§10.60	256	120	56	56	8^{120}	$(-8)^{135}$	2	10	-
McLaughlin	§10.61	275	112	30	56	2^{252}	$(-28)^{22}$	2	22	0
								3	22	1
								5	23	-
A switched version of the previous graph plus isolated point		276	140	58	84	2^{252}	$(-28)^{23}$	2	24	-
								3	23	2
								5	24	3
Mathon-Rosa	§10.62	280	117	44	52	5^{195}	$(-13)^{84}$	2	68	-
								3	42	1
$NO_7^{-\perp}(3)$	§10.66	351	126	45	45	9^{168}	$(-3)^{182}$	2	79	1
								3	27	0
$\mathsf{G}_2(4)$	§10.68	416	100	36	20	20^{65}	$(-4)^{350}$	2	38	-
								3	65	1
$P(23^2)$, $P^*(23^2)$, $P^{**}(23^2)$	§10.70	529	264	131	132	11^{264}	$(-12)^{264}$	23	144	-
$\mathsf{U}_6(2)$		693	180	51	45	15^{252}	$(-9)^{440}$	2	35	1
								3	231	-
Games	§10.75	729	112	1	20	4^{616}	$(-23)^{112}$	3	98	-
$NO_8^+(3)$	§10.78	1080	351	126	108	27^{260}	$(-9)^{819}$	2	261	1
								3	36	2
Dodecads mod **1**	§10.80	1288	792	476	504	8^{1035}	$(-36)^{252}$	2	22	0
								11	230	3
$\mathsf{U}_6(2)$ on 1408	§10.81	1408	567	246	216	39^{252}	$(-9)^{1155}$	2	22	-
								3	229	0
Suz	§10.83	1782	416	100	96	20^{780}	$(-16)^{1001}$	2	638	0
								3	66	-
$2^{11}.\mathsf{M}_{24}$, $k=276$	§10.84	2048	276	44	36	20^{759}	$(-12)^{1288}$	2	112	-
$2^{11}.\mathsf{M}_{24}$, $k=759$	§10.85	2048	759	310	264	55^{276}	$(-9)^{1771}$	2	24	-
Co_2	§10.88	2300	891	378	324	63^{275}	$(-9)^{2024}$	2	23	1
								3	275	0
Fi_{22}	§10.90	3510	693	180	126	63^{429}	$(-9)^{3080}$	2	79	1
								3	351	0
Ru	§10.91	4060	1755	730	780	15^{3276}	$(-65)^{783}$	2	29	1
								5	784	4
Fi_{22} on 14080	§10.94	14080	3159	918	648	279^{429}	$(-9)^{13650}$	2	352	-
								3	351	0

Table 9.1: p-ranks of some strongly regular graphs

Some graph families

Lattice graphs

For $n \times n$ the interesting primes are those dividing n. For $p \,|\, n$ we have $\mathrm{rk}_p(A + 2I - bJ) = 2n - 2$ for all b.

Triangular graphs

For $T(n)$ the interesting primes are those dividing $n-2$. For $p \,|\, (n-2)$, p odd, $n \geq 3$ we have $\mathrm{rk}_p(A + 2I - bJ) = n$ if $b \neq 2$, and $\mathrm{rk}_p(A + 2I - 2J) = n - 1$.

For $p = 2$, n even, $n \geq 2$ we have $\mathrm{rk}_2(A) = n - 2$ and $\mathrm{rk}_2(A + J) = n - 1$.

p-rank of Paley and Peisert graphs

Proposition 9.3.3 (Brouwer & van Eijl [126]) *Let $q = p^e$ where p is prime and $q \equiv 1 \pmod 4$, and let A be the adjacency matrix of $P(q)$, the Paley graph of order q. Then*

$$\mathrm{rk}_p(2A + I) = (\frac{p+1}{2})^e.$$

Proposition 9.3.4 (Weng, Qiu, Wang & Xiang [725]) *Let $q = p^e$ where p is prime, $p \equiv 3 \pmod 4$ and $e = 2t$ is even. Let A be the adjacency matrix of $P^*(q)$, the Peisert graph of order q. Then*

$$\mathrm{rk}_p(2A + I) = 2(3^t - 1)(\frac{p+1}{4})^{2t}.$$

(For $e > 4$ this p-rank is smaller than that of the previous proposition, so in that sense $P^*(q)$ is nicer than $P(q)$.)

More generally, [126] gives the p-ranks of arbitrary strongly regular graph with 'half case' parameters for all p not dividing v, but only $\mathrm{rk}_p(2A + I) \leq (v + 1)/2$ when $p \mid v$. Equality holds if $p \,||\, v$ (that is, $p \mid v$, $p^2 \nmid v$):

Proposition 9.3.5 (Peeters [611]) *Let A be the adjacency matrix of a strongly regular graph with 'half case' parameters $(v, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)$. If p is prime, and $p \,||\, v$, then $\mathrm{rk}_p(2A + I) = (v + 1)/2$.*

Symplectic graphs

For $\mathsf{Sp}(n, q)$, $n = 2m$ we have $r, s = -1 \pm q^{m-1}$, so the interesting primes are 2 and p, where $q = p^e$. For the p-rank: $\mathrm{rk}_p(A + I - J) = \mathrm{rk}_p(A + I - bJ) - 1$ for $b \neq 1 \pmod p$, that is $b_0 = 1$. And $\mathrm{rk}_p(A + I) = \binom{p+n-2}{n-1}^e + 1$ since $A + I$ is just the point-hyperplane incidence matrix of $\mathsf{PG}(n - 1, q)$.

In particular, for $\mathsf{Sp}(n, 2)$ we have $\mathrm{rk}_2(A + I) = n + 1$ and $\mathrm{rk}_2(J - I - A) = n$. Peeters [611] showed that the corresponding graphs are characterized by their parameters and 2-rank. Abiad & Haemers [3] constructed graphs with the same parameters and varying 2-rank. Godsil & Royle [351] show that any graph Γ of which the adjacency matrix has 2-rank $2m$ and does not have zero rows or repeated rows, can be embedded in the noncollinearity graph Σ of $\mathsf{Sp}(n, 2)$. Since $\chi(\Sigma) = 2^m + 1$ (by the existence of symplectic spreads) it follows that $\chi(\Gamma) \leq 2^m + 1$.

For the 2-rank, if p is odd: $\mathrm{rk}_2(A) = \mathrm{rk}_2(J - A)$ (one sees $\mathbf{1} \in \langle A \rangle$ since $\mathbf{1}$ is the sum of the rows indexed by $x \in L$ for a t.i. line L). If $n = 4$, then $\mathrm{rk}_2(A) = \mathrm{rk}_2(J - A) = \frac{1}{2}q(q^2 + 1) + 1$ ([33]).

Generalized quadrangles and orthogonal polar spaces

In Bagchi, Brouwer & Wilbrink [33] it is shown that $\mathrm{rk}_2\langle A \rangle = q^3 - q^2 + q + 1$ for the collinearity graph of any $\mathsf{GQ}(q, q^2)$ with odd q. Also, that $\mathrm{rk}_2\langle A \rangle = q^2 + 1$ for the $\mathsf{O}_5(q)$ generalized quadrangle with odd q. More generally, for orthogonal polar spaces with odd q we have the 2-ranks given in the table below.

	$\mathsf{O}_{2m+1}(q)$	$\mathsf{O}^{\varepsilon}_{2m}(q)$
m even	$\frac{q^{2m}-1}{q^2-1}$	$\frac{q(q^{2m-2}-1)}{q^2-1} + \varepsilon q^{m-1}$
m odd	$\frac{q^{2m}-1}{q^2-1} - 1$	$\frac{q(q^{2m-2}-1)}{q^2-1} + \varepsilon(q^{m-1} - 1)$

Intersecting flats

Sɪɴ [658] determines the p-rank of the 0-1 matrix M with rows and columns indexed by the c-flats and d-flats in a vector space of dimension $n+1$ over $\mathbb{F}_q$, where $q = p^e$, with 1-entry when they intersect nontrivially. Let $N = J - M$ be the disjointness matrix. Since $M\mathbf{1} = \mathbf{1} \pmod p$ and $N\mathbf{1} = 0 \pmod p$, we see $\mathrm{rk}_p M = 1 + \mathrm{rk}_p N$.

The formula for $\mathrm{rk}_p N$ is nicest when $c + d = n + 1$. In that case $\mathrm{rk}_p N = (\sum_{i\geq 0}(-1)^i\binom{n+1}{i}\binom{n+c(p-1)-ip}{n})^e$. For example, for lines in $\mathsf{PG}(3,q)$, adjacent when disjoint, one finds $\mathrm{rk}_p A = (\frac{1}{3}p(2p^2+1))^e$.

Binary codes

The binary codes spanned by the rows of A or $A + I$, where A is the adjacency matrix of a strongly regular graph, were investigated in [383]. (The dimension of these codes is $\mathrm{rk}_2 A$, which was studied in the above. In *loc. cit.* in some cases also the weight enumerators are given.) See also [513].

9.4 Smith normal form

Let us write $M \sim N$ for integral matrices M, N, not necessarily square, if there are integral square invertible matrices P, Q such that $N = PMQ$. Then $\sim$ is an equivalence relation. Let M be an integral matrix. The *Smith normal form* of M is a diagonal matrix $S(M)$ with $S(M) \sim M$ such that the diagonal entries $s_i := S(M)_{ii}$ satisfy $s_i \mid s_{i+1}$ for all i. These entries are uniquely determined up to sign, and satisfy $s_i = d_i/d_{i-1}$ for all i, where d_i is the g.c.d. of all minors of order i of M (so that $d_0 = 1$). The s_i are called *elementary divisors* or *invariant factors*. The p-rank $\mathrm{rk}_p(M)$ equals the number of s_i not divisible by p, and the $\mathbb{Q}$-rank $\mathrm{rk}(M)$ equals the number of nonzero s_i. In particular, $\mathrm{rk}_p(M) \leq \mathrm{rk}(M)$. If M is square of order n, then $\prod_i s_i = \det S(M) = \pm \det M$. If $p^e || \det M$, then $\mathrm{rk}_p M \geq n - e$.

Let $\langle M\rangle$ denote the row space of M over $\mathbb{Z}$. By the fundamental theorem for finitely generated abelian groups, the group $\mathbb{Z}^n/\langle M\rangle$ is isomorphic to a direct sum $\mathbb{Z}_{s_1} \oplus \cdots \oplus \mathbb{Z}_{s_m} \oplus \mathbb{Z}^s$ for certain $s_1, \ldots, s_m, s$, where $s_1|\cdots|s_m$. Since $\mathbb{Z}^n/\langle M\rangle \simeq \mathbb{Z}^n/\langle S(M)\rangle$, we see that $\mathrm{diag}(s_1, \ldots, s_m, 0^t)$ is the Smith normal form of M, when M has r rows and $n = m + s$ columns, and $t = \min(r, n) - m$.

The *Laplacian* matrix L of a graph equals $D - A$, where D is the diagonal matrix of vertex degrees, and A is the ordinary adjacency matrix. Thus, for a regular graph $L = kI - A$. The Laplacian is positive semidefinite, and the multiplicity of its eigenvalue 0 equals the number of connected components of the graph. For a connected graph one has $\mathbb{Z}^n/\langle L\rangle \simeq \mathbb{Z}_{s_1} \oplus \cdots \oplus \mathbb{Z}_{s_{n-1}} \oplus \mathbb{Z}$ with a single $\mathbb{Z}$ summand. Now the group $\mathbb{Z}_{s_1} \oplus \cdots \oplus \mathbb{Z}_{s_{n-1}}$ is called the *sandpile group* or *critical group* of the graph. The product $s_1 s_2 \ldots s_{n-1}$ equals the number of spanning trees of the graph.

In the above, $\mathbb{Z}_s$ denotes $\mathbb{Z}/s\mathbb{Z}$. In an expression $S(M) = \mathrm{diag}\,(1^a, \ldots)$ the 1's are written, but summands $\mathbb{Z}/\mathbb{Z}$ are invisible in a direct sum.

SNF and spectrum

Some detail about Smith normal form and spectrum can be found in [132], §13.8. We quote two results.

Proposition 9.4.1 *Let A be an integral square matrix with integral eigenvalue a of (geometric) multiplicity m. Then the number of invariant factors of A divisible by a is at least m.*

Proposition 9.4.2 *Let A be the adjacency matrix of a strongly regular graph with v vertices and eigenvalues k, r, s, $k > r > s$. Let p be prime, and suppose that $p \nmid v$, $p^a || k$, $p^b || r$, $p^c || s$, where $a \geq b + c$. Let e_i be the number of invariant factors s_j of A such that $p^i || s_j$. Then $e_i = 0$ for $\min(b, c) < i < \max(b, c)$ and for $b + c < i < a$ and for $i > a$. Moreover, $e_{b+c-i} = e_i$ for $0 \leq i < \min(b, c)$.*

Diagonal form

Sometimes a diagonal form of a matrix is almost as good as the Smith normal form. If D is a diagonal matrix, and $M \sim D$, then $S(M)$ is obtained by factoring the diagonal entries of D into prime powers, sorting the result for each separate prime, and multiplying again. So if $D = \text{diag}\,(4, 6, 8, 10)$, we find $S(M) = S(D) = \text{diag}\,(2, 2, 4, 120)$.

Wilson [737] proved the beautiful result that the $\binom{v}{t} \times \binom{v}{k}$ matrix W_{tk}, the 0-1 inclusion matrix of t-subsets against k-subsets of a fixed v-set, has (for $t \leq \min(k, v-k)$) the diagonal form consisting of the entries $\binom{k-i}{t-i}$ with multiplicity $\binom{v}{i} - \binom{v}{i-1}$ $(0 \leq i \leq t)$, where $\binom{v}{-1} = 0$. As an immediate corollary one finds the p-rank of W_{tk}.

A diagonal form for many related matrices is given by

Theorem 9.4.3 (Wilson [738]) *Let X be a v-set. Let M be an integral matrix whose $\binom{v}{t}$ rows are indexed by the t-subsets of X and which has the property that the set of column vectors of M is invariant under the action of the symmetric group S_v acting on the t-subsets of X. Further assume that for each column c of M there is a t-set T such that $c_S \neq 0$ implies $S \cap T = \emptyset$. Let d_i be the g.c.d. of all entries of $W_{it}M$, $i = 0, \ldots, t$. Then a diagonal form for M is given by the diagonal entries d_i with multiplicity $\binom{v}{i} - \binom{v}{i-1}$, $i = 0, 1, \ldots, t$.*

Let us say that a graph has the 'miraculous SNF property' when it has integral eigenvalues θ_i $(0 \leq i \leq v-1)$ and $\text{diag}\,(\theta_0, \ldots, \theta_{v-1})$ is a diagonal form for its adjacency matrix. For example, the Petersen graph has spectrum $3^1\ 1^5\ (-2)^4$, and $S(A) = \text{diag}\,(1^6, 2^3, 6)$.

Corollary 9.4.4 *The Kneser graph $K(v, t)$ (with the t-subsets of a fixed v-set as vertices, adjacent when they are disjoint) has the miraculous SNF property.*

One finds the diagonal form consisting of the numbers $\binom{v-t-i}{t-i}$ with multiplicities $\binom{v}{i} - \binom{v}{i-1}$ $(0 \leq i \leq t)$, The spectrum consists of the numbers $(-1)^i \binom{v-t-i}{t-i}$ with these same multiplicities.

In particular this gives the Smith normal form for the graphs $\overline{T(n)}$.

Some graph families

Complete graphs

The complete graph K_n has adjacency matrix $J_n - I_n$ (where the subscript indicates the size). We have $S(J_n + cI_n) = \text{diag}\,(1, c^{n-2}, c(c+n))$ for integral c. More generally, $S(bJ_n + cI_n) = \text{diag}\,(g, c^{n-2}, (bn + c)c/g)$ where $g = \gcd(b, c)$ and exponents denote multiplicities. See [126].

Lattice graphs

Let A be the adjacency matrix of the $n \times n$ grid graph. Then

$$S(A) = \text{diag}\,(1^{2n-2}, 2^{(n-2)^2}, (2n-4)^{2n-3}, 2(n-1)(n-2)).$$

More generally,

$$A + (c+2)I \sim \text{diag}\,(I_n, (J_n + cI_n)^{n-2}, (n+c)(2J_n + cI_n)),$$

so that for example $S(A + 2I) = \text{diag}\,(1^{2n-2}, 2n, 0^{(n-1)^2})$ and $S(A - (n-2)I) = \text{diag}\,(1^{2n-2}, n^{(n-2)^2}, 0^{2n-2})$.

The Shrikhande graph is cospectral with the 4×4 grid graph (both have spectrum $6^1\ 2^6\ (-2)^9$). Some of the Smith normal forms distinguish them.

name	$S(A)$	$S(A+2I)$	$S(A-2I)$	$\text{rk}_2 A$
4×4	$\text{diag}\,(1^6, 2^4, 4^5, 12)$	$\text{diag}\,(1^6, 8^1, 0^9)$	$\text{diag}\,(1^6, 4^4, 0^6)$	6
Shrikhande	$\text{diag}\,(1^6, 2^4, 4^5, 12)$	$\text{diag}\,(1^6, 2^1, 0^9)$	$\text{diag}\,(1^6, 2^1, 4^2, 8^1, 0^6)$	6

See [126].

Triangular graphs

Let A be the adjacency matrix of the triangular graph $T(n)$, $n \geq 2$. Then

$$S(A) = \begin{cases} \text{diag}\,(1^{n-2}, 2^{\frac{1}{2}(n-2)(n-3)}, (2n-8)^{n-2}, (n-2)(n-4)) & \text{if } n \text{ is even} \\ \text{diag}\,(1^{n-1}, 2^{\frac{1}{2}(n-1)(n-4)}, (2n-8)^{n-2}, 2(n-2)(n-4)) & \text{if } n \text{ is odd} \\ \text{diag}\,(1^2, 2) & \text{for } n = 3 \end{cases}$$

and

$$S(A + 2I) = \begin{cases} \text{diag}\,(1^{n-2}, 2^2, 0^{\frac{1}{2}n(n-3)}) & \text{if } n \text{ is even, } n \geq 4 \\ \text{diag}\,(1^{n-1}, 4^1, 0^{\frac{1}{2}n(n-3)}) & \text{if } n \text{ is odd.} \end{cases}$$

(Compare the spectrum $(2n-4)^1\ (n-4)^{n-1}\ (-2)^{\frac{1}{2}n(n-3)}$ of A.)

The three Chang graphs are cospectral with $T(8)$ (with spectrum $12^1\ 4^7\ (-2)^{20}$).

name	$S(A)$	$S(A+2I)$	$S(A-4I)$	$\text{rk}_2 A$
$T(8)$	$\text{diag}\,(1^6, 2^{15}, 8^6, 24)$	$\text{diag}\,(1^6, 2^2, 0^{20})$	$\text{diag}\,(1^6, 2^2, 6^{13}, 0^7)$	6
Chang	$\text{diag}\,(1^8, 2^{12}, 8^7, 24)$	$\text{diag}\,(1^8, 0^{20})$	$\text{diag}\,(1^8, 6^{12}, 24^1, 0^7)$	8

See [126]. The Smith normal form of the Laplacian of $\overline{T(n)}$ was computed in [297]. Values like $S(A)$ do not follow from Theorem 9.4.3 since the condition on the support of the columns is not satisfied. Wilson & Wong [739] develop a more general theory in which the value of $S(A)$ for the triangular graph follows.

Paley graphs

Consider Paley(q), where $q = 4t + 1$ is a prime power. Chandler, Sin & Xiang [190] showed that its adjacency matrix A satisfies $S(A) = (1^{2t}, t^{2t}, 2t)$. They also determined $S(L)$, where $L = 2tI - A$ is the Laplacian of this graph.

Peisert graphs

Sin [659] showed for the Peisert graphs $P^*(q)$ of order q, where $q = 4t+1$ is a prime power, that $S(A) = (1^{2t}, t^{2t}, 2t)$, the same as for the Paley graphs. There is also information about $S(L)$, where $L = 2tI - A$. Here Paley(q) and $P^*(q)$ may have different behavior. For example, for Paley(81) one has

$$\mathbb{Z}^{81}/\langle L\rangle \simeq \mathbb{Z} \oplus \mathbb{Z}_{20}^{40} \oplus (\mathbb{Z}_3^{16} \oplus \mathbb{Z}_9^{18} \oplus \mathbb{Z}_{27}^{16} \oplus \mathbb{Z}_{81}^{14}),$$

while $P^*(81)$ has

$$\mathbb{Z}^{81}/\langle L\rangle \simeq \mathbb{Z} \oplus \mathbb{Z}_{20}^{40} \oplus (\mathbb{Z}_3^{20} \oplus \mathbb{Z}_9^{10} \oplus \mathbb{Z}_{27}^{20} \oplus \mathbb{Z}_{81}^{14}),$$

where $\mathbb{Z}_s$ abbreviates $\mathbb{Z}/s\mathbb{Z}$. On the other hand, if $q = p^2$ where $p \equiv 3 \pmod 4$, then for Paley(q) and $P^*(q)$ all matrices $A + bJ + cI$ have the same spectrum and Smith normal form.

For $q = 23^2$, the three graphs Paley(q), $P^*(q)$, and the sporadic Peisert graph $P^{**}(q)$ all have the same $S(A)$, and the same $S(L) = (0^1, 1^{144}, 23^{120}, 3036^{122}, 69828^{142})$, where $3036 = 11 \cdot 12 \cdot 23$ and $69828 = 11 \cdot 12 \cdot 23^2$, that is, $\mathbb{Z}^q/\langle L\rangle \simeq \mathbb{Z} \oplus \mathbb{Z}_{132}^{264} \oplus (\mathbb{Z}_{23}^{242} \oplus \mathbb{Z}_{23^2}^{122})$.

In all cases, the p'-part of $\mathbb{Z}^q/\langle L\rangle$ is $\mathbb{Z}_t^{2t}$ for $q = 4t+1$.

Van Lint-Schrijver graphs

Recall from §7.3.1 the definition of the Van Lint-Schrijver graphs $\Gamma_{p,e,t}$. Let p be prime, and e an odd prime such that p is primitive mod e. Let $t \geq 1$. Put $q = p^{(e-1)t}$. Then $\Gamma_{p,e,t}$ is the Cayley graph with vertex set $\mathbb{F}_q$ and the set of nonzero e-th powers as difference set D. Pantangi [614] determined $S(L)$. If $e = 3$ and $p \equiv 2 \pmod 3$, then $\mathrm{rk}_p(L) = (2^{t+1} - 2)(\frac{p+1}{3})^{2t}$.

Skew lines

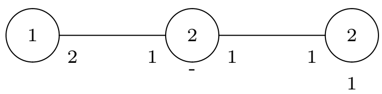
Consider the graph on the lines of $\mathsf{PG}(3, q)$, adjacent when they are skew. This graph is strongly regular with eigenvalues q^4, $-q^2$ and q, so all elementary divisors will be powers of p (where $q = p^e$). The Smith normal form was determined in [125]. For example, for $q = 2$: $S(A) = (1^6, 2^{14}, 4^8, 8^6, 16)$. For the graph on the lines in $\mathsf{PG}(n-1, q)$, see [296].

Chapter 10

Individual graph descriptions

We describe the sporadic rank 3 graphs, and further interesting graphs that have special properties not shared by the other graphs in the infinite families to which they belong. Part of the information given here was obtained using the computer algebra system GAP [333] and its package GRAPE [666] (with Nauty [555]).

10.1 The pentagon



$v = 5$

There is a unique strongly regular graph on 5 vertices. It has parameters $(v, k, \lambda, \mu) = (5, 2, 0, 1)$ and eigenvalues 2 (with multiplicity 1) and $(-1 \pm \sqrt{5})/2$ (with multiplicity 2 each). The full automorphism group is D_{10} with point stabilizer 2.

This is the pentagon, the Paley graph of order 5. It is self-complementary.

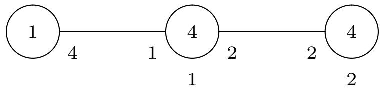
Regular two-graph

The disjoint union $K_1 + C_5$ of a single point and a pentagon is a graph in the switching class of a regular two-graph (X, Δ) (cf. p. 8 and p. 215). If the underlying set is $X = \{0, 1, 2, 3, 4, 5\}$, then the set of triples Δ can be taken to be (omitting commas and parentheses) 012, 023, 034, 045, 015, 124, 235, 134, 245, 135, which is up to isomorphism the unique 2-$(6, 3, 2)$ design. Every 4-set contains 2 coherent triples. The pentagon is the descendant of (X, Δ) (at any vertex).

Locally pentagon graphs

The unique connected locally pentagon graph is the icosahedron. Up to isomorphism, there are three connected locally icosahedron graphs, namely the point graph of the 600-cell on 120 vertices, and quotients on 60 and 40 vertices ([77]).

10.2 The 3×3 grid



$v = 9$

There is a unique primitive strongly regular graph on 9 vertices. It has parameters $(v, k, \lambda, \mu) = (9, 4, 1, 2)$ and spectrum $4^1\ 1^4\ (-2)^4$ (with exponents denoting multiplicities). The full automorphism group is $(\mathsf{S}_3 \times \mathsf{S}_3).2$ with point stabilizer D_8.

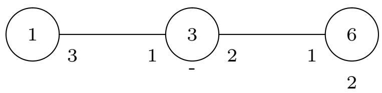
This graph is the Paley graph of order 9. It is the 3×3 grid, the line graph of $K_{3,3}$. It is the collinearity graph of the unique generalized quadrangle $\mathsf{GQ}(2, 1)$, the hyperbolic polar space $\mathsf{O}_4^+(2)$. It is also the affine graph $VO_2^+(3)$.

It is self-complementary, like any Paley graph.

There are precisely two connected locally 3×3 grid graphs, on 16 and 20 vertices, namely $\overline{4 \times 4}$ and $J(6, 3)$.

The imprimitive strongly regular graphs on 9 vertices are $3K_3$ with parameters $(9, 2, 1, 0)$ and spectrum $2^3\ (-1)^6$, and its complement $K_{3\times3}$ with parameters $(9, 6, 3, 6)$ and spectrum $6^1\ 0^6\ (-3)^2$.

10.3 The Petersen graph



$v = 10$

There is a unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (10, 3, 0, 1)$. Its spectrum is $3^1\ 1^5\ (-2)^4$. The full group of automorphisms is S_5 acting rank 3 with point stabilizer $2 \times \mathsf{S}_3$.

This graph was found by the Danish mathematician Julius Petersen (1839–1910), who constructed this graph in [618] as the smallest counterexample against the claim that a connected bridgeless cubic graph has an edge coloring with three colors.

This graph is the complement of the triangular graph $T(5)$, and not sporadic, but it plays a role in the construction of many sporadic graphs.

Cocliques

The largest cocliques have size 4. There are 5 of them, corresponding to the 5 symbols of $T(5)$. The complement of a 4-coclique is a subgraph $3K_2$. It follows that the chromatic number is 3. The edge-chromatic number is 4.

Cycles

There are 12 pentagons, 10 hexagons, 0 heptagons, 15 octagons, 20 nonagons and 0 decagons. The binary code spanned by the (edges of the) cycles is a [15,6,5]-code. The 64 code words are the zero word, the $12 + 10 + 15 + 20 = 57$ cycles, and the 6 unions of two disjoint pentagons.

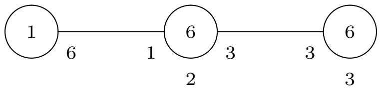
Decomposition of K_{10}

An old question is whether K_{10} can be decomposed into three edge-disjoint copies of the Petersen graph. From the spectrum one sees that the answer is No: the result of removing two edge-disjoint copies of the Petersen graph from K_{10} is connected and bipartite (cf. [132], 1.5.1).

Locally Petersen graphs

Hall [388] showed that there are precisely three connected locally Petersen graphs, namely (i) $\overline{T(7)}$ on 21 vertices, and (ii) a triple cover, on 63 vertices, distance-regular with intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$ and group $3 \cdot \mathsf{S}_7$, and (iii) a graph on 65 vertices, distance-regular with intersection array $\{10, 6, 4; 1, 2, 5\}$ with group $\mathsf{P\Sigma L}(2, 25)$ (the commuting graph of the class of nontrivial field automorphisms). This last graph is the local graph of $NO_5^{-\perp}(5)$, see §10.64.

10.4 The Paley graph on 13 vertices

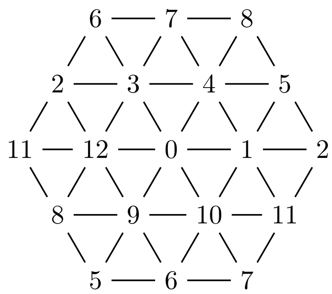


$v = 13$

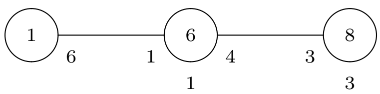
There is a unique strongly regular graph on 13 vertices, namely the Paley graph. It is the graph on $\mathbb{F}_{13}$ where two vertices are joined when their difference is a nonzero square, see §7.4.4. For unicity, see [643].

The parameters are $(v, k, \lambda, \mu) = (13, 6, 2, 3)$, and the spectrum $6\ (\frac{-1\pm\sqrt{13}}{2})^6$. As all Paley graphs, this graph is self-complementary. The full group of automorphisms is $13:6$, acting rank 3.

This graph is locally a hexagon, so it is a quotient of a tiling of the plane. (The Hoffman bound for cliques is $\sqrt{13}$, so the local graph does not have triangles.)



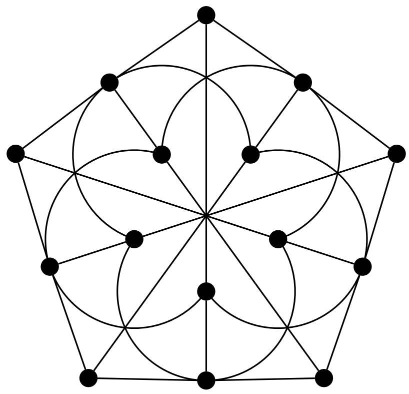
10.5 GQ(2,2)



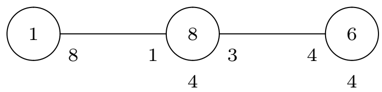
$v = 15$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (15, 6, 1, 3)$. Its spectrum is $6^1\ 1^9\ (-3)^5$. The full group of automorphisms is S_6 acting rank 3 with point stabilizer $2 \times \mathsf{S}_4$.

This graph is the collinearity graph of the unique generalized quadrangle $\mathsf{GQ}(2,2)$ with 15 points and 15 lines of size 3, drawn below. It is the symplectic polar graph $\mathsf{Sp}(4,2)$.



Complement



$v = 15$

The complementary graph is the triangular graph $T(6)$ with parameters $(v,k,\lambda,\mu) = (15,8,4,4)$ and spectrum $8^1\ 2^5\ (-2)^9$. We see that Γ has 20 maximal 3-cocliques, and 6 maximal 5-cocliques (ovoids), each vertex in two of those. Since $\mathsf{GQ}(2,2)$ is self-dual, there are also 6 spreads (1-factorizations of K_6), any line in two.

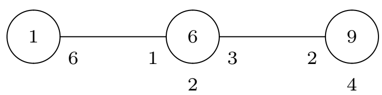
Regular sets

The graph Γ has 91 regular sets, of four types. We give the degree d and nexus e of the smallest part.

	H	index	orbitlengths	sizes	d	e	graph
a	S_5	6	5, 10	5, 10	0	3	ovoid, $\overline{K_5}$
b	$\mathsf{S}_4 \times 2$	15	3, 12	3, 12	2	1	line, K_3
c	$(\mathsf{S}_3 \times \mathsf{S}_3):2$	10	6, 9	6, 9	3	2	$K_{3,3}$
d	$\mathsf{S}_3 \times 2$	60	6, 3, 6	6, 9	3	2	$K_2 \times K_3$

Types (b), (d), (c) belong to 1, 2, 3 pairwise disjoint lines. Type (c) also belongs to an orthogonal pair of hyperbolic lines.

10.6 The Shrikhande graph



$v = 16$

Up to isomorphism, there are precisely two strongly regular graphs with parameters $(v,k,\lambda,\mu) = (16,6,2,2)$, namely the Hamming graph $H(2,4)$, that is, the 4×4 grid, the direct product of two 4-cliques, and the *Shrikhande graph*. These graphs have spectrum $6^1\ 2^6 (-2)^9$.

Construction

The Shrikhande graph arises from the 4×4 grid by switching w.r.t. a diagonal.

The Shrikhande graph is the Cayley graph for the group 4^2 with difference set $\{\pm(0,1), \pm(1,0), \pm(1,1)\}$. It is locally a hexagon, and hence a quotient of the hexagonal tiling of the plane.

```
            22 — 23 — 20 — 21 — 22
           / \  / \  / \  / \  /
         32 — 33 — 30 — 31 — 32
        / \  / \  / \  / \  /
      02 — 03 — 00 — 01 — 02
     / \  / \  / \  / \  /
   12 — 13 — 10 — 11 — 12
  / \  / \  / \  / \  /
22 — 23 — 20 — 21 — 22
```

There are precisely two Latin squares of order 4, namely the addition tables of $\mathbb{F}_4$ and of the cyclic group of order 4. The corresponding Latin square graphs are the complements of $H(2,4)$ and the Shrikhande graph, respectively.

Group

The full group is $(4\times4) : D_{12}$ of order 192 with point stabilizer D_{12}. It is sharply transitive on ordered triangles. It acts rank 4: two vertices can be identical, adjacent, ot nonadjacent where the two common neighbors form an edge or a nonedge.

Designs

As mentioned earlier (p. 191), strongly regular graphs with $\lambda = \mu$ coexist with symmetric (i.e., square) 2-(v,k,λ) designs together with a polarity without absolute points. In the present case, there are three nonisomorphic symmetric 2-(16,6,2) designs (Husain [450]). Two of these do not possess a suitable polarity. The third has two nonequivalent polarities without absolute points, giving rise to the two strongly regular graphs with parameters $(16,6,2,2)$ (Haemers [373]).

Cliques and cocliques

The Shrikhande graph has independence number 4 and chromatic number 4. Its complement has independence number 3 and chromatic number 6. Both the lattice graph $H(2,4)$ and its complement have independence number and chromatic number 4.

2-Ranks and Smith normal form

The Shrikhande graph and $H(2,4)$ have the same p-ranks, but differ somewhat in Smith normal form. If A is the adjacency matrix of $H(2,4)$, and B that of the Shrikhande graph, and $S(M)$ denotes the Smith normal form of the matrix M, then $S(A) = S(B) = \mathrm{diag}(1^6, 2^4, 4^5, 12)$, $S(A+2I) = \mathrm{diag}(1^6, 8^1, 0^9)$, $S(A-2I) = \mathrm{diag}(1^6, 4^4, 0^6)$ while $S(B+2I) = \mathrm{diag}(1^6, 2^1, 0^9)$, $S(B-2I) =$

$\mathrm{diag}(1^6, 2^1, 4^2, 8^1, 0^6)$ (see [126]). It follows that $\mathrm{rk}_2(A) = \mathrm{rk}_2(B) = 6$. Also $\mathrm{rk}_2(A + J) = \mathrm{rk}_2(B + J) = 6$.

Bipartite double

Both $H(2, 4)$ and the Shrikhande graph are $(0, 2)$-graphs, and have the folded 6-cube as bipartite double.

$v = 32$

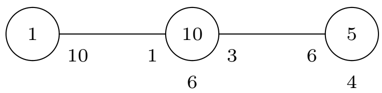
Locally Shrikhande graphs

Makhnev & Paduchikh [537] show that there are precisely two connected locally Shrikhande graphs, one on 80 vertices and a quotient on 40 vertices.

Dyck graph

There is a unique cubic symmetric (i.e., both vertex- and edge-transitive) connected graph on 32 vertices known as the *Dyck graph* ([299, 490]). It is the graph that has as vertices the triangles in the Shrikhande graph, adjacent when they share an edge. This graph is bipartite, with spectrum $(\pm 3)^1\ (\pm\sqrt{5})^6\ (\pm 1)^9$, and is uniquely determined by its spectrum. It has girth 6 and diameter 5 and full group of order 192. The two components of the distance-2 graph are copies of the Shrikhande graph. It has an embedding in a genus 3 surface as a cubic map with twelve octagonal faces.

10.7 The Clebsch graph



$v = 16$

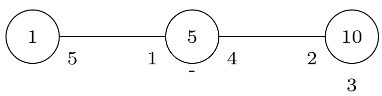
The *Clebsch graph* is the unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (16, 10, 6, 6)$. Its spectrum is $10^1\ 2^5\ (-2)^{10}$. The full group of automorphisms is $2^4 : \mathsf{S}_5$ acting rank 3, with point stabilizer S_5.

Construction

The Clebsch graph is the halved 5-cube, that is, the vertices are the binary vectors of length 5 and even weight, joined when the Hamming distance is 2.

The Clebsch graph is the local graph of the Schläfli graph (§10.10).

Complement



$v = 16$

The complement of the Clebsch graph is the folded 5-cube. That is, its vertices are the 16 cosets of $\{00000, 11111\}$, adjacent when the Hamming distance is 1. It is the graph obtained by identifying antipodes in the 5-cube.

It has parameters $(v, k, \lambda, \mu) = (16, 5, 0, 2)$. Its spectrum is $5^1\ 1^{10}\ (-3)^5$.

The complement of the Clebsch graph is also the graph on $\mathbb{F}_{16}$ where two vertices are adjacent when their difference is a cube. It follows that K_{16} is the edge-disjoint union of three copies of the complement of the Clebsch graph.

The complement of the Clebsch graph is also the graph $VO_4^-(2)$.

The second subconstituent is the Petersen graph.

Cliques and cocliques

The Clebsch graph has independence number 2 and chromatic number 8. The complement of the Clebsch graph has independence number 5 and chromatic number 4.

Regular sets

All regular sets in Γ are obtained from subgroups H of $G = \text{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit in $\overline{\Gamma}$, the complement of the Clebsch graph.

	H	index	orbitlengths	d	e	graph
a	$\mathsf{D}_8 \times \mathsf{S}_3$	40	4, 12	2	1	C_4
b	$2^2 \times \mathsf{S}_4$	20	8, 8	3	2	3-cube
c	D_{16}	120	8, 8	3	2	Wagner graph
d	2^{3+2}:3:2	10	8, 8	1	4	$4K_2$

The *Wagner graph* is the 8-gon with diagonals.

Ramsey number

Let $R(3, 3, 3)$ be the smallest number N of vertices such that for any assignment of three colors to the edges of K_N there is a monochromatic triangle. The above decomposition shows that $R(3, 3, 3) > 16$. GREENWOOD & GLEASON [365] proved that $R(3, 3, 3) = 17$. See also p. 183.

Xor-magic graphs

A connected graph on 2^n vertices is called *xor-magic* ([656]) if the vertices can be labeled with distinct n-bit numbers such that the label of each vertex is the bitwise XOR of the labels of the adjacent vertices. The complement of the Clebsch graph is xor-magic since it is the graph with vertices in $\mathbb{F}_2^4$, adjacent when the difference has weight 3 or 4, and the sum of the five neighbors of x is x again. Also the Dyck graph is xor-magic.

Name

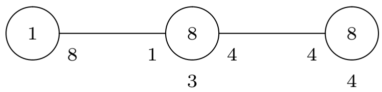
This graph was given this name by SEIDEL [642] in his paper classifying the strongly regular graphs with smallest eigenvalue -2 (for which the Seidel matrix $S = J - I - 2A$ has eigenvalue 3):

> *In terms of polytopes, the 16 vertices and 80 adjacencies of the graph $\{V, A\}$ can be identified with the 16 vertices and 80 edges of the polytope $h\gamma_5$, also denoted*

by 1_{21} *(Coxeter, Regular polytopes, 2nd ed., pp. 158, 201). This remark is due to H. S. M. Coxeter, who also points out the relation of this polytope to the 16 lines (and 80 pairs of skew lines) on Clebsch's quartic surface (cf. Clebsch (1868)). Therefore,* $\{V, A\}$ *will be called the Clebsch graph.*

The paper referred to is Clebsch [201]. Later some confusion has arisen, and some authors use the name 'Clebsch graph' for the complement of Γ.

10.8 The Paley graph on 17 vertices



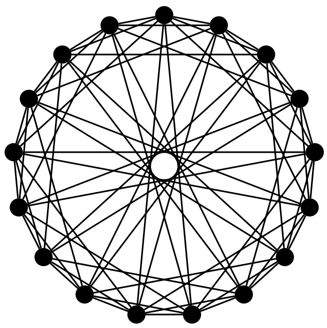
$v = 17$

There is a unique strongly regular graph on 17 vertices, namely the Paley graph. It is the graph on $\mathbb{F}_{17}$ where two vertices are joined when their difference is a nonzero square, see §7.4.4. For unicity, see [643].

The parameters are $(v, k, \lambda, \mu) = (17, 8, 3, 4)$, and the spectrum $8\ (\frac{-1\pm\sqrt{17}}{2})^8$. As all Paley graphs, this graph is self-complementary. The full group of automorphisms is $17{:}8$, acting rank 3.

There is a unique orbit of maximal cliques, namely that of the triangles. It follows that the Ramsey number $R(4,4)$ is at least 18. In fact $R(4,4) = 18$.

Since the graph is self-complementary, we only need to check that there is no K_4. Since the group is edge-transitive we only need to check that there is no K_4 on the edge 0–1. The squares mod 17 are $\pm1, \pm2, \pm4, \pm8$ so the three common neighbors of 0 and 1 are 2, 9, 16 and these are mutually nonadjacent.



The local graph of Γ is the Wagner graph (the octagon with diagonals).

Sinkovic [660] shows that this graph has no weight matrix for which the Cvetković bound would be tight: every weight matrix for this graph has at least 4 positive and at least 4 negative eigenvalues.

10.9 The Paulus-Rozenfel'd graphs

There are 4 regular two-graphs on 26 vertices. The corresponding switching classes of graphs contain 10 isomorphism classes of strongly regular graphs with parameters $(26, 10, 3, 4)$ and spectrum $10^1\ 2^{13}\ (-3)^{12}$.

Switching a point isolated yields 15 isomorphism classes of strongly regular graphs with parameters $(25, 12, 5, 6)$ and spectrum $12^1\ 2^{12}\ (-3)^{12}$. These graphs were found independently by Paulus [606], who was unable to do a complete search, and by Rozenfel'd [632], who did an exhaustive search.

Construction

A Steiner triple system $STS(13)$ has 26 blocks (triples) and the graph on the triples, adjacent when disjoint, is strongly regular with parameters $(26, 10, 3, 4)$.

A Latin square LS(5) of order 5 yields a Latin square graph $\mathrm{LS}_3(5)$ with parameters $(25, 12, 5, 6)$ (§8.4.2).

The two nonisomorphic $STS(13)$ and the two nonisomorphic LS(5) yield graphs in four distinct regular two-graphs of order 26 (for LS(5): after adding an isolated point). These are the four regular two-graphs of order 26.

Let A_i $(1 \le i \le 10)$ be the 10 graphs of order 26, and B_j $(1 \le j \le 15)$ the 15 graphs of order 25. The four regular two-graphs of order 26 contain the indicated A_i, and have the indicated 26 descendants B_j, where j^e means that B_j occurs e times.

name	**groupsize**	**graphs**	**descendants**
A	6	A_{1-5}	$1^6\ 2^6\ 3^3\ 4^3\ 5^3\ 6^3\ 7^1\ 8^1$
B	72	A_{6-7}	$9^{12}\ 10^{12}\ 11^1\ 12^1$
C	39	A_{8-9}	$13^{13}\ 14^{13}$
D	15600	A_{10}	15^{26}

Cliques and cocliques

We give the counts of maximal cliques and cocliques of various sizes, and other statistics in Tables 10.1 and 10.2 below. For the graphs of order 25 the Hoffman bound for cliques and cocliques is 5. For the graphs of order 26 the Hoffman bound for cocliques is 6.

name	**groupsize**	**orbitsizes**	**max cliques**	**max cocliques**	$\chi(\Gamma)$
A_1	1	1^{26}	3^{130}	$4^{115}\ 5^{76}\ 6^1$	6
A_2	2	$1^6\ 2^{10}$	3^{130}	$4^{116}\ 5^{76}\ 6^1$	6
A_3	2	$1^6\ 2^{10}$	$3^{122}\ 4^2$	$4^{100}\ 5^{81}\ 6^1$	6
A_4	6	$1^2\ 3^4\ 6^2$	$3^{122}\ 4^2$	$4^{104}\ 5^{81}\ 6^1$	6
A_5	6	$1^2\ 3^4\ 6^2$	$3^{98}\ 4^8$	$4^{164}\ 5^{24}\ 6^{13}$	6
A_6	4	$1^2\ 2^4\ 4^4$	$3^{90}\ 4^{10}$	$4^{136}\ 5^{70}\ 6^3$	5
A_7	6	$1^3\ 2^1\ 3^3\ 6^2$	$3^{82}\ 4^{12}$	$4^{124}\ 5^{75}\ 6^3$	5
A_8	3	$1^2\ 3^8$	$3^{126}\ 4^1$	$4^{95}\ 5^{81}\ 6^1$	6
A_9	39	13^2	$3^{78}\ 4^{13}$	$4^{104}\ 5^{39}\ 6^{13}$	6
A_{10}	120	$6^1\ 20^1$	$3^{90}\ 4^{10}$	$4^{210}\ 5^{12}\ 6^{13}$	5

Table 10.1: Strongly regular graphs with parameters (26,10,3,4)

Here A_9 is the complement of the block graph of the cyclic $STS(13)$. The other $STS(13)$ yields A_5.

The coclique counts for B_j in Table 10.2 are the clique counts for its complement. The graph B_{15} is the only self-complementary one. It is the Paley graph of order 25, and $\mathrm{LS}_3(5)$ for the cyclic LS(5). The other $\mathrm{LS}_3(5)$ is B_{12}.

For further detail, see [606], [726], [226]. Note that different authors use a different numbering of these graphs. Explicit matrices with the present numbering are given in [120].

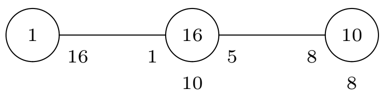
name	complement	group size	orbit sizes	max cliques	$\chi(\Gamma)$
B_1	B_2	1	1^{25}	$3^7\ 4^{74}\ 5^3$	6
B_2	B_1	1	1^{25}	$3^5\ 4^{74}\ 5^3$	6
B_3	B_4	2	$1^5\ 2^{10}$	$3^8\ 4^{72}\ 5^3$	6
B_4	B_3	2	$1^5\ 2^{10}$	$3^8\ 4^{72}\ 5^3$	6
B_5	B_6	2	$1^5\ 2^{10}$	$3^4\ 4^{74}\ 5^3$	6
B_6	B_5	2	$1^5\ 2^{10}$	$3^8\ 4^{74}\ 5^3$	6
B_7	B_8	6	$1^1\ 3^4\ 6^2$	$3^{14}\ 4^{68}\ 5^3$	6
B_8	B_7	6	$1^1\ 3^4\ 6^2$	$3^{14}\ 4^{68}\ 5^3$	6
B_9	B_{10}	6	$1^2\ 2^1\ 3^3\ 6^2$	$3^{54}\ 4^{58}\ 5^3$	6
B_{10}	B_9	6	$1^2\ 2^1\ 3^3\ 6^2$	$3^{54}\ 4^{58}\ 5^3$	6
B_{11}	B_{12}	72	$1^1\ 12^2$	$3^{36}\ 4^{64}\ 5^3$	5
B_{12}	B_{11}	72	$1^1\ 12^2$	$3^{84}\ 4^4\ 5^{15}$	6
B_{13}	B_{14}	3	$1^1\ 3^8$	$3^3\ 4^{75}\ 5^3$	6
B_{14}	B_{13}	3	$1^1\ 3^8$	$3^1\ 4^{75}\ 5^3$	6
B_{15}	B_{15}	600	transitive	$3^{100}\ 5^{15}$	5

Table 10.2: Strongly regular graphs with parameters (25,12,5,6)

p-ranks

The p-ranks of the graphs involved only depend on the regular two-graph they belong to. The seven graphs A_i and ten graphs B_j belonging to two-graphs A or C have adjacency matrices A satisfying $rk_5(A-2I+2J)=12$. The two graphs A_i and four graphs B_j belonging to two-graph B have $rk_5(A-2I+2J)=11$. The graphs A_{10} and B_{15} belonging to two-graph D have $rk_5(A-2I+2J)=9$. For all graphs B_j the value of $rk_5(A-2I+bJ)$ is independent of b. For all graphs A_i the value of $rk_5(A-2I+bJ)$ is one larger for $b \neq 2$. (Peeters [610])

10.10 The Schläfli graph

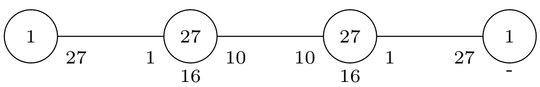


$v = 27$

The *Schläfli graph* is the unique strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (27,16,10,8)$. Its spectrum is $16^1\ 4^6\ (-2)^{20}$. The full group of automorphisms is $W(\mathsf{E}_6) = \mathsf{U}_4(2).2 = \mathsf{O}_6^-(2).2 = \mathsf{O}_5(3).2$ acting rank 3, with point stabilizer $2^4 : \mathsf{S}_5$.

It is the $\mathsf{E}_{6,1}(1)$ graph, the local graph of the $\mathsf{E}_{7,7}(1)$ (Gosset) graph. Its local graph is the Clebsch graph.

Aside: the Gosset graph



$v = 56$

The *Gosset graph* is the unique distance-regular graph with intersection array $\{27, 10, 1; 1, 10, 27\}$. It is distance-transitive, an antipodal double cover of K_{28}.

This graph can be constructed as follows. The vertices are the pairs from the 8-sets $\{1, 2, \ldots, 8\}$ and $\{1', 2', \ldots, 8'\}$. Two pairs from the same set are adjacent if they intersect in precisely one element; two pairs $\{a, b\}$ and $\{c', d'\}$ from different sets are adjacent if $\{a, b\}$ and $\{c, d\}$ are disjoint.

Construction: as local graph in the Gosset graph

The local structure of the Gosset graph at the vertex $\{7', 8'\}$ yields a construction of the Schläfli graph: the vertices are the pairs from the set $\{1, 2, \ldots, 6\}$ together with the 'double sixes' $1, 2, \ldots, 6$ (each element a of which corresponds to the vertex $\{a', 7'\}$ of the Gosset graph) and $1', 2', \ldots, 6'$ (each element a' of which corresponds to the vertex $\{a', 8'\}$ of the Gosset graph); pairs are adjacent if they intersect in a unique element, vertices from the same 6-set are always adjacent, vertices a and b' from different 6-sets are adjacent if and only if $a = b$, and finally a vertex a or a' is adjacent to a pair $\{b, c\}$ if and only if $a \notin \{b, c\}$.

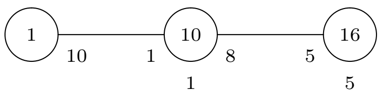
Construction: in the regular two-graph on 28 points

The regular two-graph on 28 vertices of which $\overline{T(8)}$ is a member has the Schläfli graph as descendant.

Construction: in affine 3-space over $\mathbb{F}_3$

An explicit coordinate description in affine 3-space over $\mathbb{F}_3$ goes as follows: the vertices are the ordered triples $(x, y, z) \in \mathbb{F}_3^3$ with (x_1, y_1, z_1) adjacent to (x_2, y_2, z_2) if $z_2 - z_1 \neq x_1 y_2 - y_1 x_2$ and $(x_1, y_1) \neq (x_2, y_2)$.

Complement



$v = 27$

The complement $\overline{\Gamma}$ of Γ is strongly regular with parameters $(v, k, \lambda, \mu) = (27, 10, 1, 5)$. Its spectrum is $10^1\ 1^{20}\ (-5)^6$. It is the collinearity graph of the geometry of isotropic points and totally isotropic lines in the $\mathsf{O}_6^-(2)$ geometry, the unique $\mathsf{GQ}(2, 4)$. It is also the graph on the totally isotropic lines in the $\mathsf{U}_4(2)$ geometry, adjacent when they meet.

Name

This graph was given this name by Seidel [642] in his paper classifying the strongly regular graphs with smallest eigenvalue -2 (for which the Seidel matrix $S = J - I - 2A$ has eigenvalue 3):

> *We shall refer to this graph as the Schläfli graph after its earliest describer (cf.* Coxeter *[238, p. 211]).*

Coxeter refers to Schläfli [636]. Schläfli does not construct a graph, but discusses the 27 lines on a cubic surface, earlier found by Cayley [188] and Salmon [634].

The 27 lines on a cubic surface

A generic nonsingular cubic surface in 3-dimensional projective space contains 27 lines. The graph on these 27 lines, adjacent when they meet, is the complement of Γ.

For example, the surface $X^3 + Y^3 + Z^3 + W^3 = 0$ in complex 3-dimensional projective space contains the 27 lines like $\langle(1, -a, 0, 0), (0, 0, 1, -b)\rangle$ where $a^3 = b^3 = 1$. (The values a, b can each be chosen in 3 ways, and the coordinate split $XY|ZW$ can be chosen in 3 ways, 27 choices altogether.)

Each of these 27 lines intersects 10 others, and these 10 intersect in pairs, so that each of the 27 lines is in 5 coplanar triples and there are 45 coplanar triples (that is, 45 triple tangent planes) altogether. These lines and planes form the points and lines of the generalized quadrangle $\mathsf{GQ}(2, 4)$.

Cliques, cocliques and chromatic number

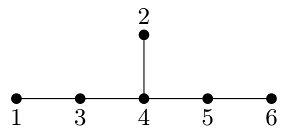
The maximal cliques in Γ have size 5 or 6, a single orbit of each, with stabilizers $2 \times \mathsf{S}_5$ and S_6, respectively. The maximal cocliques in Γ have size 3. The chromatic number of Γ is 9, and there are two essentially different ways to color Γ with 9 colors ([144]). The chromatic number of $\overline{\Gamma}$ is 6.

In terms of $\mathsf{GQ}(2, 4)$, the 5- and 6-cliques are the sets $x^\perp \cap y^\perp$ and $(x^\perp \setminus y^\perp) \cup \{y\}$ where x and y are noncollinear, and $\perp$ denotes collinearity. The 3-cocliques are the lines of $\mathsf{GQ}(2, 4)$.

Double sixes

The graph Γ has 36 subgraphs $K_2 \times K_6$ ('double sixes') forming a single orbit. The stabilizer of one is $\mathsf{S}_6 \times 2$ with vertex orbit sizes $12 + 15$. The orbits of size 15 are the subsets that carry a sub-$\mathsf{GQ}(2, 2)$ of $\mathsf{GQ}(2, 4)$. In the representation as $\mathsf{O}_6^-(2)$ these correspond to the nonisotropic points.

Apartment of E_6



E_6.

The Schläfli graph Γ is the collinearity graph of the thin geometry (apartment) of type E_6. The objects of types 1–6 are the 27 vertices, 72 6-cliques, 216 edges, 720 triangles, 216 maximal 5-cliques and 27 subgraphs of the form $\Gamma_2(x)$.

Local characterizations

The Gosset graph is the unique graph that is locally Schläfli. It is the Taylor extension $\mathsf{T}\Gamma$ of Γ. By Buekenhout & Hubaut [156], there are precisely two graphs that are locally the complement of the Schläfli graph, namely $VO_6^-(2)$ (see §10.25) and $\mathsf{T}\overline{\Gamma}$.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$2^{1+4}{:}3^2{:}2^2$	45	3, 24	0	2	3-coclique
b	$2 \times (\mathsf{S}_3 \,\mathrm{wr}\, 2)$	360	6, 9+12	2	4	$2K_3$
c	$\mathsf{S}_3 \times (\mathsf{S}_3 \,\mathrm{wr}\, 2)$	120	9, 18	4	6	3×3
d	$2 \times \mathsf{S}_6$	36	12, 15	6	8	2×6

In case (b), the group has three orbits.

More generally, the union of t pairwise disjoint 3-cocliques (lines of $\mathsf{GQ}(2,4)$) is a regular set in Γ of size $3t$, with degree $d = 2(t-1)$ and nexus $e = 2t$. Since $\mathsf{GQ}(2,4)$ admits a spread, all values of t with $1 \le t \le 8$ are admissible. Every regular set is the union of pairwise disjoint subgraphs $3K_1$ or $2K_3$.

Shannon capacity

For a graph Γ and an integer k, let $\Gamma^{\boxtimes k}$ denote the graph of which the vertices are k-tuples of vertices of Γ, where two distinct vertices $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are adjacent when for all i we have either $x_i = y_i$ or $x_i \sim y_i$. This is the graph with adjacency matrix $(A+I)^{\otimes k} - I$ where A is the adjacency matrix of Γ, and $\otimes k$ denotes k-th tensor power.

Let $\alpha(\Gamma)$ be the independence number of a graph Γ. The *Shannon capacity* $\Theta(\Gamma)$ of Γ is defined as

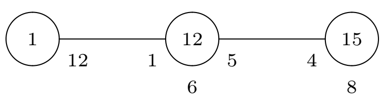
$$\Theta(\Gamma) = \sup_k \sqrt[k]{\alpha(\Gamma^{\boxtimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha(\Gamma^{\boxtimes k})}.$$

Computation of $\Theta(\Gamma)$ is very difficult, even for graphs as simple as the heptagon.

Lovász [526] gave an easily computable upper bound $\theta(\Gamma)$ for $\Theta(\Gamma)$ and used this to show that the pentagon has Shannon capacity $\sqrt{5}$. He asked: (i) Is it true that $\theta = \Theta$? (ii) Is it true that $\Theta(\Gamma \boxtimes \Delta) = \Theta(\Gamma)\Theta(\Delta)$? (iii) Is it true that $\Theta(\Gamma)\Theta(\overline{\Gamma}) \ge |V\Gamma|$?

Haemers [375] answered thrice No: Let Γ be the Schläfli graph. Then $\alpha(\Gamma) = \Theta(\Gamma) = \theta(\Gamma) = 3$ and $6 = \alpha(\overline{\Gamma}) \le \Theta(\overline{\Gamma}) \le 7 < 9 = \theta(\overline{\Gamma})$ and $\Theta(\Gamma)\Theta(\overline{\Gamma}) \le 21 < 27 = |V\Gamma| \le \Theta(\Gamma \boxtimes \overline{\Gamma})$. See also [374] and [132], §3.7.

10.11 $T(8)$ and the Chang graphs



$v = 28$

Up to isomorphism, there are precisely four strongly regular graphs with parameters $(v, k, \lambda, \mu) = (28, 12, 6, 4)$. They have spectrum $12^1\ 4^7\ (-2)^{20}$. The classification is due to Chang [191, 192].

One is the triangular graph $T(8)$, that is, the Johnson graph $J(8,2)$. The remaining three are called the *Chang graphs*. The three Chang graphs can be obtained by Seidel switching from $T(8)$ (the line graph of K_8). Namely, switch w.r.t. a set of edges that induces the following subgraph of K_8: (a) 4 pairwise disjoint edges, (b) $C_3 + C_5$, (c) an 8-cycle C_8.

The triangular graph $T(8)$ does not contain 3-claws, but the three Chang graphs do ([76]).

2-Ranks and Smith normal form

The Chang graphs can be distinguished from $T(8)$ by their p-ranks: If A is the adjacency matrix of $T(8)$ and B that of one of the Chang graphs, and $S(M)$ denotes the Smith normal form of the matrix M, then $S(A) = \mathrm{diag}(1^6, 2^{15}, 8^6, 24^1)$ (and $S(A + 2I) = \mathrm{diag}(1^6, 2^2, 0^{20})$, $S(A - 4I) = \mathrm{diag}(1^6, 2^2, 6^{13}, 0^7)$), while $S(B) = \mathrm{diag}(1^8, 2^{12}, 8^7, 24^1)$ (and $S(B + 2I) = \mathrm{diag}(1^8, 0^{20})$, $S(B - 4I) = \mathrm{diag}(1^8, 6^{12}, 24^1, 0^7)$, so that A and B have different 2-ranks 6 and 8 ([126]).

Cliques and cocliques

We give the counts of maximal cliques and cocliques of various sizes, and other statistics.

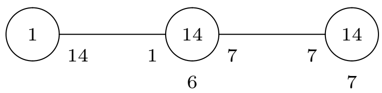
name	groupsize	max cliques	max cocliques	$\theta(\Gamma)$	$\chi(\Gamma)$
$T(8)$	40320	$3^{56}, 7^8$	4^{105}	6	7
Chang1	384	$4^{32}, 5^{24}, 6^8$	$3^{128}, 4^{73}$	7	7
Chang2	360	$4^{75}, 5^{30}, 6^3$	$3^{160}, 4^{65}$	8	7
Chang3	96	$4^{48}, 5^{48}$	$3^{160}, 4^{65}$	6	7

Here $\chi(\Gamma)$ is the chromatic number of Γ, and $\theta(\Gamma)$ is the clique covering number, the chromatic number of the complementary graph.

Connectivity

One may investigate how large a disconnecting set of a strongly regular graph must be if it does not contain a complete vertex neighborhood. Usually the answer is $2k - \lambda - 2$, the size of an edge neighborhood, but in $T(n)$ there are triangles with neighborhood of size $3n - 9$ while $2k - \lambda - 2 = 3n - 8$. It can be shown that the Chang graphs do not show this exceptional behavior ([198]).

10.12 The strongly regular graphs on 29 vertices



$v = 29$

Up to isomorphism, there are precisely 41 strongly regular graphs with parameters $(v, k, \lambda, \mu) = (29, 14, 6, 7)$. Their spectrum is $14\ (\frac{-1\pm\sqrt{29}}{2})^{14}$. These graphs are descendants of the precisely six regular 2-graphs on 30 vertices.

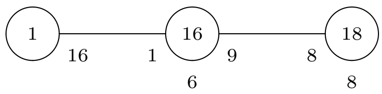
The graphs and 2-graphs were found by Arlazarov et al. and others. Later, Bussemaker and Spence independently did an exhaustive search and found that there are no further examples. See [10], [161], [669].

One of these graphs is the Paley graph Paley(29). This is the only one that is self-complementary. The remaining 40 fall into 20 complementary pairs. The

Paley graph has a group of order $29 \cdot 14 = 406$. The remaining graphs have groups of order 1 (18×), 2 (10×), 3 (10×), and 6 (2×).

The Paley graph has maximum clique and coclique sizes 4. There is one other graph with maximum clique size 4 (and the complementary graph has maximum coclique size 4). All others have maximum clique and coclique sizes 5. All graphs have chromatic number 7, except for the two without 5-cocliques; these have chromatic number 8.

10.13 The S_8 graph on 35 vertices



$v = 35$

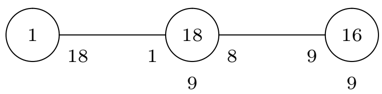
There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (35, 16, 6, 8)$. Its spectrum is $16^1\ 2^{20}\ (-4)^{14}$. The full group of automorphisms is S_8, acting rank 3 with point stabilizer $\mathsf{S}_4 \,\mathrm{wr}\, 2$.

Construction

This graph is the antipodal quotient of the Johnson graph $J(8, 4)$. Vertices are the $4 + 4$ splits of a fixed 8-set Ω, adjacent when the common refinement has shape $3 + 1 + 1 + 3$. Equivalently, take the 3-subsets of a 7-set, adjacent when they meet in 0 or 2 elements.

This graph is also the graph on the 35 lines in $\mathsf{PG}(3, 2)$, adjacent when disjoint. This graph is also the graph on the isotropic points of the $\mathsf{O}_6^+(2)$ geometry, adjacent when nonorthogonal.

Complement



$v = 35$

The complementary graph $\overline{\Gamma}$ has parameters $(v, k, \lambda, \mu) = (35, 18, 9, 9)$ and spectrum $18^1\ 3^{14}\ (-3)^{20}$. It is the graph on the triples from a 7-set, adjacent when they have precisely one element in common, or on the lines of $\mathsf{PG}(3, 2)$, adjacent when intersecting, or on the isotropic points of the $\mathsf{O}_6^+(2)$ polar space, adjacent when collinear.

Cliques, cocliques and chromatic number

The 56 maximal cliques have size 5 and form a single orbit. (These are the 56 splits of Ω with a fixed triple on one side. In the $\mathsf{PG}(3, 2)$ setting these 5-cliques are the spreads. In the $\mathsf{O}_6^+(2)$ setting they are the ovoids.) The stabilizer of a maximal clique is $\mathsf{S}_3 \times \mathsf{S}_5$.

The 30 maximal cocliques have size 7 and form a single orbit. (These are the 30 $STS(7)$'s on a fixed 7-set. In the $\mathsf{PG}(3, 2)$ setting these are the sets of 7 lines on a point or 7 lines in a plane. In the $\mathsf{O}_6^+(2)$ setting these are the maximal totally isotropic subspaces.) The stabilizer of a maximal coclique is $\mathsf{AGL}_3(2) = 2^3 : \mathsf{L}_3(2)$.

The chromatic numbers are $\chi(\Gamma) = 6$ and $\chi(\overline{\Gamma}) = 7$. In the $\mathsf{PG}(3,2)$ setting, a coloring of $\overline{\Gamma}$ with seven colors is called a *packing*, a partition of the set of 35 lines into 7 spreads. Up to isomorphism, there is a unique such packing.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e	subgraph
a	$2^3 : \mathsf{L}_3(2)$	30	7, 28	0	4	$\overline{K_7}$
b	$\mathsf{L}_3(2) : 2$	120	14, 21	4	8	$\{4,3,2;\ 1,2,4\}$
c	$\mathsf{S}_5 \times \mathsf{S}_3$	56	5, 30	4	2	K_5
d	$\mathsf{S}_5 \times \mathsf{S}_2$	168	10, 20+5	6	4	$T(5)$
e	$\mathsf{S}_3 \times (\mathsf{S}_2 \,\mathrm{wr}\, 2)$	840	4+6, 1+12+12	6	4	
f	$\mathsf{S}_6 \times \mathsf{S}_2$	28	15, 20	8	6	$T(6)$

In case (b), the 14-set of triples is a 2-(7,3,2) design. There are four such designs ([585]), and this is the unique such design without repeated blocks. The induced graph is the nonincidence graph of the Fano plane $\mathsf{PG}(2,2)$. This 14-set is the union of two 7-cocliques (as under (a)).

Case (d) is that of ten triples in a fixed 5-set. Here the group has 3 orbits.

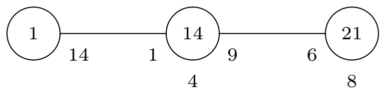
Case (e) has the unions of two disjoint 5-cliques. Here the group has 5 orbits.

The union of t pairwise disjoint 5-cliques is a regular set of size $5t$, degree $2t+2$ and nexus $2t$. Since $\chi(\overline{\Gamma}) = 7$, all values of t with $1 \le t \le 6$ are admissible. For $t = 1, 2$ this gives cases (c) and (e).

Case (f) is the union of examples of (c) and (d). In the $\mathsf{PG}(3,2)$ setting, case (f) corresponds to a linear line complex, or to the set of totally isotropic lines of the $\mathsf{Sp}_4(2)$ geometry.

Any further regular sets have size 15 or 20.

10.14 The $\mathsf{G}_2(2)$ graph on 36 vertices



$v = 36$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (36, 14, 4, 6)$. Its spectrum is $14^1\ 2^{21}\ (-4)^{14}$. The full group of automorphisms is $\mathsf{G}_2(2) = \mathsf{U}_3(3).2$ acting rank 3 with point stabilizer $\mathsf{L}_2(7).2$.

This graph is not determined by its parameters alone: there are precisely 180 nonisomorphic strongly regular graphs with parameters $(36, 14, 4, 6)$ (Spence [669], McKay & Spence [556]).

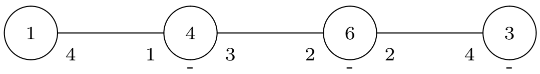
2-Ranks

The adjacency matrices of Γ and its complement both have 2-rank 8.

Peeters [611] showed that both Γ and its complement are uniquely determined by their strongly regular graph parameters and 2-rank.

Local graph and Suzuki tower

The local graph is the point-line nonincidence graph of the Fano plane.



$v = 14$

Starting from the Suzuki graph (on 1782 points) and repeatedly taking local graphs, one finds the $\mathsf{G}_2(4)$ graph on 416 vertices, the Hall-Janko graph on 100 vertices, the present graph Γ, and the point-line nonincidence graph Δ of the Fano plane. Conversely, there are precisely three connected graphs that are locally Δ (on 36, 48, and 108 vertices, see [128]), the Hall-Janko graph is the unique graph that is locally Γ, the $\mathsf{G}_2(4)$ graph is the unique graph that is locally the Hall-Janko graph, and the Suzuki graph and its triple cover are the only graphs that are locally the $\mathsf{G}_2(4)$ graph (Pasechnik [601]).

Construction: subhexagons of the $\mathsf{G}_2(2)$ generalized hexagon

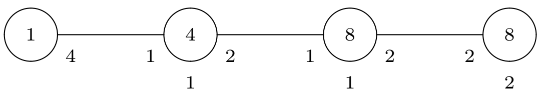
The classical $\mathsf{G}_2(2)$ generalized hexagon has 36 sub-$\mathsf{GH}(1,2)$'s. Join two of these when they have 4 points in common.

Construction: partitions into bases

As we have seen, the dual of the split Cayley hexagon $\mathsf{G}_2(2)$ can be seen in $\mathsf{PG}(2,9)$ provided with a nondegenerate Hermitian form. The set of 63 nonisotropic points has precisely 36 partitions into 21 bases, twelve on any given basis. Each partition meets 1, 14, 21 partitions in 21, 3, 9 bases, respectively. Our graph is the graph on these 36 partitions where two are adjacent when they meet in 3 bases.

Construction: 1 + 14 + 21

Take a vertex ∞, let its 14 neighbors be the 7 points and 7 lines of the Fano plane, where a point is adjacent to a line when they are not incident, and let the 21 nonneighbors of ∞ be the 21 flags of the Fano plane, where two flags are adjacent when they have no element in common, but the point of one is on the line of the other (so that the subgraph on these 21 is the distance-2 graph of the generalized hexagon that is the flag graph of the Fano plane),



$v = 21$

and finally the flag (p, L) is adjacent to the three points on L and the three lines on p. This is our graph.

$K_{4,4}$ subgraphs

There are 63 $K_{4,4}$ subgraphs, forming a single orbit. The stabilizer of one is $4^2{:}\mathsf{D}_{12}$ with vertex orbit sizes $8 + 12 + 16$. In the representation inside the generalized hexagon, these are the lines of the generalized hexagon. In the representation in $\mathsf{PG}(2,9)$ with Hermitian form, these are the orthogonal bases.

Partitions into triangles

Let us call two disjoint or equal triangles S, T in Γ 'parallel' when $\{S,T\}$ is a regular partition of the subgraph induced on $S \cup T$. Then each triangle in Γ determines a unique partition of $\mathsf{V}\Gamma$ into 12 mutually parallel triangles. There are 28 of these partitions, forming a single orbit. In the representation in $\mathsf{PG}(2,9)$ with Hermitian form, these are the isotropic points.

Cliques, cocliques and chromatic number

The maximal cliques have size 3 (since the local graph is bipartite) and form a single orbit under $\mathrm{Aut}\,\Gamma$. Since Γ has partitions into 12 triangles, the complementary graph has chromatic number 12. The maximal cocliques fall into two orbits: there are 72 7-cocliques (namely the parts of the bipartitions of the 36 local graphs) and 126 maximal 4-cocliques (namely the parts of the bipartitions of the 63 $K_{4,4}$'s). The chromatic number is 6.

Regular sets

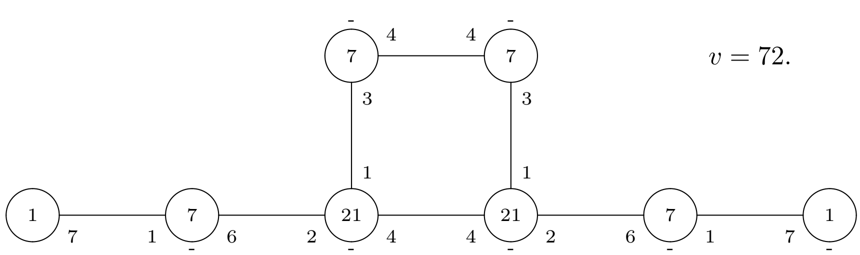
Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	3^{2+1}:4	112	18, 18	5	9
b	$4\mathsf{S}_4$:2	63	12, 24	6	4
c	3^{2+1}:D_8	56	18, 18	8	6

In case (b) the graph induced on the orbit of size 12 is the 2-coclique extension of 2×3.

Semibiplane

The graph Γ is locally bipartite. Construct a graph on 72 vertices (x, M) where x is a vertex of Γ and M one bipartite half of the neighbors of x. Call (x, M) and (y, N) adjacent when $x \in N$ and $y \in M$. The resulting graph is a bipartite (0,2)-graph (i.e., the incidence graph of a semibiplane) of diameter 5 and valency 7. Each vertex has distance 5 to a unique other point. Interchanging antipodes is not an automorphism, but identifying antipodes yields the graph Γ again. This graph has automorphism group $\mathsf{U}_3(3).2$ with point stabilizer $\mathsf{L}_2(7)$. The orbit sizes are $1 + 7 + 21 + 7 + 7 + 21 + 7 + 1$, with diagram

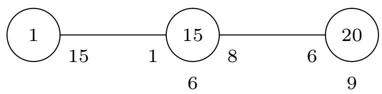


$v = 72.$

Cospectral graphs

McKay & Spence [556] found that there are 180 strongly regular graphs with parameters $(v, k, \lambda, \mu) = (36, 14, 4, 6)$. Klin, Meszka, Reichard & Rosa [492] found that four of these satisfy the 4-vertex condition, namely the above rank 3 one and three with groups of orders 64, 32, and 24. These three are the smallest non-rank 3 graphs satisfying the 4-vertex condition.

10.15 $NO_6^-(2)$



$v = 36$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (36, 15, 6, 6)$. Its spectrum is $15^1\ 3^{15}\ (-3)^{20}$. The full group of automorphisms is $\mathsf{O}_5(3):2$ acting rank 3 with point stabilizer $2 \times \mathsf{S}_6$. A construction (as $NO_6^-(2)$) was given in §3.1.2. Another construction (as $NO_5^{-\perp}(3)$) was given in §3.1.4.

The local graph is the collinearity graph of $\mathsf{GQ}(2,2)$, the complement of the triangular graph $T(6)$. The second subconstituent is the Johnson graph $J(6,3)$. This graph Γ is the local graph of $NO_6^+(3)$, see §10.35. This graph is also the 2nd subconstituent of $VO_6^-(2)$, see §10.25.

Maximal cliques have size 4, a single orbit. Maximal cocliques have sizes 3 and 5, a single orbit each.

Regular sets

Examples of regular sets are obtained from subgroups H of $\mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$2.(\mathsf{A}_4 \times \mathsf{A}_4).2^2$	45	12, 24	3	6	$3K_4$
b	$2^4:\mathsf{S}_5$	27	16, 20	5	8	folded 5-cube
c	$\mathsf{S}_3 \,\mathrm{wr}\, \mathsf{S}_3$	40	9, 27	6	3	$K_{3,3,3}$
d	$3^{2+1}:\mathsf{D}_8$	240	18, 18	9	6	

Altogether, the numbers of regular sets are as follows.

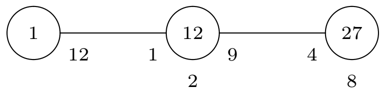
(d,e)	$(1,4)$	$(3,6)$	$(5,8)$	$(6,3)$	$(9,6)$
#	135	1485	3699	40	240

The regular sets with $(d,e) = (1,4)$ are subgraphs $4K_2$. In the $NO_6^-(2)$ representation these arise as the nonsingular parts of ovoids in the $\mathsf{Sp}_4(2)$ geometry on $p^\perp/\langle p\rangle$ for singular points p. The above 45 regular sets with $(d,e) = (3,6)$ are subgraphs $3K_4$ that arise as the nonsingular part of the union of three t.i. planes on a fixed t.s. line. There are 1440 further regular sets with $(d,e) = (3,6)$. The regular sets with $(d,e) = (6,3)$ are subgraphs $K_{3,3,3}$ that arise as unions of three pairwise orthogonal elliptic lines. In the $NO_5^{-\perp}(3)$ representation, these subgraphs arise as the perps of a singular point. Each t.s. line yields a partition of $V\Gamma$ into four $K_{3,3,3}$'s.

Locally $\mathsf{GQ}(2,2)$ graphs

Consider the $\mathsf{Sp}(6,2)$ polar graph Σ. It is strongly regular with parameters $(v,k,\lambda,\mu) = (63, 30, 13, 15)$, see §10.21. There are three graphs that are locally $\mathsf{GQ}(2,2)$, on 28, 32 and 36 vertices ([156]). They can be obtained from Σ by removing a hyperbolic quadric, a hyperplane, and an elliptic quadric, respectively. The first is $\overline{T(8)}$. The second has diameter 3, and is the Taylor extension of the $\mathsf{GQ}(2,2)$ graph. The third is our present graph Γ. See also [142].

10.16 The $\mathsf{O}_5(3)$ graphs on 40 vertices



$v = 40$

There are exactly two rank 3 strongly regular graphs with parameters $(v, k, \lambda, \mu) = (40, 12, 2, 4)$. Their spectrum is $12^1\ 2^{24}\ (-4)^{15}$. Both have full group of automorphisms $\mathsf{O}_5(3).2$. The point stabilizers are $3^3:(\mathsf{S}_4 \times 2)$ and $3_+^{1+2}:2\mathsf{S}_4$.

There are precisely two generalized quadrangles $\mathsf{GQ}(3,3)$, duals to each other ([607]). One is that on the isotropic points, the other that on the totally isotropic lines of the $\mathsf{O}_5(3)$ geometry, cf. §2.6.1. The latter is isomorphic to the generalized quadrangle on the points of the $\mathsf{Sp}_4(3)$ geometry.

Our graphs are the collinearity graphs of these generalized quadrangles. Let these graphs be Γ and Δ, where Γ is the $\mathsf{O}_5(3)$ graph, and Δ the dual $\mathsf{O}_5(3)$ graph or $\mathsf{Sp}_4(3)$ graph.

Construction inside the $\mathsf{U}_4(2)$ geometry

The graph Δ is the graph $\overline{NU_4(2)}$ on the nonisotropic points of the $\mathsf{U}_4(2)$ geometry, adjacent when orthogonal with respect to the Hermitian form (i.e., when joined by a secant line).

In this setting, Γ is the graph on the Hermitian bases, adjacent when intersecting nontrivially.

As we saw (§3.1.6), the graph $\overline{NU_n(2)}$ is locally $\overline{NU_{n-1}(2)}$. Pasechnik [603] showed that $\overline{NU_5(2)}$ is the unique graph that is locally $\mathsf{GQ}(3,3)$.

Construction inside the unique $\mathsf{GQ}(2,4)$

The vertex set of Δ consists of the Hermitian spreads of the unique $\mathsf{GQ}(2,4)$, adjacent when sharing exactly three lines (nonadjacent spreads then share exactly one line). These three lines necessarily form a regulus of a 3×3 grid, a subquadrangle of order $(2,1)$ of $\mathsf{GQ}(2,4)$.

The vertex set of Γ consists of the partitions of the point set of $\mathsf{GQ}(2,4)$ into three 3×3 grids, adjacent when they have exactly 9 lines in common (each grid of the first partition shares exactly one line with each grid of the second partition; the 9 lines form a Hermitian spread).

Cliques, cocliques and chromatic number

The maximal cliques in both cases have size 4, the lines of the generalized quadrangle. The maximal cocliques in Γ have sizes 5, 8 and 10, those in Δ have sizes 4 and 7, a single orbit in all cases. The 10-cocliques in Γ are ovoids. The chromatic numbers are $\chi(\Gamma) = 5$, $\chi(\Delta) = 6$, $\chi(\overline{\Gamma}) = 11$, $\chi(\overline{\Delta}) = 10$.

Regular sets

Examples of regular sets in Γ and Δ are obtained from subgroups H of their automorphism groups with two orbits on the vertex set. We give degree d, nexus e, and i^{n_i}, where n_i is the number of lines meeting the smallest orbit in i points, and structure for the smallest orbit.

For $\Gamma = \Gamma(\mathsf{O}_5(3))$:

	H	index	orbitlengths	d	e	line stats
a	$2 \times \mathsf{S}_6$	36	10, 30	0	4	1^{40}
b	$2^4{:}5{:}2$	324	20, 20	4	8	2^{40}
c	$3_+^{1+2}{:}2\mathsf{S}_4$	40	4, 36	3	1	$4^1\ 1^{12}\ 0^{27}$
d	$2.(\mathsf{A}_4 \times \mathsf{A}_4).2^2$	45	16, 24	6	4	$4^8\ 1^{32}$

Case (a): ovoid, the perp of a minus point.

Case (b): a *hemisystem of points* of the $\mathsf{O}_5(3)$ generalized quadrangle, i.e., a set of points intersecting each line in half of its points. Here, it is *not* the union of two ovoids. Both halves are conjugate. An explicit construction runs as follows: in the $\mathsf{GQ}(2,4)$ pick a point x and order the lines through x arbitrarily in a cyclic way (say, L_i, $i \bmod 5$). Then, referring to the $\mathsf{GQ}(2,4)$ construction of Γ, the hemisystem of points consists of all partitions of the point set of $\mathsf{GQ}(2,4)$ into those 3×3 grids one of which contains two consecutive lines L_i, L_{i+1} through x.

Case (c): 4 isotropic points on a fixed t.i. line.

Case (d): 4×4 grid, the perp of a plus point.

The union of any number of pairwise disjoint t.i. lines is a regular set (and so is its complement). Maximal sets of disjoint lines have 4 or 7 elements (exactly the sizes of the maximal cocliques of the dual $\mathsf{O}_5(3)$ graph), hence there are regular sets of size $4t$ with (degree, nexus) $= (2+t, t)$, for all $t \in \{1, 2, \ldots, 9\}$.

For $\Delta = \Gamma(\mathsf{Sp}_4(3))$:

	H	index	orbitlengths	d	e	line stats
a	$2 \times \mathsf{A}_5$	432	20, 20	4	8	2^{40}
b	$3^3{:}(\mathsf{S}_4 \times 2)$	40	4, 36	3	1	$4^1\ 1^{12}\ 0^{27}$
c	$2.(\mathsf{A}_4 \times \mathsf{A}_4).2^2$	45	8, 32	4	2	$2^{16}\ 0^{24}$
d	$\mathsf{S}_4 \times \mathsf{D}_8$	270	16, 24	6	4	$4^4\ 2^{24}\ 0^{12}$
e	$2 \times \mathsf{S}_5$	216	20, 20	7	5	$4/0^{10}\ 2/2^{10}\ 1/3^{20}$

Case (a): a *hemisystem of points* of the $\mathsf{Sp}_4(3)$ generalized quadrangle, i.e., a set of points intersecting each line in half of its points. (Cf. [42], [43], [227].) Both halves are conjugate.

These are the only regular sets with $d - e = s$.

Case (b): the 4 points on a fixed t.i. line

Case (c): the 8 points of a $K_{4,4}$ (i.e., $L \cup L^\perp$ for a hyperbolic line L).

All other regular sets with $(d, e) = (4, 2)$ are unions of two t.i. lines.

Case (d): In the $\mathsf{O}_5(3)$ geometry, 16 t.i. lines each of them on a point of a fixed conic of t.i. points spanning a plane which is the perp of an elliptic line (i.e., a line containing only nonisotropic points).

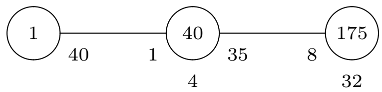
All regular sets with $(d, e) = (6, 4)$ do contain a (symplectic) t.i. line.

Case (e): In the $\mathsf{U}_4(2)$ setting, consider a subquadrangle $\mathsf{GQ}(2,2)$ of the $\mathsf{U}_4(2)$ generalized quadrangle and an ovoid O in $\mathsf{GQ}(2,2)$ (on which S_5 acts). The nonisotropic points on the lines of $\mathsf{PG}(3,4)$ joining two points of O form a regular set.

All other regular sets with $(d, e) = (7, 5)$ do contain a (symplectic) t.i. line.

The union of any number of pairwise disjoint (symplectic) t.i. lines is a regular set (and so is its complement). Hence there are regular sets of size $4t$ with (degree, nexus) $= (2+t, t)$, for all $t \in \{1, 2, \ldots, 9\}$.

Graph on the $20+20$ splits



$v = 216$

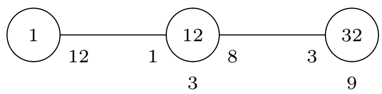
Above we saw that $\Gamma(\mathsf{Sp}_4(3))$ has 216 splits into two regular 20-sets with degree 7 and nexus 5. The group $O_5(3).2$ acts on these 216 with permutation rank 7 and subdegrees $1+5+20+30+40+60+60$. The suborbit of size 40 defines a strongly regular graph with parameters $(v,k,\lambda,\mu) = (216,40,4,8)$ and spectrum $40^1\ 4^{140}\ (-8)^{75}$. The full group of automorphisms is $\mathsf{O}_5(3).2$ acting rank 7 with point stabilizer $2 \times \mathsf{S}_5$. This graph was discovered by Crnković et al. [243]. See also [242].

2-Ranks

graph	$\mathrm{rk}_2(A)$	$\mathrm{rk}_2(J-A)$	$\mathrm{rk}_3(A+I)$	$\mathrm{rk}_3(J-I-A)$
Γ	10	10	15	14
Δ	16	16	11	10

Peeters [611] showed that given their strongly regular graph parameters, the four graphs Γ, $\overline{\Gamma}$, Δ, $\overline{\Delta}$ are uniquely determined by the values 10, 10, 11, 10 of $\mathrm{rk}_2(A)$, $\mathrm{rk}_2(A+I)$, $\mathrm{rk}_2(A+I)$, and $\mathrm{rk}_2(A)$, respectively.

10.17 The $\mathsf{U}_4(2)$ graph on 45 vertices



$v = 45$

It was shown by Coolsaet, Degraer & Spence [223] that up to isomorphism there are precisely 78 strongly regular graphs with parameters $(v,k,\lambda,\mu) = (45,12,3,3)$. Their spectrum is $12^1\ 3^{20}\ (-3)^{24}$.

There is a unique rank 3 strongly regular graph Γ with these parameters. It is $\Gamma(\mathsf{U}_4(2))$, the collinearity graph of the unique $\mathsf{GQ}(4,2)$, the dual of $\mathsf{GQ}(2,4)$ discussed in §10.10. It is also $NO_5^{+\perp}(3)$. Its full group of automorphisms is $\mathsf{O}_5(3):2$ with point stabilizer $((2^{3+2}:3^2):2):2$.

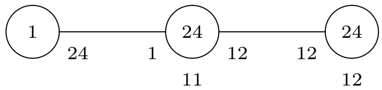
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$3_+^{1+2}:2\mathsf{S}_4$	40	9, 36	0	3	$9K_1$, ovoid
b	$3^3:(\mathsf{S}_4\times 2)$	40	18, 27	3	6	$3K_{3,3}$
c	$2^4:\mathsf{S}_5$	27	5, 40	4	1	K_5, line
d	$2\times\mathsf{S}_6$	36	15, 30	6	3	$\mathsf{GQ}(2,2)$

The union of at most five pairwise disjoint lines, or the complement thereof, gives examples with (degree, nexus)$= (3+t,t)$, for $1 \le t \le 8$.

10.18 The rank 3 conference graphs on 49 vertices



$v = 49$

There are exactly two rank 3 strongly regular graphs with parameters $(v, k, \lambda, \mu) = (49, 24, 11, 12)$. Their spectrum is $24^1\ 3^{24} (-4)^{24}$. The first, let us call it Γ_1, is the Paley graph, with full group of automorphisms $7^2:24:2$ and point stabilizer $24:2$. The second, let us call it Γ_2, is the Peisert graph, with full group of automorphisms $7^2:(3 \times \mathsf{SL}_2(3))$ and point stabilizer $3 \times \mathsf{SL}_2(3)$. Both graphs are self-complementary.

The maximal cliques of Γ_1 have sizes 5 and 7, a single orbit of each type. The maximal cliques of Γ_2 have sizes 4 and 7, a single orbit of each type. Both Γ_1 and Γ_2 have chromatic number 7, that is, there are partitions into 7-cliques and partitions into 7-cocliques. Any disjoint union of 7-cliques (7-cocliques) is a regular set with $(d, e) = (6, 3)$ (resp. $(0, 4)$).

Construction

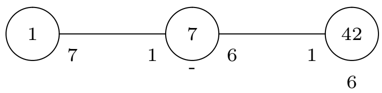
Let V be a vector space, and $H = \mathsf{P}V$ its hyperplane at infinity. Pick a subset X of H. The graph Γ with vertex set V, where $v \sim w$ when $\langle w - v \rangle \in X$ is strongly regular when the hyperplanes of H meet X in two different cardinalities (see §7.1.1). In the special case where $\dim V = 2$, hyperplanes are single points, and every choice of X (other than $\emptyset$ or H) will give a strongly regular graph. One finds nets with parameters $v = q^2$, $k = (q-1)n$, $r = q - n$, $s = -n$, $\mu = n(n-1)$, $\lambda = q + n(n-3)$, if the underlying field is $\mathbb{F}_q$ and $|X| = n$. These graphs are Latin square graphs $\mathrm{LS}_n(q)$ (see §8.4.2). They will be rank 3 when the stabilizer of X in $\mathsf{P\Gamma L}_2(q)$ acting on H has the two orbits X and $H \setminus X$.

In the special case $q = 7$, $n = 4$, the group $\mathsf{PGL}_2(q)$ has two orbits (of sizes 42 and 28) on the set $\binom{H}{4}$ of 4-sets in H. Picking X in the first orbit gives the Paley graph Γ_1. Picking X in the second orbit gives Γ_2. The stabilizers of X in these cases are D_8 and A_4, both with orbit lengths $4 + 4$ on H.

Further self-complementary graphs

Mathon [548] found all self-complementary strongly regular graphs on at most 49 vertices. With $v = 49$ there are apart from Γ_1 and Γ_2 three further examples.

10.19 The Hoffman-Singleton graph



$v = 50$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (50, 7, 0, 1)$. Its spectrum is $7^1\ 2^{28}\ (-3)^{21}$. The full group of automorphisms is $\mathsf{U}_3(5).2$ acting rank 3 with point stabilizer S_7.

This graph was found (and shown unique) by Hoffman & Singleton [436] as example of a *Moore graph*, that is a graph of diameter d and girth g where $g = 2d + 1$.

Moore graphs are regular. Apart from the odd polygons only three Moore graphs are known, namely the pentagon, the Petersen graph, and the Hoffman-Singleton graph. Any further Moore graph must be strongly regular with parameters $(v, k, \lambda, \mu) = (3250, 57, 0, 1)$. If there is such a graph, its group of automorphisms has order at most 375 ([532]). See also [132], §11.5.1.

Construction: 5×5 + 5×5

Take five pentagons P_h and five pentagrams Q_i, so that vertex j of P_h is adjacent to vertices $j-1, j+1$ of P_h and vertex j of Q_i is adjacent to vertices $j-2, j+2$ of Q_i. Now join vertex j of P_h to vertex $hi+j$ of Q_i. (All indices mod 5.)

Construction: 15 + 35

Use the identification of the 35 lines in $\mathsf{PG}(3,2)$ with the 35 triples in a 7-set where intersecting lines belong to triples meeting in precisely one element (Proposition 6.2.9). Take as vertices the 15 points and 35 lines of $\mathsf{PG}(3,2)$, let the points form a coclique, let a point be adjacent to a line when they are incident, and let two lines be adjacent when the corresponding triples are disjoint.

Construction: 20 + 30

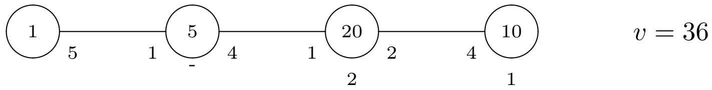
Take as vertices the 20 ternary vectors of weight 1 and the 30 ternary vectors of length 10 and weight 4 obtained by taking in the extended ternary Golay code all vectors of weight six starting 11... or 12... and deleting the first two coordinates. Join two weight 1 vectors when they have distance 1; join a weight 1 and a weight 4 vector when they have distance 3; join two weight 4 vectors when they have distance 8. This yields the Hoffman-Singleton graph, and shows that it has a partition into a subgraph $10K_2$ and two 15-cocliques.

Construction: inside the Higman-Sims graph

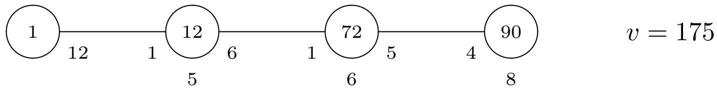
In the Steiner system $S(4, 7, 23)$, fix a symbol a. Construct the Higman-Sims graph (§10.31) on $1 + 22 + 77$ points by taking a point ∞, the 22 symbols distinct from a and the 77 blocks containing a, where blocks are adjacent when they meet in $\{a\}$ only. Now let B be a block of $S(4, 7, 23)$ not containing a. It induces a partition $(1 + 7 + 42) + (15 + 35)$ of the $1 + 22 + 77$ (the 42 blocks are those meeting B in one point, the 35 those meeting B in three points), and both $1 + 7 + 42$ and $15 + 35$ induce a Hoffman-Singleton graph.

Edges

Since $\lambda = 0$, the maximal cliques have size 2 and are the edges. There are 175 of these, forming a single orbit. The stabilizer of one is $\mathsf{A}_6.2^2$ with vertex orbit sizes $2+12+36$ and edge orbit sizes $1+12+72+90$. The subgraph of Γ induced on the 36 vertices nonadjacent to a fixed edge is the *Sylvester graph*, the unique distance-regular graph with intersection array $\{5, 4, 2; 1, 1, 4\}$.



The line graph $L(\Gamma)$ is the unique distance-regular graph with intersection array $\{12, 6, 5;\ 1, 1, 4\}$.



The graph on the edges, adjacent when they have distance 2 in the line graph, is strongly regular with parameters $(v, k, \lambda, \mu) = (175, 72, 20, 36)$.

If a regular graph has adjacency matrix A, and $v \times e$ vertex-edge incidence matrix N, and the line graph has adjacency matrix L, then $NN^\top = A + kI$ and $N^\top N = L + 2I$. The spectrum of L follows since $NN^\top$ and $N^\top N$ have the same nonzero eigenvalues. So, in the present case, the line graph $L(\Gamma)$ has spectrum $12^1\ 7^{28}\ 2^{21}\ (-2)^{125}$. Since Γ has girth 5, the distance-2 graph of $L(\Gamma)$ has adjacency matrix L_2 where $L^2 = (2k-2)I + (k-2)L + L_2$. In the present case L_2 has spectrum $72^1\ 2^{153}\ (-18)^{21}$.

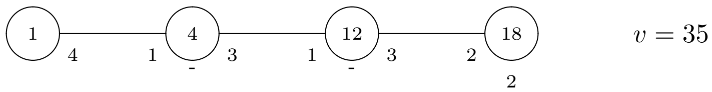
This last graph is the collinearity graph of a partial geometry $pg(5, 18, 2)$, see §8.6.1(iv). The collinearity graph of its dual, a $pg(18, 5, 2)$, is strongly regular with parameters $(v, k, \lambda, \mu) = (630, 85, 20, 10)$.

Cocliques

The largest cocliques have size 15, and there are 100 of them, forming a single orbit. The stabilizer of one is A_7, with vertex orbit sizes $15 + 35$.

The complement of a 15-coclique induces the Odd graph O_4, the unique distance-regular graph with intersection array $\{4, 3, 3;\ 1, 1, 2\}$, the graph on the triples in a 7-set, adjacent when disjoint. It has full group S_7, with point stabilizer $\mathsf{S}_3 \times \mathsf{S}_4$.

The group is twice as large as that induced by $\mathrm{Aut}\,\Gamma$ since this graph can be extended to a Hoffman-Singleton graph in two ways; both occur in the Higman-Sims graph.



A fixed 15-coclique meets 7, 35, 42, 15, 1 15-cocliques in 0, 3, 5, 8, 15 points, respectively. Meeting in 0, 5, or 15 points is an equivalence relation with two classes of size 50. The graph on the 50 15-cocliques in one equivalence class, where two 15-cocliques are adjacent when they are disjoint, is again the Hoffman-Singleton graph. The graph on the 100 15-cocliques, where two 15-cocliques are adjacent when they meet in 0 or 8 points, is the Higman-Sims graph. We see that the Higman-Sims graph has splits into two Hoffman-Singleton graphs.

Heawood and Coxeter subgraphs

Let C and D be two 15-cocliques that meet in 8 points. The stabilizer of $\{C, D\}$ has orbit sizes $8 + 14 + 28$. The induced subgraph on the orbit of size 14 is the Heawood graph, the point-line incidence graph of the Fano plane, the unique distance-regular graph with intersection array $\{3, 2, 2;\ 1, 1, 3\}$. The induced subgraph on the orbit of size 28 is the Coxeter graph, the graph that Coxeter calls 'My Graph', the unique distance-regular graph with intersection array $\{3,2,2,1;\ 1,1,1,2\}$. Both have full group $\mathsf{L}_3(2).2$.

Splits

There are 1260 pentagons, forming a single orbit. For a fixed pentagon, the 25 adjacent vertices induce $5C_5$, and the complement of this $5C_5$ also induces a $5C_5$. It follows that Γ has 126 splits into two $5C_5$ subgraphs.

For each such split, the union of a pentagon from one side and a pentagon from the other side induces a Petersen graph. Splits and Petersen graphs form the points and blocks of a unital $S(2,6,126)$ in $\mathsf{PG}(2,25)$, explaining the structure of $\mathrm{Aut}\,\Gamma$ (Benson & Losey [58]).

Chromatic number

The Hoffman-Singleton graph has chromatic number 4 and edge-chromatic number 7 (that is, its line graph has chromatic number 7). Its complement has chromatic number 25.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	A_7	100	15, 35	0	3	15-coclique
b	M_{10}	350	20, 30	1	4	$10K_2$
c	$5^{2+1}{:}(4{\times}2)$	252	25, 25	2	5	$5C_5$
d	$2\mathsf{S}_5.2$	525	10, 40	3	1	Petersen

In case (a) the subgraph induced on the orbit of size 35 is the Odd graph O_4, the graph of disjoint triples in a 7-set. In case (b) the subgraph induced on the orbit of size 30 is Tutte's 8-cage, the incidence graph of $\mathsf{GQ}(2,2)$. In case (d) the subgraph induced on the orbit of size 40 is the unique $(6,5)$-cage.

No further regular sets occur for $(d,e) \neq (4,2)$. For each regular set R with $(d,e) = (4,2)$ and hence $|R| = 20$, the complementary regular set has size 30 with $(d,e) = (5,3)$, and is a $(5,5)$-cage, see below. There are two types: the 12600 sets R that are the disjoint union of two Petersen subgraphs, and the 2625 sets R where $V\Gamma \setminus R$ is a Meringer cage.

Paths and Cycles

The group $G = \mathrm{Aut}\,\Gamma$ is transitive on ordered induced paths of length at most five. It has three orbits on ordered induced paths of length 6 (with 7 vertices). In particular, the group is 3-arc transitive (transitive on ordered paths of length 3). This group is transitive on induced 5-cycles, 6-cycles, and 7-cycles. It has two orbits on induced 8-cycles. Each hexagon is contained in a unique Petersen subgraph.

Cages

The Hoffman-Singleton graph Γ is the $(7,5)$-cage, that is, the unique smallest graph of valency 7 and girth 5. Of course every subgraph has girth at least 5. The unique $(6,5)$-cage has 40 vertices, and is found by removing the vertices of a Petersen subgraph from Γ. The $(5,5)$-cages have 30 vertices. There are four nonisomorphic examples ([750], [558]), two of which can be found inside Γ. Also $(3,6)$-cages (the incidence graph of the Fano plane) and $(3,8)$-cages (the incidence graph of $\mathsf{GQ}(2,2)$) are found in Γ. See also [123], pp. 206–210, and [314].

About the (5,5)-cages

There is some confusion concerning naming and properties of the (5,5)-cages, the main problem being that nobody knows what graph is called the Robertson-Wegner graph. The four (5,5)-cages have groups of orders 20, 30, 96, and 120. The cage with group of order 96 was discovered by Yang & Zhang [750] and rediscovered by Meringer [558]. The survey [743] knew about the remaining three. Its Figure 6 displays the cage with group of order 30, with reference 'R. M. Foster (unpublished)'. The cage with group of order 20 was discovered by Robertson [626] (upper left corner of Figure 1.1C), later mentioned in Wegner [724] (who refers to [626]), and is Figure 5 in [743]. It is the `RobertsonWegnerGraph` in Mathematica. The cage with group of order 120 was given in Robertson [626] (Figure 1.1D), and is called the Robertson-Wegner graph by many authors; it is Figure 4 in [743], and the `WongGraph` in Mathematica.

	$\lvert G\rvert$	orbit sizes	spectrum	name
a	20	5+5+10+10	$(-3)^4$ $(-2.71)^2$ $(-2.47)^2$ $(-2.12)^2$ $(-1.78)^2$ $(-1)^1$ 0.78^2 1.12^2 1.47^2 1.71^2 2^8 5^1	Robertson-Wegner graph
b	30	15+15	$(-2.71)^4$ $(-2.12)^4$ $(-1)^1$ 1.12^4 $(-1\pm\sqrt{5})^2$ 1.71^4 2^4 $(\pm\sqrt{5})^2$ 5^1	Foster cage
c	96	6+24	$(-3)^2$ $(-2)^3$ 0^1 $(-1\pm\sqrt{3})^4$ $\frac{1}{2}(-1\pm\sqrt{17})^3$ 2^9 5^1	Yang-Zhang cage / Meringer cage
d	120	10+20	$(-1)^2$ 1^5 $\frac{1}{2}(-1\pm\sqrt{21})^8$ $(\pm\sqrt{5})^3$ 5^1	Robertson cage

Case (a) is obtained from the Hoffman-Singleton graph Γ by removing two Petersen subgraphs, and as we saw in the discussion of regular sets, also case (c) is contained in Γ. Cases (b) and (d) cannot be contained in Γ because their eigenvalue $\sqrt{5}$ would contradict interlacing. In cases (a) and (d), the group size and orbit sizes were given incorrectly in [123]. A nice description of graph (d), showing its full group $\mathsf{A}_5 \times 2$, is the following. Take the 20 vertices of the dodecahedron, and the 10 4-subsets of the dodecahedron that have all internal distances 3; the adjacencies are the obvious ones: the dodecahedron is an induced subgraph of valency 3, each 4-subset is adjacent to its 4 elements and to the antipodal 4-subset.

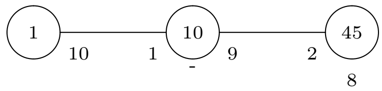
Locally Hoffman-Singleton graphs

No locally Hoffman-Singleton graphs are known. Such a graph cannot be distance-transitive or flag-transitive (Van Bon [85]) and must have diameter at most 6. See also [337].

Decomposition of K_{50}

We saw that K_{16} can be split into three edge-disjoint Clebsch graphs, and K_{10} cannot be split into three edge-disjoint Petersen graphs. It is unknown whether K_{50} can be split into seven edge-disjoint Hoffman-Singleton graphs. However, it is possible to pack six edge-disjoint Hoffman-Singleton graphs into K_{50} ([531]).

10.20 The Gewirtz graph



$v = 56$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (56, 10, 0, 2)$. Its spectrum is $10^1\ 2^{35}(-4)^{20}$. The full group of automorphisms is $G = \mathsf{L}_3(4).2^2$ (of order $2^8.3^2.5.7$) acting rank 3, with point stabilizer $\mathsf{A}_6.2^2$.

Construction

This is the graph on the $77 - 21 = 56$ blocks of the (unique) Steiner system $S(3, 6, 22)$ not containing a fixed symbol, adjacent when they are disjoint. It is also the subgraph of the Higman-Sims graph induced on the set of vertices at distance 2 from an edge (and this construction shows the full group).

From the first construction we deduce the following explicit construction. Vertices are the hyperovals of a $\mathsf{PSL}_3(4)$-orbit in $\mathsf{PG}(2, 4)$, adjacent if disjoint. Since the lines not meeting a given hyperoval form a dual hyperoval, we can see each vertex of Γ as a pair (hyperoval, dual hyperoval), which explains the doubling of the automorphism group compared to the point stabilizer of $S(3, 6, 22)$. These extra automorphisms are dualities of $\mathsf{PG}(2, 4)$.

Uniqueness

Uniqueness is due to Gewirtz [342]. For shorter uniqueness proofs, and further properties, see [131].

Cliques and cocliques

The maximal cliques are the 280 edges.

Maximal cocliques have sizes 7, 9, 10, 11, 12, 13 or 16. The table below gives the number of cocliques of each given size.

size	7	9	10	11	12	13	16
#	240	2520	43960	20160	5460	1680	42

The maximum cocliques have size 16, reaching the Hoffman bound. They form a single orbit. The stabilizer of one in G is $2^4.\mathsf{S}_5$, with vertex orbit sizes $16 + 40$. If Γ is seen as the subgraph induced on the vertices at distance 2 from an edge xy in the Higman-Sims graph Δ, these 42 16-cocliques are the intersections $\mathsf{V}\Gamma \cap \Delta(z)$ of $\mathsf{V}\Gamma$ with the point neighborhoods of the 42 neighbors z of the edge xy in Δ. In the $\mathsf{PG}(2, 4)$-setting, these 42 cocliques are the 21 sets of hyperovals sharing a common point and the 21 sets of hyperovals avoiding a common line of $\mathsf{PG}(2, 4)$.

Chromatic number

Γ has chromatic number 4. Its complement has chromatic number 28.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$2^4{:}\mathsf{S}_5$	42	16, 40	0	4	16-coclique
b	M_{10}	112	20, 36	1	5	$10K_2$
c	$2^{2+4}.3.2^2$	105	24, 32	2	6	$6C_4$
d	$\mathsf{L}_2(7)$	480	28, 28	3	7	$\{3,2,2,1;1,1,1,2\}$
e	$2 \times \mathsf{L}_2(7){:}2$	120	14, 42	4	2	$\{4,3,2;1,2,4\}$

In case (b), the subgraph induced on the 36-set is the Sylvester graph.

Each quadrangle is contained in a unique subgraph $6C_4$.

Case (d) is that of splits into two Coxeter graphs. These splits can be seen inside the Higman-Sims graph. It has splits into two Hoffman-Singleton graphs. Choosing an edge that meets both sides we find that the subgraph of the Higman-Sims graph far away from that edge is split into two Coxeter graphs.

Case (e) has the co-Heawood graph, the bipartite nonincidence graph of the Fano plane.

There are no further examples of regular sets with $d - e = s$.

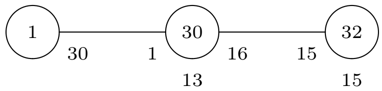
Biplane

If A is the adjacency matrix of Γ, then $A+I$ is the point-block incidence matrix of a biplane 2-(56,11,2) (due to Hall, Lane & Wales [400]). Up to isomorphism, there are five biplanes with these parameters ([483]).

Hill cap

The Gewirtz graph is an induced subgraph of the $\mathsf{O}_6^-(3)$ graph on 112 vertices (see below), and hence can be seen as a set of points in $\mathsf{PG}(5,3)$, a subset of an elliptic quadric. Viewed in this way, it is a *cap*, a set of points no three of which are collinear, and in fact is the unique largest possible cap in $\mathsf{PG}(5,3)$. (Note that lines meet the quadric in at most two points, unless they are contained in the quadric. Hence three collinear points determine a triangle in the graph, but the Gewirtz graph does not have triangles.) It follows that the vertex set of Γ, viewed as subgraph of the $\mathsf{O}_6^-(3)$ graph, defines a hemisystem of points of the $\mathsf{O}_6^-(3)$ generalized quadrangle.

10.21 $\mathsf{Sp}_6(2)$



$v = 63$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (63, 30, 13, 15)$. Its spectrum is $30^1\ 3^{35}\ (-5)^{27}$. The full group of automorphisms is $\mathsf{Sp}_6(2)$ acting rank 3 with point stabilizer $2^5{:}\mathsf{S}_6$. It is the collinearity graph of the polar space $\mathsf{Sp}_6(2)$, cf. §2.5.

The maximal cliques have size 7 and form a single orbit. They are the totally isotropic planes. The maximal cocliques have size 3, 5 or 7, a single orbit each. Those of size 3 are the hyperbolic lines. Those of size 5 are elliptic quadrics in the perp of a hyperbolic line. Those of size 7 are the 7-cocliques in the $\overline{T(8)}$ subgraphs (see below). The chromatic numbers of this graph and its complement are $\chi(\Gamma) = 11$ and $\chi(\overline{\Gamma}) = 9$.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$\mathsf{O}_5(3):2$	28	27, 36	10	15	$\mathsf{GQ}(2,4)$
b	$2^6:\mathsf{L}_3(2)$	135	7, 56	6	3	K_7
c	S_8	36	28, 35	15	12	$\overline{T(8)}$

These are all regular sets with $(d, e) = (6, 3)$, but there are many further regular sets with $(d, e) = (10, 15), (9, 6), (12, 9), (15, 12)$. For example, partial spreads provide examples with $d - e = 3$. No other pairs (d, e) occur.

Cospectral graphs

Ihringer [451] finds 13505292 different graphs cospectral with Γ by applying GM-switching (§8.13.1) to it at most five times in succession. No doubt there are many further graphs with these parameters.

10.22 The $\mathsf{G}_2(2)$ graph on 63 vertices

There are precisely two generalized hexagons of order 2, duals of each other (cf. p. 108). The distance 1-or-2 graph of each of these is strongly regular, with parameters $(v, k, \lambda, \mu) = (63, 30, 13, 15)$ (cf. Proposition 1.3.12). In this way we obtain two graphs. The rank 3 one was discussed above. Here we look at the other one, which is rank 4. Its full group of automorphisms is $\mathsf{G}_2(2)$, with point stabilizer $4 \cdot \mathsf{S}_4 : 2$ with orbits of sizes $1 + 6 + 24 + 32$ (and it is the only strongly regular graph with these parameters of which the full group acts rank 4).

Construction

This graph Γ is the graph $\overline{NU_3(3)}$: the vertices are the nonisotropic points in $\mathsf{PG}(2, 9)$ provided with a nondegenerate hermitian form, adjacent when joined by a secant. Its complement is the graph on $V\Gamma$ where vertices are adjacent when joined by a tangent. If Δ_1 is the graph on $V\Gamma$ where vertices are adjacent when orthogonal, then Δ_1 is the collinearity graph of a generalized hexagon of order 2, and $\Gamma = \Delta_1 \cup \Delta_2$, where Δ_2 is the distance-2 graph of Δ_1. This Δ_2 is the 2nd subconstituent of the Hall-Janko graph on 100 vertices.

Cliques, cocliques and chromatic number

The maximal cliques have size 4 or 7, a single orbit of each. The 63 maximal 7-cliques C_x each consist of a vertex x and the six orthogonal vertices. The maximal cocliques have size 5 or 9, a single orbit of each. The 28 maximal 9-cocliques are the tangents. The chromatic numbers of this graph and its complement are $\chi(\Gamma) = 11$ and $\chi(\overline{\Gamma}) = 9$. A partition into nine 7-cliques is given by the nine sets C_x where x runs over the vertices on a fixed tangent.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$[2^4.3^3]$	28	9, 54	0	5	9-coclique
b	$\mathsf{L}_3(2):2$	36	21, 42	12	9	$\mathsf{GH}(2,1)_{1,2}$

Subgraphs of type (b) are the distance 1-or-2 graphs of sub-$\mathsf{GH}(2,1)$'s in the $\mathsf{GH}(2,2)$, that is, are subgraphs of Γ induced by the point set of a $\mathsf{GH}(2,1)$.

10.23 The block graph of the smallest Ree unital

Above in §10.21 and §10.22 we described the two strongly regular graphs with parameters $(v,k,\lambda,\mu) = (63,30,13,15)$ and a group of automorphisms acting primitively. There are many further strongly regular graphs with these parameters, most of them ugly. Maybe the nicest one is the complement of the block graph of the smallest Ree unital, described below.

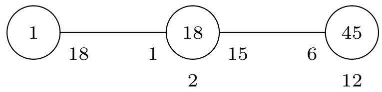
Given any Steiner system $S(2,m,u)$, the *block graph* is the graph on the blocks, adjacent when they meet. This graph is strongly regular, with parameters given in §8.5.4A. In the particular case of a $S(2,4,28)$ this block graph has parameters $(63,32,16,16)$, so that the adjacency matrices are square 2-(63,32,16) designs. (There are many further such designs.)

Let $q = 3^{2m+1}$, $m \geq 0$. The *Ree unital* of order q (Lüneburg [528]) is a unital ($S(2,q+1,q^3+1)$ design) on which the Ree group ${}^2\mathsf{G}_2(q)$ of order $(q-1)q^3(q^3+1)$ acts 2-transitively. It is not embedded (in $\mathsf{PG}(2,q^2)$).

A unital of order 3 is a Steiner system $S(2,4,28)$. The two examples of such Steiner systems with a doubly transitive group are the Hermitian unital and the Ree unital. The 4466 examples with a nontrivial group were given in [498]. The $2+4+4+8 = 18$ examples embedded in a projective plane of order 9 (there are four: the Desarguesian, Hall, dual Hall, and Hughes planes) were found in [616]. The 6 resolvable $S(2,4,28)$ (and 7 nonisomorphic resolutions) were found in [484]. The 68806 examples with a blocking set were found in [7].

The Ree unital of order 3 can be embedded as a $(0,4)$-set (a maximal arc) in $\mathsf{PG}(2,8)$. We find that it has 45 spreads (falling into two orbits, of sizes $9+36$), corresponding to the 45 exterior points, and 10 resolutions (falling into two orbits, of sizes $1+9$), corresponding to the 10 exterior lines. The graph Γ on the blocks, adjacent when they are disjoint is the graph on the involutions of $\mathsf{L}_2(8)$, adjacent when the product has order 2 or 7. The graph Γ has maximal cliques of sizes 4, 5 and 7. Those of size 7 are the spreads. It has maximal cocliques of sizes 5 and 9. Those of size 9 are the sets of 9 blocks on a given point. The full group of automorphisms is $\mathsf{P\Gamma L}_2(8)$ seen in its natural action on the fixed exterior line L. Its action on Γ is imprimitive: the 9 spreads determined by the points of L form a system of imprimitivity.

10.24 GQ(3,5) and the hexacode



$v = 64$

Haemers & Spence [384] showed that there are exactly 167 strongly regular graphs with parameters $(v, k, \lambda, \mu) = (64, 18, 2, 6)$. The spectrum is 18^1 2^{45} $(-6)^{18}$. Precisely one of these is rank 3, let us call it Γ. Its full group of automorphisms is $2^6 : 3.\mathsf{S}_6$ with point stabilizer $3.\mathsf{S}_6$. This is the collinearity graph of the unique $\mathsf{GQ}(3, 5)$.

Construction

Take the 64 words of the hexacode, and join two words when their distance is 6. Or take the points of $\mathsf{AG}(3, 4)$, and join two points when the joining line hits the $\mathsf{PG}(2, 4)$ plane at infinity in a fixed hyperoval (cf. §3.4.6).

Cliques, cocliques and chromatic number

The maximal cliques of Γ are the 96 lines of $\mathsf{GQ}(3, 5)$. They have size 4 and meet the Hoffman bound. There are 24 cocliques of size 16, meeting the Hoffman bound. These correspond to the planes in $\mathsf{AG}(3, 4)$ hitting the plane at infinity in an external line of the hyperoval. The chromatic numbers are $\chi(\Gamma) = 4$ and $\chi(\overline{\Gamma}) = 16$.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$2^4 : (\mathsf{A}_5 : \mathsf{S}_3)$	24	16, 48	0	6	16-coclique
b	$2^5 : \mathsf{S}_5$	36	32, 32	6	12	$2\,\overline{K_{16}}$
c	$(2^{4+2} : 3) : 2$	360	32, 32	6	12	

These are all regular sets with $d - e = s$.

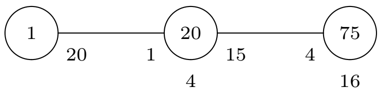
	H	index	orbitlengths	d	e	graph
d	$2^2 : (3 : \mathsf{S}_5)$	96	4, 60	3	1	K_4
e	$2^4 : (3\mathsf{S}_4 : 2)$	60	16, 48	6	4	4×4
f	$[2^8.3]$	180	16, 48	6	4	
g	$\mathsf{S}_4 \times D_8$	720	16, 48	6	4	
h	$\mathsf{A}_5 : D_8$	288	24, 40	8	6	
i	$[2^9.3]$	90	32, 32	10	8	
j	$[2^6.3]$	720	32, 32	10	8	
k	$(D_8 \times D_8) : 2$	1080	32, 32	10	8	
l	$4^2 : 4$	2160	32, 32	10	8	

Case (d) is that of a line of the GQ. Case (e) is that of a plane with a secant at infinity. Case (i) is that of two planes with a common secant at infinity. Every union of t pairwise disjoint lines is a regular set with $(d, e) = (t + 2, t)$.

2-Ranks

Peeters [611] showed that Γ is the unique strongly regular graph with its parameters and satisfying $\mathrm{rk}_2(A) = 14$. Similarly, $\overline{\Gamma}$ is the unique strongly regular graph with its parameters and satisfying $\mathrm{rk}_2(A+I) = 14$.

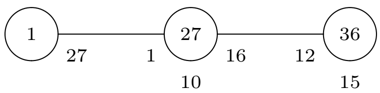
Dual



$v = 96$

The dual generalized quadrangle $\mathsf{GQ}(5,3)$ has 96 points and 64 lines, and the same automorphism group, acting rank 4. The collinearity graph Δ has parameters $(v,k,\lambda,\mu) = (96,20,4,4)$ and spectrum $20^1\ 4^{45}\ (-4)^{50}$. Maximal cliques have size 6 (they are the lines). Maximal cocliques have sizes 10–14 and 16. There are 5 orbits of ovoids (16-cocliques) corresponding to the 5 orbits of spreads in $\mathsf{GQ}(3,5)$. The chromatic numbers are $\chi(\Delta) = 6$ and $\chi(\overline{\Delta}) = 16$.

10.25 $VO_6^-(2)$



$v = 64$

There is a unique rank 3 strongly regular graph with parameters $(v,k,\lambda,\mu) = (64,27,10,12)$. Its spectrum is $27^1\ 3^{36}\ (-5)^{27}$. The full group of automorphisms is $2^6:(\mathsf{O}_6^-(2):2)$ acting rank 3 with point stabilizer $\mathsf{O}_5(3):2$. A construction (as $VO_6^-(2)$) was given in §3.3.1.

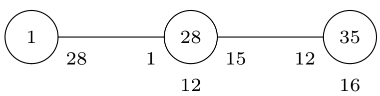
The local graph is the complement of the Schläfli graph, the collinearity graph of $\mathsf{GQ}(2,4)$. The graph induced on the second subconstituent is $NO_6^-(2)$, strongly regular with parameters $(v,k,\lambda,\mu) = (36,15,6,6)$ (see §10.15). The vertices of the first subconstituent *not* adjacent to a fixed vertex of the second subconstituent form a sub-$\mathsf{GQ}(2,2)$ of $\mathsf{GQ}(2,4)$.

Maximal cliques have size 4, and form a single orbit. Maximal cocliques have sizes 4 or 6, a single orbit each. For the chromatic numbers of the graph Γ and its complement, we have $\chi(\Gamma) = 11$, $\chi(\overline{\Gamma}) = 16$.

From the local structure as given in §3.6 it is clear that Γ does not contain $K_5 - e$ (K_5 minus an edge) and $\overline{\Gamma}$ does not contain $K_7 - e$, giving a lower bound $R(K_5 - e, K_7 - e) \geq 65$ for the corresponding Ramsey number. In fact $R(K_5 - e, K_7 - e) = 65$ ([516], [712]).

Ihringer [451] found 8613977 graphs cospectral with Γ by applying GM-switching.

10.26 The halved folded 8-cube and $VO_6^+(2)$



$v = 64$

There is a unique rank 3 strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (64,28,12,12)$. Its spectrum is $28^1\ 4^{28}\ (-4)^{35}$. The full group of automorphisms is $2^6:\mathsf{S}_8$ acting rank 3 with point stabilizer S_8. It can be constructed as the halved folded 8-cube.

The complementary graph $\overline{\Gamma}$ has parameters $(v, k, \lambda, \mu) = (64, 35, 18, 20)$ and spectrum $35^1\ 3^{35}\ (-5)^{28}$. A construction (as $VO_6^+(2)$) was given in §3.3.1.

The local graph is the triangular graph $T(8)$. The graph induced on the 2nd subconstituent is strongly regular with parameters $(v, k, \lambda, \mu) = (35, 16, 6, 8)$, the complement of $\mathsf{O}_6^+(2)$ (see also §10.13).

Maximal cliques have sizes 4 or 8, a single orbit each. Maximal cocliques have size 8, a single orbit. For the chromatic numbers of the graph Γ and its complement, we have $\chi(\Gamma) = \chi(\overline{\Gamma}) = 8$.

Regular sets

Represent the vertices by vectors of even weight in $\mathbb{F}_2^8$, identifying two vectors when they differ by $\mathbf{1}$. For $i = 2, 4$ consider the split with an odd/even weight in the first i bits. This yields examples (l) and (e) below. For $i = 3$, normalize by taking even weight in the first i bits, and split into weight 0/2. This is example (j). The $[8, 4, 4]$ Hamming code modulo $\mathbf{1}$ yields example (a). The set of 8 unit vectors (shifted over an odd weight vector) yields example (i).

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

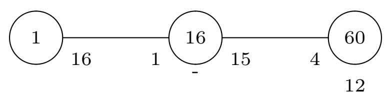
	H	index	orbitlengths	d	e	graph
a	$2^3 : (2^3 : \mathsf{L}_3(2))$	240	8, 56	0	4	$\overline{K_8}$
b	$[2^{11}.3]$	420	16, 48	4	8	
c	$[2^7.3]$	6720	16, 48	4	8	
d	$A_5 : D_8$	5376	24, 40	8	12	
e	$2^5 \times (\mathsf{S}_4 \,\mathrm{wr}\, 2)$	70	32, 32	12	16	
f	$[2^9.3]$	1680	32, 32	12	16	
g	$[2^9]$	5040	32, 32	12	16	
h	$[2^6.3]$	13440	32, 32	12	16	
i	S_8	64	8, 56	7	3	K_8
j	$\mathsf{S}_3 \times (2^4 : \mathsf{S}_5)$	224	16, 48	10	6	Clebsch
k	$2 \times (((\mathsf{A}_4 \times \mathsf{A}_4) : 2) : 2) : 2$	1120	16, 48	10	6	
l	$2 \times (2^5 : \mathsf{S}_6)$	56	32, 32	16	12	
m	$[2^{10}.3]$	840	32, 32	16	12	
n	$[2^9.3]$	1680	32, 32	16	12	
o	$[2^6.3]$	13440	32, 32	16	12	

This is complete for $(d, e) = (7, 3), (10, 6)$, not for $(d, e) = (13, 9), (16, 12)$. Apart from these, there are no further examples with $d - e = r$.

Cospectral graphs

Ihringer [451] found 11063360 graphs cospectral with Γ by applying GM-switching.

10.27 The M_{22} graph on 77 vertices



$v = 77$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (77, 16, 0, 4)$. Its spectrum is $16^1\ 2^{55}\ (-6)^{21}$. The full group of automorphisms is $\mathsf{M}_{22}.2$ acting rank 3 with point stabilizer $2^4 : \mathsf{S}_6$.

The existence of this graph is folklore. An early description (using an explicit list of 77 blocks) was given in Mesner [560], pp. 75–83. Uniqueness is due to Brouwer [111].

Γ is the second subconstituent of the Higman-Sims graph (§10.31).

Construction

Take the 77 blocks of $S(3, 6, 22)$ as vertices, where two blocks are adjacent when they are disjoint.

(Since S(3,6,22) has two block intersection numbers, 0 and 2, this is a special case of the construction of a strongly regular graph from a quasi-symmetric design.)

Cliques and cocliques

Since $\lambda = 0$, the maximal cliques have size 2 and are the edges. The largest cocliques have size 21. There are 22 of those, corresponding to the 22 points of $S(3, 6, 22)$. On the 56 vertices outside a 21-coclique, Γ induces the Gewirtz graph.

Maximal cocliques have sizes 7, 10, 11, 13, 14, 16 or 21. The table below gives the number of cocliques of each given size.

size	7	10	11	13	14	16	21
#	330	216832	149184	43120	330	1309	22

The smallest maximal cocliques have size 7 and stabilizer $2 \times 2^3{:}\mathsf{L}_3(2)$. In the Steiner system $S(5, 8, 24)$, let a and b be two fixed symbols, such that our $S(3, 6, 22)$ is the derived design at $\{a, b\}$. There are 330 octads that contain neither a nor b, and each induces a $7 + 56 + 14$ partition of $V\Gamma$, corresponding to intersection size $0, 2, 4$. The parts of sizes 7 and 14 are maximal cocliques.

Chromatic number

The chromatic number of Γ is 5. That of $\overline{\Gamma}$ is 39.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

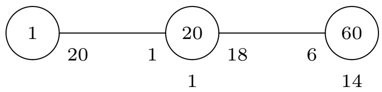
	H	index	orbitlengths	d	e	graph
a	$\mathsf{L}_3(4){:}2$	22	21, 56	0	6	21-coclique
b	A_7	352	35, 42	4	10	$\{4, 3, 3; 1, 1, 2\}$
c	$\mathsf{L}_2(11){:}2$	672	22, 55	6	4	$\{6, 5, 3; 1, 3, 6\}$

In case (b) the induced subgraph on the short orbit is the Odd graph O_4.

In case (c) the induced subgraph on the short orbit is the incidence graph of the unique 2-(11,6,3) design, the complement of the 2-(11,5,2) biplane.

There are no further regular sets with $d - e = s$.

10.28 The Brouwer-Haemers graph



$v = 81$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (81, 20, 1, 6)$. Its spectrum is $20^1\ 2^{60}\ (-7)^{20}$. The full group of automorphisms is $3^4 : ((2 \times \mathsf{S}_6).2)$ acting rank 3 with point stabilizer $(2 \times \mathsf{S}_6).2$.

This graph was known already to Mesner [560]. Brouwer & Haemers [130] showed uniqueness, and gave seven different descriptions of this graph. Uniqueness is also an easy corollary of Ivanov & Shpectorov [458] (see Theorem 3.4.1).

Construction: fourth power difference set

Take the finite field $\mathbb{F}_{81}$, where two elements are adjacent when they differ by a fourth power. (This construction shows the affine group $A\Gamma L(1, 81)$ of order $3^4 \cdot 80 \cdot 4$, acting rank 4.)

Construction: affine orthogonal graph

This is the affine orthogonal graph $VO_4^-(3)$, cf. §3.3.1. (This construction shows the full group: we have $\mathsf{O}_4^-(3) \simeq \mathsf{A}_6$, which has index 2 in $\mathsf{PGO}_4^-(3) \simeq \mathsf{S}_6$, which has index 2 in $\mathsf{GO}_4^-(3) \simeq 2 \times \mathsf{S}_6$, which again has index 2 in the group preserving the form up to a constant.)

A nice symmetric representation is found by taking $\mathbf{1}^\perp/\mathbf{1}$ in $\mathbb{F}_3^6$ provided with the 'sum of squares', i.e., weight, quadratic form, where two vertices are adjacent when their difference has weight 3. And instead of taking $\mathbf{1}^\perp$ (i.e., sum 0), we can also take sum 1, or sum 2.

Equivalently, take the points of $AG(4, 3)$, adjacent when the joining line hits a fixed elliptic quadric in the hyperplane at infinity.

Construction: Hermitian forms graph

This graph is the Hermitian forms graph on $\mathbb{F}_9^2$, cf. §3.4.4.

Construction: from the ternary Golay code

This graph is the coset graph of the truncated ternary Golay code.

Construction: in the $\mathsf{O}_6^-(3)$ graph

This graph is the 2nd subconstituent of the $\mathsf{O}_6^-(3)$ graph on 112 vertices, the collinearity graph of the unique $\mathsf{GQ}(3, 9)$, cf. §10.34.

Cliques, cocliques and chromatic number

Since $\lambda = 1$, maximal cliques have size 3, and there are 270 lines of size 3, ten on each point. The group acts rank 5 on the lines, distinguishing the relations (i) identity, (ii) meeting, (iii) disjoint with three transversals, (iv) disjoint with two transversals, (v) disjoint without transversals, with subdegrees 1, 27, 18, 216, 8. The union (i)+(v) is an equivalence relation, partitioning the lines into 30 sets of size 9 that are concurrent in $\mathsf{GQ}(3,9)$. It follows that Γ has a unique embedding into $\mathsf{GQ}(3,9)$.

Sizes and counts of maximal cocliques:

size	6	9	10	11	12	15
#	324	68445	338580	87480	21060	324

The 15-cocliques form a single orbit, with stabilizer S_6. This stabilizer has three vertex orbits, of sizes $15 + 60 + 6$, where such orbits of size 6 are the maximal 6-cocliques. Such cocliques are most easily seen in the representation as ternary vectors of length 6 with sum 1, modulo **1**. Each vertex has a unique representative of weight 1, 2, or 3, and there are $6 + 15 + 60$ such vectors.

This graph has chromatic number 7 (E. van Dam). Its complement has chromatic number 27 (that is, there are spreads of lines).

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$3^{1+4}.4.2^3$	30	27, 54	2	9	$9K_3$
b	M_{10}	324	36, 45	5	12	{5,4,2; 1,1,4}
c	$3^2.(4\times 2).2^3$	405	9, 72	4	2	$K_3 \times K_3$
d	$2 \times (3^3{:}2^2{:}3).2^2$	90	27, 54	8	6	

In case (a) the short orbit corresponds to the vertices adjacent to a fixed vertex of the first subconstituent in the $\mathsf{O}_6^-(3)$ construction.

In case (b) the induced subgraph on the short orbit is Sylvester's double six graph.

In case (c) the 9 points are the points of a 4×4 grid noncollinear to a fixed point ∞, where Γ is the 2nd subconstituent (w.r.t. ∞) of $\mathsf{GQ}(3,9)$.

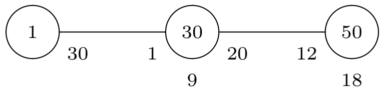
In case (d), in the $AG(4,3)$ construction: the 27 points are those of $AG(3,3)$, adjacent when the joining line hits a fixed conic at infinity.

There are no further regular sets with $d - e = s$.

Second subconstituent

The second subconstituent Δ of Γ has spectrum $14^1\ 2^{40}\ (-4)^{10}\ (-6)^9$. The automorphism group of Δ is $(2^2 \times \mathsf{S}_6).2$, twice as large as the point stabilizer of the automorphism group of Γ. This graph is uniquely determined by its spectrum ([79]).

10.29 $VNO_4^-(3)$ and the Van Lint-Schrijver partial geometry



$v = 81$

There is a unique edge-transitive graph Γ with parameters $(v, k, \lambda, \mu) = (81, 30, 9, 12)$. Its spectrum is $30^1\ 3^{50}\ (-6)^{30}$. The full group of automorphisms is $3^4 : (2 \times \mathsf{S}_6)$ acting rank 4 with point stabilizer $2 \times \mathsf{S}_6$.

We met this graph as $VNO_4^-(3)$. The sporadic part is that it is also the collinearity graph of a partial geometry $pg(6, 6, 2)$, see §8.6.1. The partial geometry has full group $3^4 : \mathsf{S}_6$ ([181]).

Projective two-weight codes

As a special case of the Delsarte correspondence (§7.1.2) we find a 1-1-1 correspondence between subsets X of $\mathsf{PG}(3, 3)$ such that each plane meets it in either 3 or 6 points, and projective $[n, k, d]_q = [15, 4, 9]_3$ codes with weights 9 and 12, and strongly regular graphs with the parameters of Γ defined on $\mathbb{F}_3^4$ by a difference set D of size 30 with $D = -D$.

There are precisely two such graphs, namely Γ, and a graph Δ with group $3^4 : (2 \times (3^2 : 4))$ acting rank 6. There are precisely three $[15, 4, 9]_3$ codes, namely two projective codes (with weight enumerator $1 + 50X^9 + 30X^{12}$) and a single non-projective one (with weight enumerator $1 + 52X^9 + 26X^{12} + 2X^{15}$) ([408], [103], [311]).

Cliques and cocliques

Maximal cliques in Γ have sizes 3 and 6, a single orbit of each. The orbit of 6-cliques (of size 162) splits into two orbits of size 81 under a subgroup of index 2 in $\mathrm{Aut}\,\Gamma$, and vertices together with one such orbit form a $pg(6, 6, 2)$.

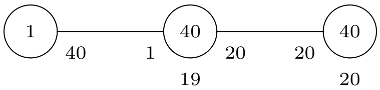
Maximal cocliques in Γ have sizes 7, 9, and 11. Maximal cliques in Δ have sizes 3 and 4. Maximal cocliques in Δ have sizes 6–9.

Cospectral graphs

Krčadinac [499] constructed a different $pg(6, 6, 2)$, with full automorphism group $3^3 : (3^2 : 4)$. Its collinearity graph has the same full group, and has 108 6-cliques. Both $pg(6, 6, 2)$ geometries are self-dual.

Almost simultaneously, Crnković, Švob & Tonchev [245], looking for graphs invariant under a subgroup of the group of the known examples (namely Γ and Δ above), found twelve further graphs cospectral with Γ, one of which is the Krčadinac example. Ihringer [451] found 3770759 examples using WQH-switching (§8.13.2).

10.30 The rank 3 conference graphs on 81 vertices



$v = 81$

There are exactly two rank 3 strongly regular graphs with parameters $(v, k, \lambda, \mu) = (81, 40, 19, 20)$. Their spectrum is $40^1\ 4^{40} (-5)^{40}$. The first, let us call it Γ_1, is the Paley graph, with full group of automorphisms $3^4 : 40 : 4$ and point stabilizer $40 : 4$. The second, let us call it Γ_2, is the Peisert graph, with full group of automorphisms $3^4 : (\mathsf{SL}_2(5) : 2^2)$ and point stabilizer $\mathsf{SL}_2(5) : 2^2$. Both graphs are self-complementary.

The maximal cliques of Γ_1 and Γ_2 have sizes 5 and 9. The orbit sizes are:

Γ_1	9	5	5	5	Γ_2	9	5
#	45	648	3240	6480	#	90	3240

Both Γ_1 and Γ_2 have chromatic number 9, that is, there are partitions into 9-cliques and partitions into 9-cocliques.

Construction

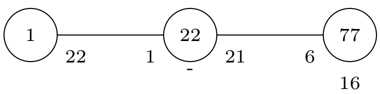
The graph Γ_1 is the Paley graph: the vertex set is $\mathbb{F}_{81}$ and two vertices are adjacent when their difference is a square. The graph Γ_2 is unusual in that it is not determined by a set of directions in $\mathsf{AG}(2, 9)$ (see Theorem 11.4.3). Instead, for both graphs the vertex set can be taken to be $\mathbb{F}_3^4$, where two vertices are adjacent when the line joining them hits the $\mathsf{PG}(3, 3)$ at infinity in a suitable set of size 20 obtained as the union of two disjoint elliptic quadrics.

Any 10-cap in $\mathsf{PG}(3, 3)$ is an elliptic quadric (ovoid), preserved by $\mathsf{PGO}_4^-(3) \simeq \mathsf{A}_6.2^2$. Up to collineation there are three pairs of disjoint elliptic quadrics, and the union of such a pair is a 20-set that meets all planes in either 5 or 8 points. Two of the examples give rise to our graphs Γ_1 and Γ_2. The third example gives a rank 5 graph Γ_3. The three cases can be distinguished by counting common tangents to the two ovoids (0, 20, and 16, respectively), or lines contained in the union (5, 10, and 9, respectively), or by the number of ways to split the 20-set into two ovoids (1, 6, and 2, respectively) or by the group stabilizing the 20-set ($20 : 4$, $2 \times \mathsf{S}_5$, and $4^2 : 2 : 2$, respectively). These three examples (of two disjoint ovoids) occur in two partitions of $\mathsf{PG}(3, 3)$ into four ovoids, of which one is a pencil. See also [302], [152].

Further examples

Hurkens & Seidel [449] construct 26 distinct conference matrices of order 82, which give rise to 175 distinct strongly regular graphs with parameters $(v, k, \lambda, \mu) = (81, 40, 19, 20)$.

10.31 The Higman-Sims graph



$v = 100$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (100, 22, 0, 6)$. Its spectrum is $22^1\ 2^{77}\ (-8)^{22}$. The full group of automorphisms is $\mathsf{HS}.2$ acting rank 3 with point stabilizer $\mathsf{M}_{22}.2$.

This graph was found by HIGMAN & SIMS [425], and uniqueness was proved by GEWIRTZ [341]. Earlier, this graph had been constructed, and uniqueness was shown, by MESNER [559, 560]. (Mesner was interested in the graph, and did not determine the group of automorphisms. Higman and Sims used the graph to construct a new sporadic group.)

Construction: 1 + 22 + 77

Take a symbol ∞, the 22 points, and the 77 blocks of $S(3,6,22)$ as the $1+22+77 = 100$ vertices. Let ∞ be adjacent to the points, let a point be adjacent to the blocks containing it, and let two blocks be adjacent when they are disjoint.

Construction: 50 + 50

The Higman-Sims graph is the graph with as vertices the 100 15-cocliques of the Hoffman-Singleton graph, adjacent when they meet in 0 or 8 points.

Or, equivalently, the Higman-Sims graph is the graph with as vertices the 50 vertices of the Hoffman-Singleton graph, and the 50 15-cocliques in one class, with obvious adjacencies.

Leech lattice construction

Fix the two Leech lattice vectors $v_1 = \frac{1}{\sqrt{8}}(51\,11\ldots1)$ and $v_2 = \frac{1}{\sqrt{8}}(15\,11\ldots1)$. Take the 100 norm 4 vectors with inner product 3 with both, adjacent when their inner product is 1. The $1+22+77$ vertices have the shapes $\frac{1}{\sqrt{8}}(44\,00\ldots0)$, $\frac{1}{\sqrt{8}}(11\,1^{21}(-3))$ and $\frac{1}{\sqrt{8}}(22\,0^{16}2^6)$.

Properties

Since $\lambda = 0$, the maximal cliques have size 2 and are the edges. The largest cocliques have size 22 and are the point neighborhoods. The chromatic number is 6. The chromatic number of $\overline{\Gamma}$ is 50.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$\mathsf{U}_3(5)$	704	50, 50	7	15
b	$2_+^{1+6} : \mathsf{S}_5$	5775	20, 80	6	4
c	$\mathsf{S}_8 \times 2$	1100	30, 70	8	6
d	$(2 \times \mathsf{A}_6.2^2).2$	15400	40, 60	10	8
e	$5^2{:}5{:}(4 \times 2){:}2$	44352	50, 50	12	10

Case (a) corresponds to the split 50+50 above; the induced subgraph on an orbit is the Hoffman-Singleton graph.

In case (b) the subgraph induced on the short orbit is the 2-coclique extension of the Petersen graph.

In case (c) the subgraph induced on the short orbit is the point-plane nonincidence graph of $\mathsf{PG}(3,2)$. The subgraph induced on the long orbit is the graph on the 4-subsets of an 8-set, adjacent when they meet in a single element.

There are no further regular sets with $d - e = s$.

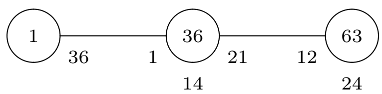
Cayley graph

The group HS.2 has (nonabelian) subgroups $5^2 : 4$ and $5 \times (5:4)$ of order 100 that act regularly on the vertices of Γ. Thus, Γ is a Cayley graph.

Spin model

In *knot theory*, one studies knots embedded in $\mathbb{R}^3$, with projections in $\mathbb{R}^2$ provided with over/under indications. Two knots are equivalent if and only if the projections can be connected by a series of *Reidemeister moves.* In order to distinguish inequivalent knots, one uses objects that are invariant under Reidemeister moves, such as the *Kauffman polynomial* ([486]). A new invariant using the formalism of statistical mechanics was defined by Jones [468]. Jaeger [462] translated the requirements of these 'spin models' into association scheme terms, and discovered that a new knot invariant can be defined using the Higman-Sims graph.

10.32 The Hall-Janko graph



$v = 100$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (100, 36, 14, 12)$. Its spectrum is $36^1\ 6^{36}\ (-4)^{63}$. The full group of automorphisms is HJ.2 acting rank 3 with point stabilizer $\mathsf{G}_2(2) = \mathsf{U}_3(3).2$. The existence of the group and the rank 3 permutation representation was established by Hall & Wales [401].

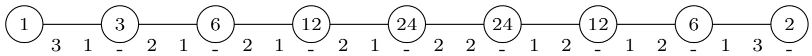
This graph is not determined by its parameters alone: the Latin square graphs $\mathrm{LS}_4(10)$ (constructed from a pair of orthogonal Latin squares of order 10) have the same parameters, but cannot be isomorphic. This graph is the unique connected graph that is locally the $\mathsf{G}_2(2)$ graph on 36 vertices (Pasechnik [601]).

Construction: 1 + 36 + 63

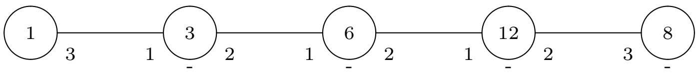
In the projective plane $\mathsf{PG}(2,9)$ provided with a nondegenerate Hermitian form, one has a unital with 28 points, and 63 nonisotropic points. The plane has $63 \cdot 6 \cdot 1/6 = 63$ orthogonal bases, and the 63 points and 63 bases are the points and lines of the dual of the classical $\mathsf{GH}(2,2)$. Any apartment (hexagon) in this $\mathsf{GH}(2,2)$ determines a unique sub-$\mathsf{GH}(2,1)$ (with 14 lines and 21 points) and we find 36 $\mathsf{GH}(2,1)$'s in this way. These either coincide, or meet in a line and the lines meeting it (4 lines and 9 points in common), or meet in two intersecting lines (2 lines and 5 points in common), and these intersections occur with frequencies 1, 14, 21.

The graph Γ is obtained by taking a symbol ∞, the 36 $\mathsf{GH}(2,1)$'s and the 63 points of the $\mathsf{GH}(2,2)$ as vertices, where ∞ is adjacent to the $\mathsf{GH}(2,1)$'s, two $\mathsf{GH}(2,1)$'s are adjacent when they have 4 lines in common, a $\mathsf{GH}(2,1)$ is adjacent to a point when it contains that point, and two points are adjacent when they have distance 2 in the $\mathsf{GH}(2,2)$ (i.e., when they are not orthogonal and the joining line is not a tangent).

Construction: 10 + 90



$v = 90$



$v = 30$

Construct the graph Γ using two ingredients: the Foster graph F on 90 vertices, and the Moebius plane $S(3,4,10)$. The Foster graph is the unique distance-regular graph with intersection array $\{3,2,2,2,2,1,1,1;1,1,1,1,2,2,2,3\}$, has group $3.\mathsf{A}_6.2^2$, and is an antipodal 3-cover of the unique distance-regular graph with intersection array $\{3,2,2,2;1,1,1,3\}$ on 30 vertices, the incidence graph of $\mathsf{GQ}(2,2)$, and also the graph on the 30 circles (blocks) of $S(3,4,10)$, adjacent when disjoint. Let π be the folding map.

The 100 vertices of Γ are the 90 vertices of F and the 10 points of the point set X of $S(3,4,10)$. The set X is a 10-coclique in Γ, two vertices of F are adjacent in Γ when they have distance 3, 6, 7 or 8 in F, and the point x in X is adjacent to y in F when x is in the block $\pi(y)$.

Construction: Cohen-Tits near octagon

The group $\mathsf{HJ}.2$ is the full automorphism group of the Cohen-Tits near octagon Δ of order (2, 4), see §10.68. Moreover, Δ contains subgeometries isomorphic to the dual of the split Cayley hexagon $\mathsf{G}_2(2)$. The vertices of the graph Γ are the dual split Cayley hexagons of order (2, 2) contained in Δ as a subgeometry, adjacent when they intersect in a subhexagon of order (2, 1) (and not adjacent when they intersect in the seven points equal or collinear to a given point). See [289].

Cliques and cocliques

All maximal cliques in Γ have size 4, since the local graphs have maximal cliques of size 3. The chromatic number of $\overline{\Gamma}$ is 25. Maximum cocliques in Γ have size 10, reaching the Hoffman bound. The chromatic number of Γ is 10.

Maximal cocliques have sizes 4, 6, 7, 10 and fall into five orbits (there are two orbits of maximal 7-cocliques). The group is transitive on 2-cocliques (nonedges) but has two orbits on 3-cocliques. Below we give for each coclique C how many triples from C belong to these two orbits (called A and B).

size	4	6	7	7	10
#	1575	100800	25200	3600	280
A	0	18	32	28	120
B	4	2	3	7	0

The maximal cocliques of size 4 are the $100 \cdot 63/4 = 1575$ sets consisting of a vertex and a line in the $\mathsf{GH}(2,2)$ far from that vertex. The maximal cocliques of size 7 in the 2nd orbit are the $2 \cdot 100 \cdot 36/2$ halves of the Heawood graph on the common neighbors of two adjacent vertices.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$3.\mathsf{A}_6.2^2$	280	10, 90	0	4
b	$(\mathsf{A}_4 \times \mathsf{A}_5):2$	840	40, 60	12	16
c	$5^2:\mathsf{D}_{12}$	4032	50, 50	16	20
d	$2_-^{1+4}.\mathsf{S}_5$	315	20, 80	12	6

Case (a) corresponds to a coclique of size 10. The union of t such disjoint cocliques (or the complement of the union of $(10-t)$ disjoint such cocliques) is again a regular set of size $10t$ with (degree, nexus)$= (4(t-1), 4t)$. Since the vertex set can be partitioned into ten cocliques of size 10, this occurs for all t with $1 \le t \le 9$.

HJ on 280 points

Γ has 280 10-cocliques, called *decads*, on which HJ acts as a rank 4 group with valencies (subdegrees) $n_0 = 1$, $n_1 = 36$, $n_2 = 108$, $n_3 = 135$. Decads in relations R_1 or R_2 are disjoint. Decads in relation R_3 meet in 2 points. The union of two decads in relation R_1 induces in Γ the extended bipartite double of the Petersen graph, of diameter 3. The union of two decads in relation R_2 induces in Γ a graph of valency 4 and diameter 4.

The intersection matrices of the association scheme are

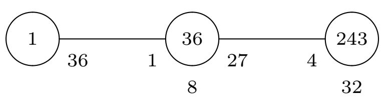
$$(p^i_{0j}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (p^i_{1j}) = \begin{pmatrix} 0 & 36 & 0 & 0 \\ 1 & 8 & 12 & 15 \\ 0 & 4 & 12 & 20 \\ 0 & 4 & 16 & 16 \end{pmatrix},$$

$$(p^i_{2j}) = \begin{pmatrix} 0 & 0 & 108 & 0 \\ 0 & 12 & 36 & 60 \\ 1 & 12 & 40 & 55 \\ 0 & 16 & 44 & 48 \end{pmatrix}, \quad (p^i_{3j}) = \begin{pmatrix} 0 & 0 & 0 & 135 \\ 0 & 15 & 60 & 60 \\ 0 & 20 & 55 & 60 \\ 1 & 16 & 48 & 70 \end{pmatrix}$$

and the eigenmatrices are

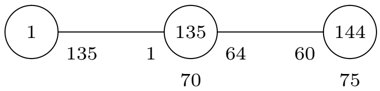
$$P = \begin{pmatrix} 1 & 36 & 108 & 135 \\ 1 & -4 & -12 & 15 \\ 1 & 8 & -4 & -5 \\ 1 & -4 & 8 & -5 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & 63 & 90 & 126 \\ 1 & -7 & 20 & -14 \\ 1 & -7 & -\frac{10}{3} & \frac{28}{3} \\ 1 & 7 & -\frac{10}{3} & -\frac{14}{3} \end{pmatrix}.$$

Let D be the set of decads. Then (D, R_1) is strongly regular with parameters $(v,k,\lambda,\mu) = (280, 36, 8, 4)$ and spectrum $36^1\ 8^{90}\ (-4)^{189}$.



$v = 280$

And (D, R_3) is strongly regular with parameters $(v,k,\lambda,\mu) = (280, 135, 70, 60)$ and spectrum $135^1\ 15^{63}\ (-5)^{216}$.



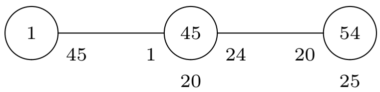
$v = 280$

Both graphs have full group $\mathsf{HJ}.2$, acting rank 4, with point stabilizer $3.\mathsf{A}_6.2^2$. The former graph satisfies the 4-vertex condition. Its μ-graphs are 4-cycles. Each edge is contained in a unique K_4. The latter graph belongs to a regular two-graph. It has a descendant with parameters $(v, k, \lambda, \mu) = (279, 150, 85, 75)$. See also [29], [457].

Partitions into decads

$\mathsf{V}\Gamma$ has $1008 + 12096$ partitions into 10 decads, falling into two orbits. Let us call those in the orbit of size 1008 *nice*, the others *ugly*. The stabilizer of a nice partition is $(\mathsf{A}_5 \times \mathsf{D}_{10}).2$, transitive on the 100 vertices, with orbit sizes $10 + 120 + 150$ on the 280 decads. The stabilizer of an ugly partition is $5^2 : 4$, transitive on the 100 vertices, with orbit sizes $10 + 10 + 10 + 50 + 50 + 50 + 100$ on the 280 decads. It stabilizes three ugly partitions, and the union of such a triple has stabilizer $5^2 : (4 \times \mathsf{S}_3)$, transitive on the 100 vertices, with orbit sizes $30 + 100 + 150$ on the 280 decads. There are 2016 such triples, forming a single orbit.

Fix a partition Π of $\mathsf{V}\Gamma$ into ten decads, and construct a new graph Δ by turning the elements of Π into cliques. Then Δ is strongly regular with parameters $(v, k, \lambda, \mu) = (100, 45, 20, 20)$ and spectrum $45^1\ 5^{45}\ (-5)^{54}$.



$v = 100$

The two choices for Π yield nonisomorphic graphs. See also [29] and [471].

The adjacency matrix for these graphs is the point-block incidence matrix for a square 2-(100,45,20) design.

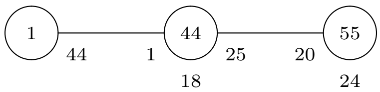
Cayley graph

We saw that $\mathsf{HJ}.2$ has a (nonabelian) subgroup $5^2 : 4$ of order 100 that acts regularly on the vertices of Γ. Thus, Γ is a Cayley graph.

Splits

$\mathsf{V}\Gamma$ has splits into two halves, where each half is in three different ways the union of five decads. There are 2016 of these splits, forming a single orbit. The stabilizer of one is $5^2 : (4 \times \mathsf{S}_3)$, transitive on the 100 vertices.

The Jørgensen-Klin graph

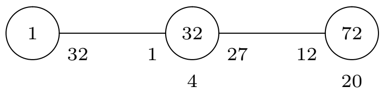


$v = 100$

Jørgensen & Klin [471] constructed a strongly regular graph with parameters $(v, k, \lambda, \mu) = (100, 44, 18, 20)$ and spectrum $44^1\ 4^{55}\ (-6)^{44}$.

Graphs with these parameters can be constructed as follows. Start with a 50 + 50 split $\{S, T\}$ of $V\Gamma$, and refine it to a partition Π of $V\Gamma$ into ten decads. Construct a strongly regular graph Δ with parameters $(100, 45, 20, 20)$ as above, by turning the elements of Π into cliques. Next, switch with respect to S, which induces a regular subgraph of degree 25 in Δ. The result is a strongly regular graph with parameters $(100, 55, 30, 30)$. The complementary graph has parameters $(100, 44, 18, 20)$.

10.33 The 105 flags of PG(2,4)



$v = 105$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (105, 32, 4, 12)$. Its spectrum is $32^1\ 2^{84}\ (-10)^{20}$. Construction is due to Goethals & Seidel [355], uniqueness to Coolsaet [221]. The full group is $\mathrm{Aut}\,\mathsf{L}_3(4)$ acting rank 4 with orbit sizes $1 + 32 + (8 + 64)$.

Construction

Take the 105 point-line flags of $\mathsf{PG}(2, 4)$, and let $(p, L) \sim (q, M)$ when $p \neq q$, $L \neq M$ and (p on M or q on L). This is the distance-2 graph of the unique $\mathsf{GH}(4, 1)$.

This graph is the second subconstituent of the second subconstituent of the McLaughlin graph, see §10.48.

Cliques and cocliques

The graph is locally bipartite, so maximal cliques have size 3. The chromatic number of $\overline{\Gamma}$ is 35. Maximal cocliques have sizes 5, 8, 9, 11, 14, 20. The chromatic number of Γ is 6.

There is a unique orbit (of size 42) of cocliques of size 20. An example is the collection of flags (q, M) with q on a fixed line L, and $M \neq L$. There is a unique orbit (of size 42) of maximal cocliques of size 5. An example is the collection of flags (p, L) with p on a fixed line L.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

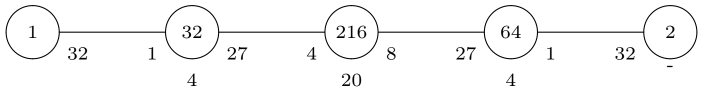
	H	index	orbitlengths	d	e
a	$\mathsf{A}_6.2^2$	168	45, 60	8	18
b	$7{:}6 \times \mathsf{S}_3$	960	42, 63	14	12

Case (a) is the set of flags of $\mathsf{PG}(2, 4)$ whose point does not belong to a fixed hyperoval and whose line intersects the same hyperoval in exactly two points. The induced graph is the distance 2 graph of the unique $\mathsf{GO}(2, 1)$.

For case (b), consider a Singer cycle g (an automorphism of $\mathsf{PG}(2, 4)$ of order 21, acting cyclically on the points and lines). Then g^3 has three orbits, partitioning the point set of $\mathsf{PG}(2,4)$ into three Fano planes. Each line L hits one of these Fano planes, say π_L, in 3 points (and the other two in a single point). The flag (P,L) belongs to the orbit of size 63 when P lies in $L \cap \pi_L$.

There are no further examples of regular sets with $d-e=s$.

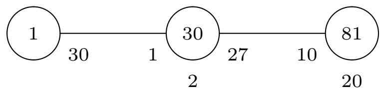
Triple cover



$v = 315$

There is a unique distance-regular (but not distance-transitive) graph with intersection array $\{32,27,8,1;1,4,27,32\}$, an antipodal 3-cover of Γ. It was constructed in Soicher [664], and uniqueness is due to Soicher [665].

10.34 The $\mathsf{O}_6^-(3)$ graph on 112 vertices



$v = 112$

There is a unique strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (112,30,2,10)$. Its spectrum is $30^1\ 2^{90}\ (-10)^{21}$. The full group of automorphisms is $\mathsf{U}_4(3).\mathsf{D}_8$ (of order $2^{10}\cdot 3^6\cdot 5\cdot 7$) acting rank 3 with point stabilizer $3^4:((2\times \mathsf{A}_6).2^2)$.

Construction

Let $V = \mathbb{F}_3^6$, provided with a nondegenerate quadratic form of non-maximal Witt index. The graph Γ is the graph on the points of the corresponding elliptic quadric in $\mathsf{P}V$, adjacent when collinear, i.e., when orthogonal. This graph is the collinearity graph of a generalized quadrangle $\mathsf{GQ}(3,9)$. The group is the group $\mathsf{PGO}_6^-(3).2$ of linear transformations of $\mathsf{P}V$ that preserve the elliptic quadric.

Uniqueness

Cameron, Goethals & Seidel [178] showed that any strongly regular graph with the parameters of Γ must be the collinearity graph of a $\mathsf{GQ}(3,9)$. Dixmier & Zara [293, 294] showed the uniqueness of the generalized quadrangle with parameters $\mathsf{GQ}(3,9)$.

Hemisystems, Gewirtz subgraphs and splits

A *hemisystem of points* in $\mathsf{GQ}(3,9)$ is a subset of the point set that meets every line in half of its points, i.e., in 2 points. Segre [640] found that there are 648 hemisystems, 324 complementary pairs, forming a single orbit. The hemisystems are precisely the Gewirtz subgraphs.

A fixed hemisystem meets any hemisystem in 0, 16, 20, 24, 28, 32, 36, 40 or 56 points (with frequencies 1, 42, 56, 105, 240, 105, 56, 42, 1, respectively). Meeting in 20, 32 or 56 points is an equivalence relation with four equivalence classes ($\mathsf{O}_6^-(3)$ orbits).

The graph Γ is the first subconstituent of the McLaughlin graph Λ (§10.61). The full automorphism group of Γ is four times as large as the vertex stabilizer in Λ because only hemisystems of a single equivalence class occur as μ-graphs in Λ.

Cocliques

Maximal cocliques have sizes 7, 10, 11, 12, 13, 16. We give the counts. In case there is just a single orbit of m-cocliques, we give the stabilizer S and the orbits of the stabilizer on that m-coclique.

size	7	10	11	12	13	16
#	5184	766584	3447360	816480	181440	2268
S	S_7			$2^2 \times \mathsf{D}_8$	$\mathsf{S}_3 \times \mathsf{S}_4$	$2^4 : \mathsf{S}_6$
orbits	tra			4^3	$1+12$	tra

The maximal 7-cocliques form a single orbit with stabilizer S_7. They can be seen by viewing the orthogonal geometry as elliptic hyperplane in the $\mathsf{O}_7(3)$ geometry. That latter geometry can be described using the form $\sum_{i=1}^{7} X_i^2$, and the point $\mathbf{1}$ is elliptic. In $\mathbf{1}^\perp$ we see $112 = 7 + 35 + 70$ points (7: 1111110; 35: 1110000; 70: 1112220), where the 7-set is a maximal coclique and the 35-set induces the Odd graph O_4, the unique distance-regular graph with intersection array $\{4, 3, 3; 1, 1, 2\}$.

The maximal 16-cocliques form a single orbit with stabilizer $2^4 : \mathsf{S}_6$. They can be seen by choosing the quadratic form to be $Q(x) = \sum_{i=1}^{6} X_i^2$. The set of 32 isotropic points without zero coordinate induces the unique distance-regular graph with intersection array $\{10, 9, 4; 1, 6, 10\}$, the distance-3 graph of the folded 6-cube. This graph is bipartite and the two parts of its bipartition are 16-cocliques.

Cliques and chromatic number

The maximal cliques are the lines of $\mathsf{GQ}(3, 9)$ and have size 4. The chromatic number of Γ is 8. That of $\overline{\Gamma}$ is 28. (That is, $\mathsf{GQ}(3, 9)$ has spreads.)

Regular sets

Easy examples of regular sets are arbitrary unions of pairwise disjoint lines of $\mathsf{GQ}(3, 9)$ (and since there exist spreads this yields regular sets of size $4t$ with (degree, nexus)= $(t + 2, t)$ for $0 < t < 28$). Further (transitive) examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$\mathsf{L}_3(4)$:2	648	56, 56	10	20
b	$3_+^{1+4}.2_-^{1+4}.\mathsf{D}_{12}$	280	4, 108	3	1
c	$4(\mathsf{S}_4 \times \mathsf{S}_4).2^2$	2835	16, 96	6	4
d	$7 : (3 \times \mathsf{D}_8)$	155520	28, 84	9	7
e	$2^5.\mathsf{S}_6$	1134	32, 80	10	8
f	$2 \times \mathsf{U}_4(2)$:2	252	40, 72	12	10
g	$4^3(2 \times \mathsf{S}_4)$	8505	48, 64	14	12
h	$2 \times \mathsf{L}_3(2)$:2	38880	56, 56	16	14

Case (a) corresponds to a hemisystem of points of $\mathsf{GQ}(3, 9)$.

Case (b) corresponds to a single line of GQ$(3,9)$.

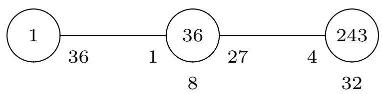
Case (c) corresponds to a 4×4 grid, also the union of four disjoint lines.

Case (f) corresponds to a sub-GQ$(3,3)$ of GQ$(3,9)$ (which does not correspond to a union of disjoint lines since this subquadrangle does not admit spreads).

Case (g): a Hermitian spread of GQ$(3,9)$ can be structured as a linear space by defining blocks as the reguli of 4×4 grid. This linear space is then isomorphic to the unital consisting of the isotropic points of the $\mathsf{U}_3(3)$ geometry, where blocks are the intersections with secant lines in the corresponding projective plane PG$(2,9)$. A Hermitian base in PG$(3,9)$ defines three secants which contain in total twelve points of the unital. These correspond to twelve disjoint lines of GQ$(3,9)$ (as part of the Hermitian spread). Their union gives the 48 points of the smallest orbit of case (g).

There are no further examples of regular sets with $d - e = s$.

Dual generalized quadrangle



$v = 280$

The collinearity graph of the dual generalized quadrangle GQ$(9,3)$ is the unique rank 3 strongly regular with parameters $(v,k,\lambda,\mu) = (280,36,8,4)$. Its spectrum is $36^1\ 8^{90}\ (-4)^{189}$. The full group of automorphisms is $\mathsf{U}_4(3).\mathsf{D}_8$ acting rank 3 with point stabilizer $3_+^{1+4}.2_-^{1+4}.\mathsf{D}_{12}$. This graph Δ is not uniquely determined by its parameters alone, we saw a graph with the same parameters and full group HJ.2, acting rank 4.

The maximal cliques have size 10 and form a single orbit, they are the lines of the generalized quadrangle. The largest cocliques have size 28 and are the ovoids. In Corollary 2.7.4 we saw that $\chi(\Delta) = 10$.

Examples of regular sets in Δ are obtained from subgroups H of $G = \mathrm{Aut}\,\Delta$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$\mathsf{U}_3(3):\mathsf{D}_8$	540	28, 252	0	4
b	$3^4:(2\times\mathsf{A}_6).2^2$	112	10, 270	9	1
c	$2\times O_5(3).2$	126	40, 240	12	4
d	S_7	1296	70, 210	15	7
e	$2^4:\mathsf{A}_6:2^2$	567	120, 160	20	12

Case (a) corresponds to a Hermitian ovoid of GQ$(9,3)$ (the Hermitian spread of GQ$(3,9)$ mentioned in case (g) for Γ).

Case (b) corresponds to a single line of GQ$(9,3)$.

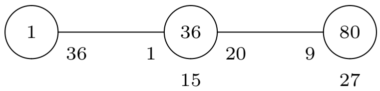
Case (c) corresponds to a subquadrangle GQ$(3,3)$ of GQ$(9,3)$.

Case (d) corresponds to a maximal coclique of size 7 in Γ and hence to the union of seven pairwise disjoint lines of GQ$(9,3)$.

Case (e) corresponds to a maximal coclique of size 16 in Γ and hence to the (complement of the) union of sixteen pairwise disjoint lines of GQ$(9,3)$.

In general, the (complement of the) union of pairwise disjoint lines is always a regular set of size $10t$, for some $t \in \{1,2,\ldots,27\}$, and (degree, nexus)$=(t+8,t)$.

10.35 $NO_6^+(3)$



$v = 117$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (117, 36, 15, 9)$. Its spectrum is $36^1\ 9^{26}\ (-3)^{90}$. The full group of automorphisms is $\mathsf{PGO}_6^+(3) = \mathsf{L}_4(3):2$ with point stabilizer $2 \times \mathsf{O}_5(3):2$.

Construction

This is the graph on one orbit of nonisotropic points in the $\mathsf{O}_6^+(3)$ geometry, adjacent when orthogonal, i.e., when joined by an elliptic line, cf. §3.1.3.

This is also the graph on the antiflags of $\mathsf{PG}(2,3)$, two antiflags (x, L) and (y, M) adjacent if either $x \in M$ and $y \in L$, or $\{x, y\} \cap (L \cup M) = \emptyset$, $L \neq M$, and $L \cap M \notin xy$.

Local graph

The local graph is $NO_6^-(2)$, strongly regular with parameters $(v, k, \lambda, \mu) = (36, 15, 6, 6)$, see §10.15. This is the graph on the orbit of nonisotropic points in the $\mathsf{O}_5(3)$ geometry that have perps that are elliptic hyperplanes, adjacent when orthogonal, cf. §3.1.4. Its full automorphism group is $\mathsf{O}_5(3):2$, acting rank 3 with point stabilizer $2 \times \mathsf{S}_6$. The graph Γ is uniquely determined by its local graph (Hall & Shult [395], Theorem 3).

Cliques, cocliques and chromatic number

The maximal cliques in Γ have size 5 and form a single orbit. They have stabilizer $2 \times (2^4:\mathsf{S}_5)$. For the quadratic form $q(x) = x_1x_2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$, a 5-clique is given by $\{e_1 + e_2, e_3, e_4, e_5, e_6\}$.

Maximal cocliques have sizes 5, 6, 7 and 9. There are two orbits of 9-cocliques, reaching the Hoffman bound. One type is that of the sets $C(L)$ of vertices contained in $L^\perp$, where L is a totally isotropic line. See §3.1.3.

Γ has chromatic number 13. A partition of the vertex set into 13 sets $C(L)$ is obtained by taking the 13 lines L in a totally isotropic plane.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$3^{1+4}:(2\mathsf{S}_4 \times 2)$	520	9, 108	0	3
b	$3^4:2(\mathsf{A}_4 \times \mathsf{A}_4).2^2$	130	36, 81	9	12
c	$2 \times (O_5(3):2)$	117	45, 72	12	15
d	$\mathsf{A}_6.2^2$	8424	45, 72	12	15

Case (a) is that of the 9-cocliques of type $C(L)$ where L is a totally isotropic line.

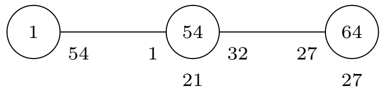
In case (b) the partition is induced by an isotropic point z. (For 81 vertices x the line xz is hyperbolic, for 36 it is a tangent.)

In case (c) the partition is induced by a nonisotropic point of the other kind.

In case (d) the partition can be obtained by viewing $V = \mathbb{F}_3^6$ as $\mathbb{F}_9^3$ and picking the quadratic form $\operatorname{tr} q(x)$ on V, where $q(x)$ is a nondegenerate quadratic form on $\mathbb{F}_9^3$ that takes a nonsquare value for the 36 interior points of the corresponding conic in $\mathsf{PG}(2,9)$. The 117 points with $\operatorname{tr} q(x) = 1$ split into 72 with $q(x)$ a square and 45 with $q(x)$ a nonsquare.

There are no regular sets with $d - e = r$.

10.36 The $\mathsf{O}_8^-(2)$ graph on 119 vertices



$v = 119$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (119, 54, 21, 27)$. Its spectrum is $54^1\ 3^{84}\ (-9)^{34}$. The full group of automorphisms is $\mathsf{O}_8^-(2):2$ acting rank 3 with point stabilizer $2^6:\mathsf{O}_6^-(2):2$.

Cliques and cocliques

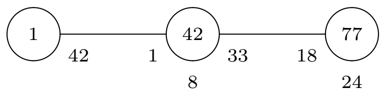
Maximal cliques have size 7 and form a single orbit. They are the totally isotropic subspaces. Maximal cocliques have sizes 5 and 7, a single orbit each.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \operatorname{Aut}\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$\mathsf{L}_2(16):4$	24192	51, 68	18	27
b	$[2^9]:(\mathsf{S}_3 \times \mathsf{L}_3(2))$	765	7, 112	6	3
c	$\mathsf{S}_8 \times \mathsf{S}_3$	1632	35, 84	18	15
d	$2 \times \mathsf{O}_7(2)$	136	56, 63	27	24

10.37 The $\mathsf{L}_3(4).2^2$ graph on 120 vertices



$v = 120$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (120, 42, 8, 18)$. Its spectrum is $42^1\ 2^{99}\ (-12)^{20}$. The full group of automorphisms is $\mathsf{L}(3,4):2^2$ acting rank 4, with point stabilizer $2 \times (L(3,2):2)$. Existence is due to GOETHALS & SEIDEL [354]. Uniqueness is due to DEGRAER & COOLSAET [274].

Maximum cliques have size 3. Maximum cocliques have size 16. The graph and its complement have chromatic numbers $\chi(\Gamma) = 8$ and $\chi(\overline{\Gamma}) = 40$.

Construction

Take the 120 heptads in $S(4,7,23)$ that miss two given symbols, adjacent when they meet in a single point. (See §8.5.4D.) This graph is an induced subgraph of the M_{22} graph on 176 vertices (§10.51).

Equivalently, look at the Fano subplanes of $\mathsf{PG}(2,4)$. The number of common points of two Fano subplanes (0, 1, 2, 3, 4 or 7) equals the number of common lines. Having an odd number of points in common is an equivalence relation with three classes of size 120. Our graph is the graph on the Fano subplanes in one class, adjacent when they have a single point (or, equivalently, a single line) in common. We see that the group of the graph is twice that what is inherited from M_{22} (namely $\mathsf{P\Sigma L}_3(4)$), since also a polarity of $\mathsf{PG}(2,4)$ acts.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$2^4:\mathsf{S}_5$	42	40, 80	6	18
b	A_6	224	60, 60	15	27
c	$\mathsf{A}_6.2^2$	56	30, 90	12	10

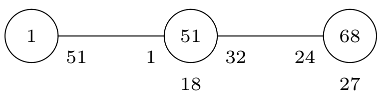
Case (a) is the split determined by one of the 21 symbols (points of $\mathsf{PG}(2,4)$) or one of the 21 lines of $\mathsf{PG}(2,4)$. The graph induced on the 40-orbit is the complement of a 16-coclique in the Gewirtz graph.

Case (b) is the split determined by the intersection size (1 or 3) with one of the 112 heptads that contains precisely one of the two given symbols. This graph is subgraph of the M_{22} graph on 176 vertices in two different ways, and in each such embedding the 56 exterior vertices determine such a split.

There are no further regular sets with $d - e = s$.

In case (c) the 30 vertices induce the bipartite nonincidence graph of points and lines of $\mathsf{GQ}(2,2)$.

10.38 $NO_5^-(4)$



$v = 120$

There is a unique rank 3 strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (120,51,18,24)$. Its spectrum is $51^1\ 3^{85}\ (-9)^{34}$. The full group of automorphisms is $\mathsf{O}_5(4):2$ acting rank 3, with point stabilizer $\mathsf{L}_2(16):4$.

This is $NO_5^-(4)$, cf. §3.1.4. Maximal cliques have size 4 (2 orbits). Maximal cocliques have sizes 6 (3 orbits), 7 or 8 (1 orbit each).

Construction in $PG(2,16)$

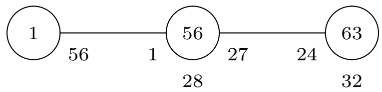
The group $\mathsf{P\Sigma L}_2(16)$ acts on the 120 exterior lines of a hyperoval in $PG(2,16)$, giving a 3-class association scheme with valencies 1, 17, 34, 68. Merging relations R_1 and R_2 yields the graph Γ (which has a much larger group).

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$2^6:(3\times \mathsf{S}_5)$	85	24, 96	3	12
b	$2^6:[2^2.3^2]$	850	48, 72	15	24
c	$\mathsf{A}_5\times \mathsf{S}_5$	272	60, 60	21	30
d	$2\times \mathsf{A}_5$	16320	60, 60	21	30
e	$5^2:((4\times 2):2)$	4896	20, 100	11	8
f	$2\times(((2^4:5):4):2)$	1530	40, 80	19	16
g	$(\mathsf{A}_5\times \mathsf{A}_5):2$	272	60, 60	27	24
h	S_6	2720	60, 60	27	24
i	$2^2\times \mathsf{A}_5$	8160	60, 60	27	24
j	$(5:4)\times \mathsf{S}_3$	16320	60, 60	27	24

10.39 $\overline{NO_8^+(2)}$



$v = 120$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (120, 56, 28, 24)$. Its spectrum is $56^1\ 8^{35}\ (-4)^{84}$. The full group of automorphisms is $\mathsf{O}_8^+(2):2$ acting rank 3, with point stabilizer $\mathsf{Sp}(6, 2)\times 2$.

Construction: nonisotropic points in the $\mathsf{O}_8^+(2)$ geometry

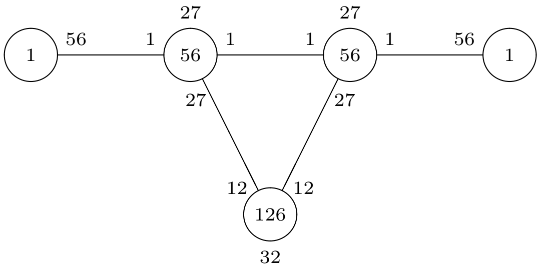
The $\mathsf{O}_8^+(2)$ geometry has 255 projective points, 135 isotropic, 120 nonisotropic. Γ is the graph on the nonisotropic points, adjacent when not orthogonal, that is, when the connecting line is an elliptic line. (Cf. §3.1.2.)

Construction: split Cayley hexagons on $\mathsf{O}_7(2)$

The graph Γ is the graph on the 120 standard representations of the split Cayley hexagon $\mathsf{G}_2(2)$ on $\mathsf{O}_7(2)$, adjacent when having exactly nine lines in common (the nine lines of a Hermitian spread in both). (Cf. §4.8.) This provides a rank 3 representation of Γ with automorphism group $\mathsf{Sp}_6(2)$ and point stabilizer $\mathsf{G}_2(2) \cong \mathsf{U}_3(3) : 2$.

Construction: the E_8 root system

Let Φ be the root system of type E_8. It has 240 vectors, and spans 120 lines in $\mathbb{R}^8$. The graph Γ is the graph on these 120 lines, where lines are adjacent when not orthogonal. The root system graph of E_8 (with vertex set Φ, where two roots are adjacent when their angle is $\frac{\pi}{3}$) is a double cover of Γ.



Construction: from the local graph

The local graph of Γ is the Gosset graph (see §10.10) with 28 extra edges joining vertices at original distance 3. The graph induced on the vertices at distance 2 from a fixed vertex in Γ is a quotient of the $\mathsf{E}_{7,1}(1)$ graph. This yields the following combinatorial description of Γ.

Label a vertex ∞ and consider three copies of an 8-set, say $W = \{1, 2, \ldots, 8\}$, $W' = \{1', 2', \ldots, 8'\}$ and $W'' = \{1'', 2'', \ldots, 8''\}$. Then the other 119 vertices are the unordered pairs from these three sets together with the 4|4 splits of W''. Two pairs from the same set are adjacent if they share exactly one element. Two 4|4 splits are adjacent if the individual subsets intersect in an odd number of elements. Let $a, b, c, d \in \{1, 2, \ldots, 8\}$. Then the pairs $\{a, b\}$ and $\{c', d'\}$ are adjacent if $|\{a, b\} \cap \{c, d\}| \in \{0, 2\}$, while $\{a, b\}$ or $\{a', b'\}$ are adjacent to $\{c'', d''\}$ if $|\{a, b\} \cap \{c, d\}| = 1$. Further, $\{a, b\}$ or $\{a', b'\}$ are adjacent to a 4|4 splitting if a and b are contained in the same subset of the splitting, while $\{c'', d''\}$ is adjacent to a 4|4 split if c and d are in distinct subsets of the splitting. Finally, ∞ is adjacent to all pairs of W and W'.

In this construction, $W \cup W''$ is a regular set of size 56, with degree 24 and nexus 28, see case (h) below under Regular sets.

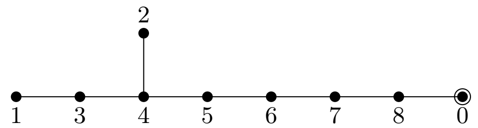
The above construction is a direct consequence of the following construction of the root system graph of E_8.

The Gosset graph (see §10.10) contains 126 $K_{6\times2}$ subgraphs. Using the construction in §10.10, 56 of these subgraphs are given by the pairs containing a fixed element $a \in \{1, 2, \ldots, 8\}$, but not a fixed element $b \in \{1, 2, \ldots, 8\} \setminus \{a\}$, or containing the element b' and not a', and the 70 others are given by the pairs of a 4-set $\{a, b, c, d\} \subseteq \{1, 2, \ldots, 8\}$ together with the pairs of the 4-set $\{1', 2', \ldots, 8'\} \setminus \{a', b', c', d'\}$. Calling two $K_{6\times2}$ subgraphs adjacent if they intersect in a 6-clique, we obtain a graph with 126 vertices, isomorphic to the $\mathsf{E}_{7,1}(1)$ graph.

Then the root system graph $\Gamma(\mathsf{E}_8)$ of E_8 has the following combinatorial construction: Let Γ'_1 and Γ'_2 be two copies of the Gosset graph Γ' (where we denote the vertices of Γ'_1 and Γ'_2 corresponding to the vertex $v \in \Gamma''$ by v_1 and v_2, respectively). Let Γ'' be the $\mathsf{E}_{7,1}(1)$ graph with vertices identified with $K_{6\times2}$ subgraphs of Γ'. Let ∞_1 and ∞_2 be two 1-vertex graphs. Then $\Gamma(\mathsf{E}_8)$ is the union of these two 1-vertex graphs, the copies Γ'_1 and Γ'_2 of the Gosset graph, and the $\mathsf{E}_{7,1}(1)$ graph Γ'', with the following extra edges: ∞_i is adjacent to every vertex of Γ'_i, $i = 1, 2$; every vertex v_1 of Γ'_1 is adjacent to the corresponding vertex v_2 of Γ'_2; a vertex v_i of Γ'_i is adjacent to a vertex v'' of Γ'' if v is a vertex of v'' (recall that v'' is a subgraph of Γ').

Cliques, cocliques and chromatic number

Maximal cliques have sizes 3, 7, 8, a single orbit of each type. In terms of E_8, those of size 3 are the triples of coplanar lines, while the 7-cliques and 8-cliques are the objects of types 3 and 2 in the E_8 geometry. (Cf. [123], Theorem 10.2.10.) In the E_8 geometry, geodesic hexagons (i.e., hexagons with the property that the distance between its points is the same whether measured in the hexagon or the geometry) correspond to maximal cliques of size 3 (after identification of opposite points in the hexagon). The objects of type 1 are the $K_{7\times 2}$ subgraphs. Each maximal K_7 lies in a unique $K_1 + K_7$ and is contained in precisely two $K_{7\times 2}$ subgraphs. Nonmaximal cliques have sizes 0–7, a single orbit of each size.



$\tilde{\mathsf{E}}_8$.

In terms of $\mathsf{O}_8^+(2)$, the maximal cliques of size 3 are the elliptic lines.

In $\mathbb{F}_2^8$, consider the quadratic form $Q(x) = \sum_{i<j} x_i x_j$. We have $Q(x) = \binom{\mathrm{wt}(x)}{2}$ and $B(x,y) = \mathrm{wt}(x)\mathrm{wt}(y) - \sum_i x_i y_i$. The nonisotropic points are the $28+56+28+8 = 120$ points x with $\mathrm{wt}(x) \equiv 2, 3 \pmod 4$, and the 8 points of weight 7 form an 8-clique.

The maximal cocliques have size 8, reaching the Hoffman bound. They form a single orbit. The chromatic number is 15.

Let π be a totally isotropic plane. Then $\pi^\perp/\pi$ is a hyperbolic line. Its two isotropic points correspond to the two maximal isotropic subspaces on π, one of each kind. The third point corresponds to a 4-space of which the 8 nonisotropic points form an 8-coclique. The 15 planes in a totally isotropic 4-space yield 15 pairwise disjoint 8-cocliques.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$[2^{10}] : \mathsf{L}_3(2)$	2025	8, 112	0	4
b	$\mathsf{S}_3 \,\mathrm{wr}\, \mathsf{S}_4$	11200	12, 108	2	6
c	$\mathsf{S}_5 \,\mathrm{wr}\, 2$	12096	20, 100	6	10
d	$[2^{10}] : (\mathsf{S}_3 \times \mathsf{S}_3 \times \mathsf{S}_3)$	1575	24, 96	8	12
e	S_9	960	36, 84	14	18
f	$(2^5 : \mathsf{A}_5) : 2^2$	45360	40, 80	16	20
g	$(\mathsf{A}_4 \times \mathsf{A}_4) : [2^5]$	75600	48, 72	20	24
h	$2^6 : \mathsf{S}_8$	135	56, 64	24	28
i	$\mathsf{S}_5 \times \mathsf{S}_3$	483840	60, 60	26	30
j	$(\mathsf{A}_5 \times \mathsf{A}_5) : 2$	48384	60, 60	32	24

Case (a): These are the 8-cocliques, discussed above.

Case (b): In $\mathbb{F}_4^4$, let $q(x) = \mathrm{wt}(x)$. If we represent $\mathbb{F}_4$ by $\{000, 011, 101, 110\}$, then the weight of a single digit is $x_1x_2 + x_1x_3 + x_2x_3$, that is, is a binary quadratic form, and we see that $\mathbb{F}_4^4$ with $q(x)$ is an $\mathsf{O}_8^+(2)$ geometry, the orthogonal sum of four elliptic lines. The nonisotropic points are the 12 vectors of weight 1 and the 108 of weight 3. The subgraph induced on the 12-set is $4K_3$.

Case (c): View $V = \mathbb{F}_2^8$ as the orthogonal direct sum of two $\mathbb{F}_2^4$ provided with elliptic quadric. The graph induced on the 20 is $2T(5)$.

Case (d): These are the splits induced by the totally isotropic lines: each t.i. line is orthogonal to 24 nonisotropic points. The subgraph induced on the 24-set is $2K_{4,4,4}$.

Case (e): Take the quadratic form $\sum_{i<j} x_i x_j$ on the hyperplane $\mathbf{1}^\perp$ in a 9-dimensional vector space over $\mathbb{F}_2$. The nonisotropic points are the 36 vectors

of weight 2 and the 84 vectors of weight 6. The subgraph induced on the 36-set is the triangular graph $T(9)$.

Case (f): The subgraph induced on the 40-set here is an antipodal double cover of $K_{5\times 4}$ of diameter 3.

Case (h): These are the splits induced by the isotropic points: each isotropic point is orthogonal to 56 nonisotropic points. The subgraph induced on the 56-set is the 2-coclique extension of the triangular graph $T(8)$. The subgraph induced on the 64-set is strongly regular with parameters (64, 28, 12, 12), the complement of $VO_6^+(2)$ (see §3.3.1).

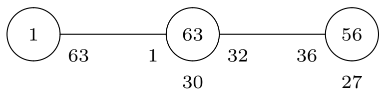
Case (j): Consider the quadratic form $Q(x) = x_1x_2 + x_3x_4$ on $\mathbb{F}_4^4$. Then $\operatorname{tr} Q$ is a nondegenerate hyperbolic quadric on $\mathbb{F}_2^8$. The 135 isotropic points have $\operatorname{tr} Q(x) = 0$, that is, $Q(x) \in \{0,1\}$ (namely, 75 with $Q(x) = 0$ and 60 with $Q(x) = 1$). The 120 nonisotropic points have $Q(x) \in \{\omega, \omega^2\}$ and are split into two 60-sets according to the value of Q.

Cases (i), (j): In case (j) the two halves are interchanged by an automorphism, so that the group preserving the split is twice as large. In case (i) the two halves are nonisomorphic.

Cayley graph

This graph is a Cayley graph for S_5.

Complement



$v = 120$

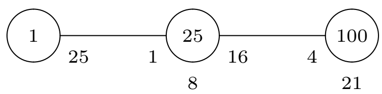
The complementary graph $\overline{\Gamma}$ of the graph described above is a Fischer graph. It is strongly regular with parameters $(v, k, \lambda, \mu) = (120, 63, 30, 36)$. Its spectrum is $63^1\ 3^{84}\ (-9)^{35}$. Its local graph is the $\mathsf{Sp}_6(2)$ graph, see §10.21.

An alternative construction of $\overline{\Gamma}$ is given by considering all projective lines $\mathsf{PG}(1,8)$ on a 9-set in one of the two A_9-orbits, adjacent when sharing a Sylow 2-subgroup (that is, a translation group). This provides a rank 3 representation of $\overline{\Gamma}$ (and hence also of Γ) with automorphism group A_9 and point stabilizer $\operatorname{Aut} \mathsf{PG}(1,8) \cong \mathsf{P\Gamma L}_2(8)$.

A cospectral rank 4 graph

The distance 1-or-3 graph of the Johnson graph $J(10,3)$ is a rank 4 strongly regular graph with parameters $(v, k, \lambda, \mu) = (120, 56, 28, 24)$ and full group of automorphisms S_{10} with point stabilizer $\mathsf{S}_7 \times \mathsf{S}_3$. The suborbit sizes are $1 + 21 + 35 + 63$. It is cospectral with the above $\overline{NO_8^+(2)}$ graph.

10.40 The S_{10} graph on 126 vertices



$v = 126$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (126, 25, 8, 4)$. Its spectrum is $25^1\ 7^{35}\ (-3)^{90}$. The full group of automorphisms is S_{10}, with point stabilizer $\mathsf{S}_5 \,\mathrm{wr}\, 2$. This graph is locally 5×5.

Construction

Take the 5+5 splits of a fixed 10-set, adjacent when the common refinement has shape $4+1+1+4$. Equivalently, take the 4-subsets of a 9-set, adjacent when they meet in 0 or 3 elements. This is the antipodal quotient of the Johnson graph $J(10, 5)$.

Cliques and cocliques

Maximal cliques have size 6 and form a single orbit. They consist of the splits containing a fixed 4-set. Maximal cocliques have sizes 7–12, with unique orbits of maximal cocliques of sizes 7, 8, and 12.

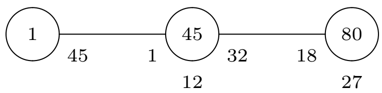
Take the representation of the graph Γ by 4-subsets of a 9-set. A maximal 7-clique is obtained by adjoining a fixed element to the 7 lines of the Fano plane.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$\mathsf{M}_{10}.2$	2520	36, 90	5	8
b	$3^2 : \mathsf{Q}_8 : 3 : 2$	8400	54, 72	9	12
c	$\mathsf{S}_7 \times \mathsf{S}_3$	120	21, 105	10	3
d	$\mathsf{S}_8 \times \mathsf{S}_2$	45	56, 70	15	8

10.41 $NO_6^-(3)$



$v = 126$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (126, 45, 12, 18)$. Its spectrum is $45^1\ 3^{90}\ (-9)^{35}$. The full group of automorphisms is $U_4(3){:}2^2_{122}$ acting rank 3 with point stabilizer $2 \times (\mathsf{O}_5(3){:}2)$.

This is the graph $NO_6^-(3)$, the graph on one class of nonisotropic points in the $\mathsf{O}_6^-(3)$ geometry, adjacent when orthogonal.

The maximal cliques all have size 6 and form a single orbit. (They are the orthonormal bases.) The vertex set has a partition into maximal cliques, so that $\chi(\overline{\Gamma}) = 21$. The maximal cocliques have sizes 9 (two orbits) or 10, 11, 15 (a single orbit each).

The local graph is strongly regular with parameters $(v, k, \lambda, \mu) = (45, 12, 3, 3)$. It is the collinearity graph of the unique $\mathsf{GQ}(4, 2)$.

This graph Γ is the local graph of $NO_7^{-\perp}(3)$, cf. §10.66, and that latter graph is the unique connected locally Γ graph (Pasechnik [599]).

This graph is the μ-graph of the Fi_{22} graph, cf. §10.90.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

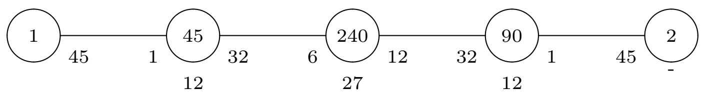
	H	index	orbitlengths	d	e	graph
a	$2^5:\mathsf{S}_6$	567	6, 120	5	2	K_6
b	$3_+^{1+4}.4\mathsf{S}_4:2$	280	18, 108	9	6	$K_{9,9}$
c	S_7	2592	21, 105	10	7	$T(7)$
d	S_7	2592	21, 105	10	7	$\overline{T(7)}$
e	$2^5:\mathsf{S}_6$	567	30, 96	13	10	
f	$2\times(\mathsf{O}_5(3):2)$	126	36, 90	15	12	$NO_6^-(2)$
g	$\mathsf{A}_6.2^2$	9072	36, 90	15	12	
h	$2\times(\mathsf{L}_3(2):2)$	19440	42, 84	17	14	
i	$3^4:(2\times\mathsf{S}_6)$	112	45, 81	18	15	

In case (e) the graph on the small orbit is the 2-clique extension of the collinearity graph of $\mathsf{GQ}(2,2)$.

In case (i) the graph on the small orbit is the 3-coclique extension of the collinearity graph of $\mathsf{GQ}(2,2)$. The maximal cocliques of size 15 arise as the 3-coclique extension of an ovoid in $\mathsf{GQ}(2,2)$. The graph induced on the large orbit is the strongly regular graph $VNO_4^-(3)$ with parameters $(v,k,\lambda,\mu) = (81,30,9,12)$ (cf. §3.3.2). It is the collinearity graph of a $pg(6,6,2)$, cf. §8.6.1.

Triple cover

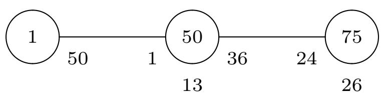
This graph has a distance-transitive antipodal 3-cover with diagram



$v = 378$

constructed by Blokhuis & Brouwer (cf. [123], p. 399). Jurišić & Koolen [473] showed that this cover is the unique distance-regular graph with these parameters.

10.42 The Goethals graph on 126 vertices



$v = 126$

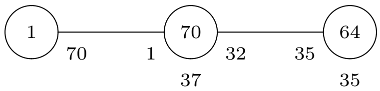
There is a unique strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (126,50,13,24)$. Its spectrum is $50^1\ 2^{105}\ (-13)^{20}$. The full group of automorphisms is S_7 acting rank 7, with point stabilizer $2\times 5{:}4$. Existence is due to Goethals (cf. [380]). Uniqueness is due to Coolsaet & Degraer [222].

Maximum cliques have size 4. Maximum cocliques have size 12.

Construction

As we saw above (§10.19), the graph on the edges of the Hoffman-Singleton graph H, adjacent when they have distance 2 in the line graph $L(H)$, is strongly regular with parameters $(v, k, \lambda, \mu) = (175, 72, 20, 36)$. Take for Γ the induced subgraph of this latter graph on the set of 126 edges at distance 2 from a fixed vertex of H. For a proof, see [137]. For a construction via switching in the Hermitian 2-graph on 126 points, see [380].

10.43 The $\mathsf{O}_8^+(2)$ graph on 135 vertices



$v = 135$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (135, 70, 37, 35)$. Its spectrum is $70^1\ 7^{50}\ (-5)^{84}$. The full group of automorphisms is $\mathsf{O}_8^+(2):2$ acting rank 3, with point stabilizer $2^6:\mathsf{S}_8$. This is the hyperbolic orthogonal graph $\Gamma(\mathsf{O}_8^+(2))$.

Construction: isotropic points in the $\mathsf{O}_8^+(2)$ geometry

The $\mathsf{O}_8^+(2)$ geometry has 255 projective points, 135 isotropic, 120 nonisotropic. Γ is the graph on the isotropic points, adjacent when orthogonal.

Cliques, cocliques and chromatic number

All maximal cliques have size 15. They are the maximal totally isotropic subspaces of the geometry. Maximal cocliques have size 5 or 9, a single orbit of each size, with cocliques stabilized by $\mathsf{S}_5 \times \mathsf{S}_5$ and S_9, respectively. The cocliques of size 9 are the ovoids.

The maximal cocliques of size 5 arise as follows. Let $V = V_1 \perp V_2$ where each V_i is a 4-space with elliptic quadratic form. Then each V_i has 5 isotropic points, and V has $135 = 5 + 5 + 25 + 100$ isotropic points (of shapes $0i$ and $i0$ and ij and mn, where i, j denote isotropic points and m, n nonisotropic points). We see subgraphs $K_{5,5}$ and $\overline{5 \times 5}$.

The chromatic number of $\overline{\Gamma}$ is 9. The chromatic number of Γ is 17.

The chromatic number of Γ is larger than 15: $\mathsf{V}\Gamma$ does not have a partition into 15 ovoids. Indeed, the maximum number of pairwise disjoint ovoids is 12. Every set of 12 pairwise disjoint ovoids leaves a copy of the complement of the Schläfli graph, which has chromatic number 6, so that $\chi(\Gamma) \leq 18$. L. H. Soicher showed that in fact $\chi(\Gamma) = 17$.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	S_9	960	9, 126	0	5
b	$\mathsf{S}_3 \times O_6^-(2):2$	1120	27, 108	10	15
c	$\mathsf{S}_3 \operatorname{wr} \mathsf{S}_4$	11200	54, 81	25	30
d	$\mathsf{O}_7(2).2$	120	63, 72	30	35
e	$2^6:\mathsf{A}_8$	270	15, 120	14	7
f	S_8	8640	30, 105	21	14
g	$\mathsf{S}_6 \times \mathsf{S}_3$	80640	45, 90	28	21
h	$(\mathsf{A}_5 \times \mathsf{A}_5):2^2$	24192	60, 75	35	28

Case (a): Take the quadratic form $\sum_{i<j} x_i x_j$ on the hyperplane $\mathbf{1}^\perp$ in a 9-dimensional vector space over $\mathbb{F}_2$. The isotropic points are the 126 vectors of weight 4 and the 9 vectors of weight 8. The 9 vectors of weight 8 form an ovoid in the $\mathsf{O}_8^+(2)$ polar space.

Case (b): Consider the geometry as the orthogonal direct sum of an elliptic line and a $\mathsf{O}_6^-(2)$ geometry. The 27-set induces $\Gamma(O_6^-(2))$.

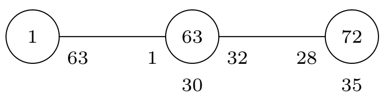
Case (c): In $\mathbb{F}_4^4$, let $q(x) = \mathrm{wt}(x)$. As before, we see that $\mathbb{F}_4^4$ with $q(x)$ is an $\mathsf{O}_8^+(2)$ geometry, the orthogonal sum of four elliptic lines. The isotropic points are the 54 vectors of weight 2 and the 81 of weight 4.

Case (d): These are the splits induced by the nonisotropic points: each nonisotropic point is orthogonal to 63 isotropic points.

Case (e): These are the maximal totally isotropic subspaces.

Case (h): Consider the quadratic form $Q(x) = x_1x_2 + x_3x_4$ on $\mathbb{F}_4^4$. Then $\operatorname{tr} Q$ is a nondegenerate hyperbolic quadric on $\mathbb{F}_2^8$. The 135 isotropic points have $\operatorname{tr} Q(x) = 0$, that is, $Q(x) \in \{0, 1\}$, namely, 75 with $Q(x) = 0$ and 60 with $Q(x) = 1$.

10.44 $NO_8^-(2)$



$v = 136$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (136, 63, 30, 28)$. Its spectrum is $63^1\ 7^{51}\ (-5)^{84}$. The full group of automorphisms is $\mathsf{O}_8^-(2):2$ acting rank 3 with point stabilizer $2 \times \mathsf{O}_7(2)$.

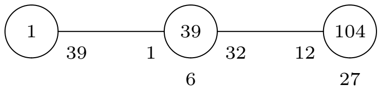
Maximal cliques have size 8, a single orbit. Maximal cocliques have sizes 3 and 7, a single orbit each.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \operatorname{Aut}\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$[2^9]:(L_3(2) \times \mathsf{S}_3)$	765	24, 112	7	12
b	$[2^9]:(\mathsf{S}_5 \times \mathsf{S}_3)$	1071	40, 96	15	20
c	$2^6:(O_5(3):2)$	119	64, 72	27	32
d	$17:8$	2903040	68, 68	29	34
e	$L_2(16):2$	48384	68, 68	35	28

10.45 The $\mathsf{L}_3(3)$ graph on 144 vertices



$v = 144$

There is a strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (144, 39, 6, 12)$, with full automorphism group $\mathsf{L}_3(3)$:2, acting rank 6, with point stabilizer 13:6. Its spectrum is $39^1\ 3^{104}\ (-9)^{39}$. This graph was discovered by FARADŽEV, KLIN & MUZYCHUK [315].

Construction — Singer cycles and imaginary triangles

We need 144 objects on which $G = \mathsf{PGL}_3(3)$ acts, and an adjacency relation. One choice for the objects is that of the 144 subgroups of order 13 in $\mathsf{PGL}_3(3)$. The normalizer $N = N_G(C)$ of such a subgroup C has order 39. Acting by conjugation, it has orbits of lengths $1^1\ 13^5\ 39^2$ on the 144 objects, and precisely one of the two orbits of size 39 is suitable as set of neighbors of C.

Let q be the power of a prime, and $r = q^m$. Then $\mathbb{F}_r$ can be regarded as an m-dimensional vector space over $\mathbb{F}_q$, and multiplication by a constant is a linear transformation. One sees that $\mathsf{PG}(m-1, q)$ admits *Singer cycles*, linear transformations of order $\frac{q^m-1}{q-1}$ that act regularly on the points and hyperplanes.

Now let $m = 3$, and fix a $\mathsf{PG}(2, q)$ subplane π_0 of the projective plane $\pi = \mathsf{PG}(2, q^3)$. The group $\mathsf{PGL}_3(q)$ has three orbits on the points of π, namely that of the q^2+q+1 points of π_0, that of the $q(q^2-1)(q^2+q+1)$ points of $\pi \setminus \pi_0$ on a line of π_0, and that of the $q^3(q^2-1)(q-1)$ remaining points. The field automorphism $x \mapsto x^q$ that fixes π_0 partitions these remaining points into $\frac{1}{3}q^3(q^2-1)(q-1)$ triples, known as *imaginary triangles*. The subgroup of $\mathsf{PGL}_3(q)$ pointwise fixing an imaginary triangle is generated by a Singer cycle.

For $q = 3$, the 144 imaginary triangles can be taken as the vertices of Γ.

Maximal cliques and cocliques

The graph induced on the common neighbors of two adjacent vertices is $2K_1 + 2K_2$. Consequently, the maximal cliques have sizes 3 and 4. Maximal cocliques have sizes 9–16 and 18. There is a single orbit of 9-cocliques and a single orbit of 18-cocliques.

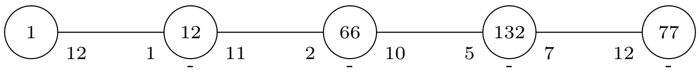
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$3^2 : 8$	156	72, 72	15	24
b	$(3^2 : 3) : \mathsf{D}_8$	52	36, 108	12	9
c	$(3^2 : \mathsf{Q}_8) : 3$	52	72, 72	21	18

This graph is a Cayley graph: the (nonabelian) group $A\Gamma L(1, 9)$ of order 144 acts regularly on Γ.

10.46 Three $\mathsf{M}_{12}.2$ graphs on 144 vertices

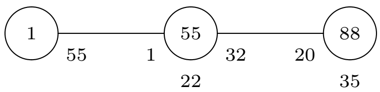


$v = 288$

The *Leonard graph* is the unique distance-regular graph with intersection array $\{12, 11, 10, 7; 1, 2, 5, 12\}$. Existence is due to LEONARD [515], uniqueness to BROUWER [115]. The full group of automorphisms is $\mathsf{M}_{12}.2$ and has two orbits on the vertex set. The two halved graphs are nonisomorphic strongly regular graphs with parameters $(v, k, \lambda, \mu) = (144, 66, 30, 30)$, and full group $\mathsf{M}_{12}.2$, known as the two halved Leonard graphs.

The group $\mathsf{M}_{12}.2$ has two distinct primitive permutation representations on 144 points, both rank 4 with suborbit sizes $1 + 22 + 55 + 66$, and the two graphs of valency 66 are the halved Leonard graphs. For the first of these two representations, also the other two suborbits define strongly regular graphs, and we find graphs with parameters $(v, k, \lambda, \mu) = (144, 22, 10, 2)$, $(144, 55, 22, 20)$. The former is the 12×12 grid, the latter has full group $\mathsf{M}_{12}.2$.

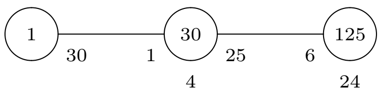
For more detail, see [123], §11.4F and [621], pp. 48, 49.



$v = 144$

This valency 55 graph satisfies the 5-vertex condition. Its μ-graphs have valency 9.

10.47 The $\mathsf{O}_5(5)$ graphs on 156 vertices



$v = 156$

There are exactly two rank 3 strongly regular graphs with parameters (v, k, λ, μ) $= (156, 30, 4, 6)$. Their spectrum is $30^1\ 4^{90}\ (-6)^{65}$. Both have full group of automorphisms $\mathsf{O}_5(5).2$. The point stabilizers are $5^3 : (\mathsf{S}_5 \times 4)$ and $5_+^{1+2} : 4\mathsf{S}_5$.

These two graphs, let us call them Γ and Δ, are the collinearity graphs of the two known generalized quadrangles $\mathsf{GQ}(5, 5)$. One is that on the isotropic points, the other, its dual, that on the totally isotropic lines of the $\mathsf{O}_5(5)$ geometry, cf. §2.6.1. The latter is isomorphic to the generalized quadrangle on the points of the $\mathsf{Sp}_4(5)$ geometry.

Maximal cliques and cocliques

In both graphs, the maximal cliques are the lines (of size 6). In Γ maximal cocliques have sizes 13–20, 22, 24, 26 (with a single orbit for sizes 19, 20, 24, 26). Those of size 26 are the ovoids, and any two ovoids meet in 1, 6 or 26 points. The chromatic number is $\chi(\Gamma) = 7$. In Δ maximal cocliques have sizes 6, 11, 12, 14–18, with a single orbit for size 6.

Regular sets

Examples of regular sets in Γ and Δ are obtained from subgroups H of their automorphism groups with two orbits on the vertex set. We give degree d, nexus e, and i^{n_i}, where n_i is the number of lines meeting the smallest orbit in i points.

For $\Gamma = \Gamma(\mathsf{O}_5(5))$:

	H	index	orbitlengths	d	e	line stats
a	$2 \times (\mathsf{O}_4^-(5):2)$	300	26, 130	0	6	1^{156}
b	$5_+^{1+2}:4\mathsf{S}_5$	156	6, 150	5	1	$6^1\ 1^{30}\ 0^{125}$
c	$2.(\mathsf{A}_5 \times \mathsf{A}_5).2^2$	325	36, 120	10	6	$6^{12}\ 1^{144}$

These sets are ovoids, lines, and hyperbolic quadrics 6×6, respectively.

For $\Delta = \Gamma(\mathsf{Sp}_4(5))$:

	H	index	orbitlengths	d	e	line stats
a	$5^3:(\mathsf{S}_5 \times 4)$	156	6, 150	5	1	$6^1\ 1^{30}\ 0^{125}$
b	$2.(\mathsf{A}_5 \times \mathsf{A}_5).2^2$	325	12, 144	6	2	$2^{36}\ 0^{120}$
c	$\mathsf{S}_5 \times \mathsf{S}_3 \times 2$	6500	36, 120	10	6	$6^6\ 2^{90}\ 0^{60}$
d	S_6	13000	36, 120	10	6	$6^6\ 2^{90}\ 0^{60}$
e	$2^4:\mathsf{S}_5$	4875	60, 96	14	10	$6^{20}\ 2^{120}\ 0^{16}$

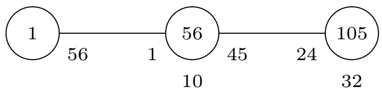
Case (a): these are the lines, and induce K_6.

Case (b): these are the unions $L \cup L^\perp$ for hyperbolic lines L, and induce $K_{6,6}$. The maximal 6-cocliques are precisely the hyperbolic lines.

Cases (c), (d): A *BLT* (Bader-Lunardon-Thas) *set* in the $\mathsf{O}_5(q)$ generalized quadrangle, where q is odd, is a set S of $q+1$ points, no two collinear, such that every point outside S is collinear with 0 or 2 points of S. In the dual $\mathsf{Sp}_4(q)$ generalized quadrangle this becomes a set of $q+1$ pairwise disjoint lines L_i such that every other line meets either 0 or 2 of them. The union $X = \bigcup L_i$ of this set of lines is a regular $(q+1)^2$-set of degree $d = 2q$ and nexus $e = (q+1)/2$. For $q = 5$ there are up to isomorphism two examples, the linear one and the FTW (Fisher-Thas-Walker) one ([269]), yielding examples (c) and (d), respectively.

Case (e): this is most easily seen in the dual $\mathsf{O}_5(q)$ generalized quadrangle. For the quadratic form $\sum X_i^2$, there are 20, 120, 16 isotropic points of weight 2, 4, 5, respectively. These 16 form a (non-maximal) coclique.

10.48 The $\mathsf{U}_4(3)$ graph on 162 vertices



$v = 162$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (162, 56, 10, 24)$. Its spectrum is $56^1\ 2^{140} (-16)^{21}$. The full group of automorphisms is $\mathsf{U}_4(3).(2^2)_{133}$ (of order $2^9 \cdot 3^6 \cdot 5 \cdot 7$) acting rank 3, with point stabilizer $\mathsf{L}_3(4) : 2^2$. Uniqueness is due to Cameron, Goethals & Seidel [178]. This graph is the second subconstituent of the McLaughlin graph. Both subconstituents of this graph are also strongly regular (§10.20, §10.33). This graph is a subgraph of the Suzuki graph (§10.83).

Construction

Γ can be constructed as $1+56+105$ by taking a point ∞, one of the three orbits of hyperovals (of size 56) in $\mathsf{PG}(2,4)$, and the 105 flags (p,L) of $\mathsf{PG}(2,4)$. Here ∞ is adjacent to the 56 hyperovals; two hyperovals are adjacent when disjoint; a hyperoval O is adjacent to a flag (p,L) when $p \notin O$ and $L \cap O \neq \emptyset$; two flags (p,L) and (q,M) are adjacent when $p \neq q$, and $L \neq M$, and $p \in M$ or $q \in L$.

Cliques, cocliques and chromatic number

Since the local graph does not have triangles, all maximal cliques have size 3. Since Γ is the 2nd subconstituent of the McLaughlin graph, maximum cocliques have size 21. The chromatic number of Γ is 10 (Soicher). That of $\overline{\Gamma}$ is 54.

Splits

This graph has 112 splits into two Brouwer-Haemers graphs. (Such splits can be enumerated by searching the 21-dimensional eigenspace for eigenvectors that are 1 on the subgraph and -1 on the complement. The result is that these 112 are the only splits of Γ into two subgraphs of valency 20.) Split halves occur as intersections with point neighborhoods in a McLaughlin graph.

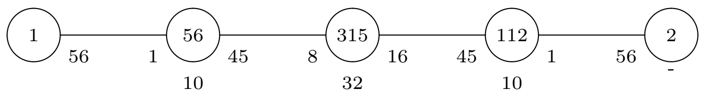
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$3^4{:}\mathsf{M}_{10}$	224	81, 81	20	36
b	$2^{1+4}.3^2.2^{1+3}$	2835	18, 144	8	6
c	$2 \times \mathsf{U}_3(3){:}2$	540	36, 126	14	12
d	$\mathsf{L}_3(4){:}2^2$	162	42, 120	16	14
e	$3^{1+4}.4.2^4$	840	54, 108	20	18
f	$\mathsf{S}_3 \times (\mathsf{S}_3{\times}\mathsf{S}_3){:}2$	30240	54, 108	20	18
g	$2 \times \mathsf{A}_6.2^2$	4536	72, 90	26	24

There are no further regular sets with $d - e = s$.

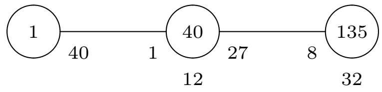
Triple cover



$v = 486$

There is a unique distance-regular graph with intersection array $\{56, 45, 16, 1; 1, 8, 45, 56\}$, a triple cover of Γ. It was constructed by Soicher [664]. It is distance-transitive with full group $3.\mathsf{U}_4(3).2^2$ with point stabilizer $\mathsf{L}_3(4).2^2$. Its second subconstituent is also distance-regular (but not distance-transitive), see §10.33.

10.49 The nonisotropic points of $\mathsf{U}_5(2)$



$v = 176$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (176, 40, 12, 8)$. Its spectrum is $40^1\ 8^{55}\ (-4)^{120}$. The full group of automorphisms is $\mathsf{P\Gamma U}_5(2)$ acting rank 3 with point stabilizer $\mathsf{U}_4(2):\mathsf{S}_3$. This is $\overline{NU_5(2)}$, the graph on the nonisotropic points in the $\mathsf{U}_5(2)$ geometry, adjacent when orthogonal, that is, when joined by a secant. The local graph is $\overline{NU_4(2)}$ (§10.16).

The maximal cliques have size 5 (a single orbit). They are the orthogonal bases. The maximal cocliques have sizes 9–13 (many orbits) and 16 (two orbits). Most maximal cocliques are messy, but there is a single nice orbit of 16-cocliques where the tangent lines induce the structure of $\mathsf{AG}(2, 4)$, namely the perps of the totally isotropic lines.

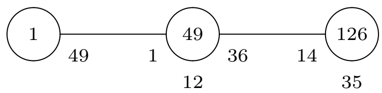
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$[2^8]:(\mathsf{A}_5 \times \mathsf{S}_3)$	297	16, 160	0	4
b	$[2^{11}.3^4]$	165	48, 128	8	12

These are the sets of vertices in $L^\perp$ and in $p^\perp$ for t.i. lines L and points p, respectively.

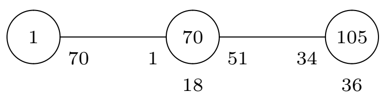
10.50 A polarity of Higman's symmetric design



$v = 176$

In Higman [426], a square 2-(176,50,14) design is constructed that has HS as automorphism group acting 2-transitively on points and blocks. In Smith [663] the following description is given: let the points be the octads from $S(5, 8, 24)$ that start 10..., and the blocks the octads that start 01.... Let the point B and the block C be incident when $|B \cap C| \in \{0, 4\}$. This design has a polarity for which all points are absolute. This means that this design has a symmetric point-block incidence matrix A with 1's on the diagonal. Now $A - I$ is the adjacency matrix of a strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (176, 49, 12, 14)$. Its spectrum is $49^1\ 5^{98}\ (-7)^{77}$. The full automorphism group of Γ is S_8 with vertex orbits of sizes 8 and 168 ([110]).

10.51 The M_{22} graph on 176 vertices



$v = 176$

There is a unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (176, 70, 18, 34)$. Its spectrum is 70^1 2^{154} $(-18)^{21}$. The full group of automorphisms is M_{22} acting rank 3 with point stabilizer A_7. Uniqueness is due to Degraer & Coolsaet [274].

Construction

The Steiner system $S(4, 7, 23)$ has 253 blocks, 77 on each point. The residual design is a quasi-symmetric 3-(22,7,4) design, with 176 blocks, and block intersection numbers 1 and 3. Call two blocks adjacent when they meet in 1 point.

This graph is an induced subgraph of the M_{23} graph (§10.56) and of the McLaughlin graph (§10.61).

Cliques, cocliques and chromatic number

All 9240 maximal cliques have size 4. The group G is transitive on i-cliques for $0 \le i \le 4$. Each triangle is contained in a unique 4-clique.

Maximal cocliques have sizes 7–13, 15, 16. We describe the three orbits of 16-cocliques. Let Ω be a 24-set, and fix an $S(5, 8, 24)$ on Ω. Fix $\alpha, \beta \in \Omega$, and view the 176 vertices as the octads containing α and missing β. There are three orbits of 16-cocliques: the 231 sets of vertices containing two fixed elements $\gamma, \delta \in \Omega \setminus \{\alpha, \beta\}$, the 462 sets of vertices containing a fixed element $\gamma \in \Omega \setminus \{\alpha, \beta\}$ and disjoint from a fixed octad B on $\{\alpha, \beta, \gamma\}$, and the 1155 sets of vertices missing two fixed elements $\gamma, \delta \in \Omega \setminus \{\alpha, \beta\}$ and meeting a fixed octad B on $\{\alpha, \beta, \gamma, \delta\}$ in 4 points.

The chromatic number is 12 (Soicher [756]).

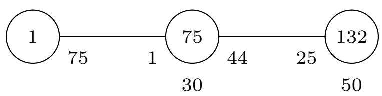
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$\mathsf{L}_3(4)$	22	56, 120	10	28
b	$2^4 : \mathsf{A}_6$	77	80, 96	22	40

There are no further regular sets with $d - e = s$.

10.52 The nonisotropic points of $\mathsf{U}_3(4)$

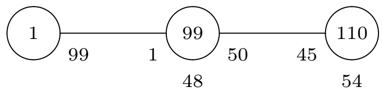


$v = 208$

There is a unique rank 4 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (208, 75, 30, 25)$. Its spectrum is 75^1 10^{64} $(-5)^{143}$. The full automorphism group is $\mathsf{P\Gamma U}_3(4)$, acting rank 4 with point stabilizer $(\mathsf{D}_{10} \times \mathsf{A}_5) \cdot 2$ and suborbit sizes $1 + 12 + 75 + 120$. It is the block graph of a unital $S(2, 5, 65)$ in $\mathsf{PG}(2, 16)$. Equivalently, it is the graph on the nonisotropic points in that projective plane, adjacent when joined by a tangent. Maximal cliques have size 6 or 16 (reaching the Hoffman bound). Maximal cocliques have sizes 5 (a single orbit), 7–9, and 13 (three orbits, reaching the Hoffman bound).

At least 1778 nonisomorphic systems $S(2,5,65)$ are known, and these have nonisomorphic block graphs, all with the above parameters. Apart from the hermitian unital all have small groups (of size at most 1200). Of these, 42 are embeddable in some projective plane of order 16 (and 13 different planes occur). For some examples, see [672], [500], [368].

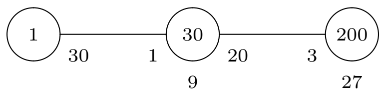
10.53 A rank 16 representation of S_7



$v = 210$

Klin et al. [494] showed as application of the computer algebra package COCO that the rank 16 scheme of S_7 on the cosets of $\mathsf{A}_4 \times 2$ has a subscheme that is a strongly regular graph with parameters $(v,k,\lambda,\mu) = (210,99,48,45)$. Its spectrum is $99^1\ 9^{77}\ (-6)^{132}$. The full automorphism group is S_7, acting rank 16 with point stabilizer $\mathsf{A}_4 \times 2$. Maximal cliques have sizes 5–8 and 10. Maximal cocliques have sizes 5–9 and 12 (reaching the Hoffman bound).

10.54 The Cameron graph



$v = 231$

There is a rank 4 strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (231,30,9,3)$. Its spectrum is $30^1\ 9^{55}\ (-3)^{175}$. It is the unique strongly regular graph with these parameters that is a gamma space with lines of size 3. The full automorphism group is $\mathsf{M}_{22}.2$, acting rank 4 with point stabilizer $2^5{:}\mathsf{S}_5$. Construction is due to Cameron (cf. [178], Example 7.8), uniqueness to Brouwer [113].

Construction

Consider a Steiner system $S(3,6,22)$ on the 22-set S. Let the vertices of Γ be the 231 pairs of symbols from S, where two vertices are adjacent when the pairs are disjoint and contained in a common block.

Gamma space

This graph is the collinearity graph of a partial linear space with lines of size 3, namely the triples of pairs that partition a block of the Steiner system. This geometry is a gamma space: given a line L, each point outside L is collinear to 0, 1, or 3 points of L. It has Fano subplanes, 10 on each point and 2 on each line. The 15 lines and 10 planes on a fixed point form the edges and vertices of the Petersen graph. We see a $\mathsf{GQ}(2,2)$ subgeometry on each block.

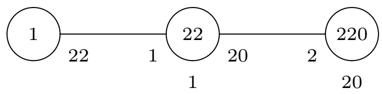
Cliques and cocliques

The maximal cliques in Γ all have size 7 and form a single orbit with clique stabilizer $2 \times 2^3 : \mathsf{L}_3(2)$. They are the Fano subplanes of the gamma space. Examples of cocliques of size 21, which is the Hoffman bound, are given by the sets of pairs containing a fixed symbol.

Triple cover

This graph has a triple cover on 693 vertices with full group $3.\mathsf{M}_{22}.2$.

10.55 The Berlekamp-Van Lint-Seidel graph



$v = 243$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (243, 22, 1, 2)$. Its spectrum is $22^1\ 4^{132} (-5)^{110}$. The full group of automorphisms is $3^5 : (2 \times \mathsf{M}_{11})$ acting rank 3 with point stabilizer $2 \times \mathsf{M}_{11}$. This graph was constructed in Berlekamp, Van Lint & Seidel [59] (with one construction per author). For example, it is the coset graph of the perfect ternary Golay code.

Dual

The Delsarte dual (cf. §7.1.3) Δ of Γ is a rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (243, 110, 37, 60)$ and spectrum $110^1\ 2^{220} (-25)^{22}$.

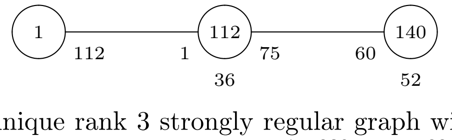
Koolen-Riebeek graph

The complement of Γ (on 243 vertices with valency 220) is the halved graph of the bipartite Koolen-Riebeek graph (distance-regular with intersection array $\{45, 44, 36, 5; 1, 9, 40, 45\}$) constructed in [136].

Generalization

More generally, one may look at strongly regular graphs with $\lambda = 1$ and $\mu = 2$. Such a graph must have $(2r + 1) \,|\, 63$, so that $r \in \{1, 3, 4, 10, 31\}$ and v is one of 9, 99, 243, 6273, 494019. The only known examples have $v = 9$ or $v = 243$.

10.56 The M_{23} graph



$v = 253$

There is a unique rank 3 strongly regular graph with parameters $(v, k, \lambda, \mu) = (253, 112, 36, 60)$. Its spectrum is $112^1\ 2^{230}\ (-26)^{22}$. The full group of automorphisms is $G = \mathsf{M}_{23}$ acting rank 3 with point stabilizer $2^4.\mathsf{A}_7$.

Construction

Take the blocks of the Steiner system $S(4,7,23)$ as vertices, and call them adjacent when they meet in 1 point.

Cliques, cocliques and chromatic number

All 212520 maximal cliques have size 4. The group G is transitive on i-cliques for $0 \le i \le 4$. Each triangle is contained in 5 4-cliques.

Maximal cocliques have sizes 8, 10–17, 21.

There is a unique orbit of 21-cocliques, consisting of the 253 sets of 21 blocks containing a fixed pair of symbols.

There is a unique orbit of 8-cocliques. (It can be seen from the description of the extended binary Golay code C using two Hamming codes (see §6.1.2). In the notation used there, take $x = \mathbf{1}$ and $a = b + \mathbf{1}$, to see the existence of code words $(b, b+\mathbf{1}, 0)$ for all $b \in H$. Take the 8 vectors b starting with 1.)

The chromatic number is 15 (Soicher [756]).

Splits 77+176

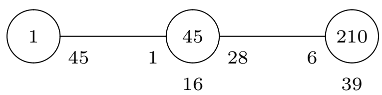
The 77 blocks on any given point induce the graph with parameters $(v,k,\lambda,\mu) = (77,16,0,4)$, see §10.27. The remaining 176 blocks induce the graph with parameters $(v,k,\lambda,\mu) = (176,70,18,34)$ described in §10.51.

Apart from these, there are no further regular sets with $d - e = s$.

Cayley graph

M_{23} has a (nonabelian) subgroup $23\!:\!11$ of order 253 that acts regularly on the vertices of Γ. Thus, Γ is a Cayley graph.

10.57 $2^8.\mathsf{S}_{10}$ and $2^8.(\mathsf{A}_8 \times \mathsf{S}_3)$



$v = 256$

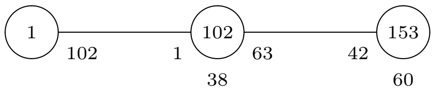
There are exactly two rank 3 strongly regular graphs with parameters $(v,k,\lambda,\mu) = (256,45,16,6)$. Their spectrum is $45^1\ 13^{45}\ (-3)^{210}$.

The first, let us call it Γ_1, has full group of automorphisms $2^8\!:\!\mathsf{S}_{10}$ and point stabilizer S_{10}. It is the graph on $\mathbf{1}^\perp/\langle\mathbf{1}\rangle$ in 2^{10}, where two vectors are adjacent when they differ in two places, a bipartite half of the folded 10-cube. Its local graph is the triangular graph $T(10)$.

The second, let us call it Γ_2, has full group of automorphisms $2^8\!:\!(\mathsf{A}_8 \times \mathsf{S}_3)$. It is the graph $H_2(4,2)$ (§3.4.1). Already $2^8\!:\!(\mathsf{A}_7 \times 3)$ acts rank 3 on Γ_2.

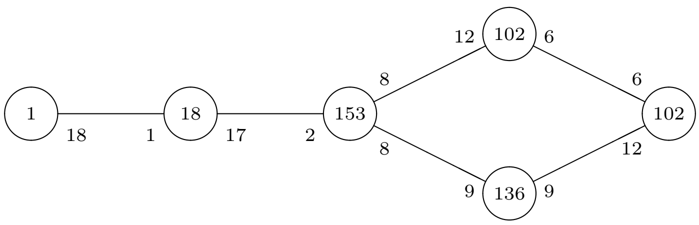
Maximal cliques in Γ_1 have sizes 4 or 10, a single orbit of each type. Maximal cliques in Γ_2 have sizes 4 or 16, a single orbit of each type.

10.58 $2^8.\mathsf{L}_2(17)$



$v = 256$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (256, 102, 38, 42)$. Its spectrum is $102^1\ 6^{153}\ (-10)^{102}$. The full group of automorphisms is $2^8 : \mathsf{L}_2(17)$ acting rank 3 with point stabilizer $\mathsf{L}_2(17)$. Maximal cliques have sizes 4, 5 and 8, a unique orbit of each type. Maximal cocliques have sizes 7–13, 16 and 18, with unique orbits for sizes 12, 13, 16, 18. The vertex set of Γ has a partition into 8-cliques, so $\chi(\overline{\Gamma}) = 32$.

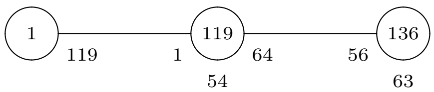


$v = 512$

A *semibiplane* is a connected bipartite graph such that any two vertices have 0 or 2 common neighbors. The graph $\overline{\Gamma}$ is the halved graph of a semibiplane of valency 18 with full group of automorphisms $2^9 : \mathsf{L}_2(17)$ acting rank 6 with point stabilizer $\mathsf{L}_2(17)$.

A strongly regular graph with the parameters of Γ can be obtained from the Van Lint-Schrijver construction (apply Theorem 7.3.2 with $(p, q, e, f, l, t, u) = (2, 2^8, 5, 8, 2, 2, 2)$), but that graph is not rank 3.

10.59 $VO_8^-(2)$

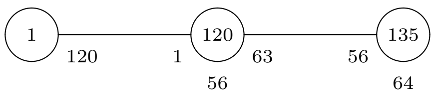


$v = 256$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (256, 119, 54, 56)$. Its spectrum is $119^1\ 7^{136}\ (-9)^{119}$. The full group of automorphisms is $2^8 : \mathsf{SO}_8^-(2)$ acting rank 3 with point stabilizer $\mathsf{SO}_8^-(2)$.

The maximal cliques have size 8, a single orbit. The maximal cocliques have sizes 4 and 8, a single orbit of each. The chromatic number of $\overline{\Gamma}$ is $\chi(\overline{\Gamma}) = 32$.

10.60 $\overline{VO_8^+(2)}$

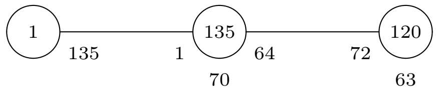


$v = 256$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (256, 120, 56, 56)$. Its spectrum is $120^1\ 8^{120}\ (-8)^{135}$. The full group of automorphisms is $2^8 : \mathsf{SO}_8^+(2)$ acting rank 3 with point stabilizer $\mathsf{SO}_8^+(2)$. Its local graph is the $\mathsf{O}_8^+(2)$ graph on 120 vertices, see §10.39.

The maximal cliques have sizes 4, 8 and 9, a single orbit of each. The maximal cocliques have size 16, a single orbit. The chromatic number of Γ is $\chi(\Gamma) = 16$.

Complement



$v = 256$

The complement of Γ is $VO_8^+(2)$, see §3.3.1. Its local graph is the $\mathsf{O}_8^+(2)$ graph on 135 vertices, see §10.43.

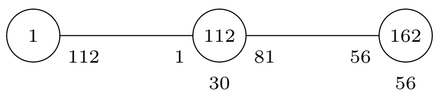
Rank 3 action of $2^8 : \mathsf{A}_9$

The polar space $\mathsf{O}_8^+(2)$ has an ovoid (see §2.6.7) on which the group A_9 acts transitively. This can be seen by taking the quadratic form $\sum_{i<j} x_i x_j$ on the hyperplane $\mathbf{1}^\perp$ in a 9-dimensional vector space over $\mathbb{F}_2$ (see also Case (e) of the regular sets in §10.39). The ovoid consisting of the base points (points with exactly one nonzero coordinate) qualifies.

Applying triality we obtain a spread S of the polar space $\mathsf{O}_8^+(2)$ (that is, a partition of the point set into nine solids) on which A_9 acts in the natural way. The stabilizer in A_9 of a solid $\Sigma \in S$ is $\mathsf{A}_8 \simeq \mathsf{PSL}_4(2)$, acting naturally on Σ. Hence A_9 acts transitively on the point set of $\mathsf{O}_8^+(2)$. Moreover, it also acts transitively on the nonisotropic points. Hence $2^8 : \mathsf{A}_9 \le 2^8 : \mathsf{O}_8^+(2)$ also acts rank 3 on Γ.

The nonisotropic points are the 36 vectors of weight 2 and the 84 vectors of weight 6. The subgraph induced on the 36-set is the triangular graph $T(9)$.

10.61 The McLaughlin graph

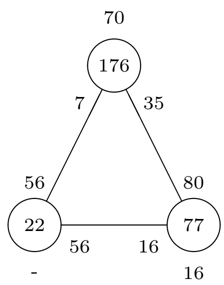


$v = 275$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (275, 112, 30, 56)$. Its spectrum is $112^1\ 2^{252}\ (-28)^{22}$. The full group of automorphisms is $G = \mathsf{McL}.2$ acting rank 3 with point stabilizer $\mathsf{U}_4(3){:}2$. This graph was constructed in McLaughlin [557]. Uniqueness is due to Goethals & Seidel [356].

Construction: 22 + 77 + 176

Take the Steiner system $S(4, 7, 23)$ with $1+22$ points and $253 = 77+176$ blocks, where the first 77 are those containing the first point. Use p, B, C to denote one of the 22, 77, 176 objects, and let $\sim$ denote adjacency. Make the 22 points a coclique, let $p \sim B$ when $p \notin B$, let $B \sim B'$ when B, B' meet in 1 point, let $p \sim C$ when $p \in C$, let $B \sim C$ when B, C meet in 3 points, let $C \sim C'$ when C, C' meet in 1 point. This yields Γ.



$v = 275.$

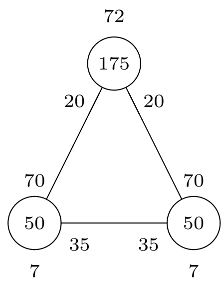
We see the M_{22} graphs on 77 and 176 vertices as subgraphs.

Leech lattice construction

This same $22 + 77 + 176$ construction is visible in the Leech lattice. Fix the two Leech lattice vectors $v_1 = \frac{1}{\sqrt{8}}(51\,11\ldots1)$ and $v_2 = \frac{1}{\sqrt{8}}(44\,00\ldots0)$. Take the 275 norm 4 vectors x with $(x, v_1) = 3$ and $(x, v_2) = 1$, adjacent when their inner product is 1. The $22 + 77 + 176$ vertices have the shapes $\frac{1}{\sqrt{8}}(11\,1^{21}(-3))$, $\frac{1}{\sqrt{8}}(3(-1)\,1^{16}(-1)^6)$ and $\frac{1}{\sqrt{8}}(20\,2^7 0^{15})$.

Construction: 50 + 50 + 175

The graph Γ has a regular partition into two Hoffman-Singleton graphs and a copy of the graph on the edges of that graph, adjacent when they are disjoint and lie in the same pentagon. Each part is strongly regular. The stabilizer of the partition is $U_3(5){:}2$, where the outer 2 interchanges the two parts of size 50. (See also below under Regular Sets.)



$v = 275.$

Conversely, Γ can be constructed in this setup: let the objects be the 100 15-cocliques and the 175 edges of the Hoffman-Singleton graph. Let two edges be adjacent when they are disjoint and lie in a pentagon. Let two 15-cocliques be adjacent when they meet in 0 or 3 points. Let an edge xy be adjacent to the 15-coclique C when $x, y \notin C$. This yields Γ. See also [455].

Construction: 1 + 112 + 162

Let Γ_1 be the collinearity graph of the unique $GQ(3, 9)$. It contains 648 hemisystems of points, i.e., subgraphs isomorphic to the Gewirtz graph, falling into four equivalence classes ($O_6^-(3)$ orbits, see §10.34). Let Γ_2 be the graph on one equivalence class, two hemisystems being adjacent when their intersection has size 20. Let ∞ be a new vertex. Then Γ is the union of $\{\infty\}$, Γ_1 and Γ_2, with the following additional edges: ∞ is joined to every vertex of Γ_1 and a vertex v_1 of Γ_1 is adjacent to the vertex v_2 of Γ_2 if the hemisystem v_2 contains the point v_1. This construction is due to Cossidente & Penttila [234].

Local graph

Γ is locally the collinearity graph of the unique $\mathsf{GQ}(3,9)$, see §10.34. It is the unique such graph, by Pasechnik [605].

Cliques and cocliques

All 15400 maximal cliques have size 5. The group G is transitive on i-cliques for $0 \leq i \leq 5$. Since in the local graph each edge is contained in a unique 4-clique, here each triangle is contained in a unique 5-clique. The stabilizer of a 5-clique is transitive on the 270 vertices outside.

Maximal cocliques have sizes 7, 10, 11, 13, 16, 22. There are 4050 22-cocliques, forming a single orbit. The stabilizer of one is M_{22}, and has orbits of sizes 22 + 77 + 176, giving rise to the above construction. The group G is transitive on 3-cocliques. Each 3-coclique is contained in eight 22-cocliques.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$3_+^{1+4}{:}4\mathsf{S}_5$	15400	5, 270	4	2
b	$2.\mathsf{S}_8$	22275	35, 240	16	14
c	$\mathsf{U}_3(5){:}2$	7128	100, 175	42	40
d	$5_+^{1+2}{:}24{:}2$	299376	125, 150	52	50

In case (b) the graph induced on the orbit of size 35 is the Odd graph O_4.

In case (c) the graph induced on the orbit of size 175 is the (rank 4) graph on the edges of the Hoffman-Singleton graph, adjacent when disjoint and in the same pentagon, strongly regular with parameters $(v, k, \lambda, \mu) = (175, 72, 20, 36)$ and spectrum $72^1\ 2^{153}\ (-18)^{21}$.

In case (d) the graph induced on the orbit of size 125 is the (rank 5) graph obtained from Taylor's unitary two-graph for $q = 5$ by switching a point isolated (cf. §8.10.1), strongly regular with parameters $(v, k, \lambda, \mu) = (125, 52, 15, 26)$ and spectrum $52^1\ 2^{104}\ (-13)^{20}$.

There are no regular sets with $d - e = s$.

Chromatic number

This graph has chromatic number 15 or 16 (Soicher). There is a partition of $\mathsf{V}\Gamma$ into 55 5-cliques, so that $\chi(\overline{\Gamma}) = 55$ ([385]).

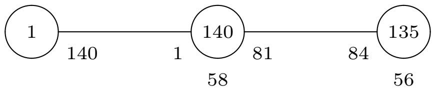
No partial geometry

The parameters of this graph are those of the collinearity graph of a putative $pg(5, 28, 2)$ partial geometry. However, Östergård & Soicher [598] showed that there is no such partial geometry.

Uniqueness and two-graph

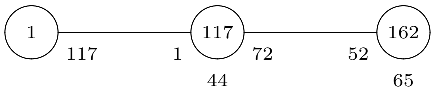
Goethals & Seidel [356] showed the uniqueness (up to complementation) of the (nontrivial) regular two-graph Ω on 276 vertices (cf. §8.10.1). Its automorphism group is Conway's group .3, acting 2-transitively. Our graph Γ is the descendant of Ω at an arbitrary vertex (obtained by switching that vertex isolated). Conversely, Ω is the two-graph that has $K_1 + \Gamma$ in its switching class.

The two-graph Ω also has (many, see [593]) strongly regular graphs in its switching class, with parameters $(v, k, \lambda, \mu) = (276, 140, 58, 84)$ and spectrum $140^1\ 2^{252}\ (-28)^{23}$.



$v = 276$

10.62 The Mathon-Rosa graph



$v = 280$

There is a strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (280, 117, 44, 52)$. Its spectrum is $117^1\ 5^{195}(-13)^{84}$. The full group of automorphisms is S_9 acting rank 5 with point stabilizer $\mathsf{S}_3 \operatorname{wr} \mathsf{S}_3$. This graph was discovered by Mathon & Rosa [550] (and also by Ivanov, Klin & Faradžev [457]).

Construction

The group S_9 acts transitively on the set of 280 partitions of a 9-set into three 3-sets. The point stabilizer is $\mathsf{S}_3 \operatorname{wr} \mathsf{S}_3$, with orbit sizes 1, 27, 36, 54, 162. Two partitions are in one of these relations when their common refinement has 3, 5, 9, 6, 7 parts, respectively. The orbit of size 162 defines $\overline{\Gamma}$.

The eigenmatrix of the 4-class association scheme, with the relations in the given order, is

$$P = \begin{pmatrix} 1 & 27 & 36 & 54 & 162 \\ 1 & 11 & -12 & 6 & -6 \\ 1 & 6 & 8 & -9 & -6 \\ 1 & -3 & 2 & 6 & -6 \\ 1 & -3 & -4 & -6 & 12 \end{pmatrix}.$$

The multiplicities are, in the order of the rows of P: 1, 27, 48, 120, 84.

Maximal cliques and cocliques

The largest cliques have size 10. The largest cocliques have size 28. Both meet the Hoffman bound. Maximal cliques have sizes 5–8 and 10. Maximal cocliques have sizes 8–14, 16, and 28.

The set of all partitions containing a fixed triple is a 10-clique (and there are other 10-cliques as well). Up to isomorphism there is a unique 28-coclique. The group $\mathsf{P\Gamma L}_2(8)$ acts 2-transitively on the set of 28 subgroups of order 3 contained in $\mathsf{PGL}_2(8)$. Each such subgroup determines a partition 3^3, and this set of 28 partitions is a coclique in Γ.

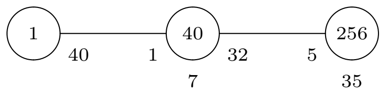
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$\mathsf{L}_2(8):3$	240	28, 252	0	13
b	$2^3:\mathsf{L}_3(2)$	270	112, 168	39	52
c	$2 \times \mathsf{S}_7$	36	70, 210	33	28

Godsil & Meagher [349] starts asking for the largest cocliques in (X, R_2). The Delsarte clique bound is $1 + 36/12 = 4$ and hence the coclique bound is $280/4 = 70$. An example meeting the clique bound is the set of parallel classes in $\mathsf{AG}(2,3)$. Examples meeting the coclique bound are the sets of partitions for which two given elements belong to the same part of the partition (this is case (c) here). Godsil & Newman [350] show that there are no further examples.

10.63 The lines of $\mathsf{U}_5(2)$



$v = 297$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (297, 40, 7, 5)$. Its spectrum is $40^1\ 7^{120}\ (-5)^{176}$. The full group of automorphisms is $\mathsf{P\Gamma U}_5(2)$ acting rank 3 with point stabilizer $2^{4+4}:(3\times\mathsf{A}_5):2$. This is the graph on the totally isotropic lines in the $\mathsf{U}_5(2)$ geometry, adjacent when they meet. This graph carries the structure of a $\mathsf{GQ}(8,4)$.

The maximal cliques have size 9 (a single orbit). These are the sets of 9 t.i. lines on an isotropic point. They reach the Hoffman bound. The largest cocliques have size 29. (In particular, the $\mathsf{U}_5(2)$ geometry does not have spreads.)

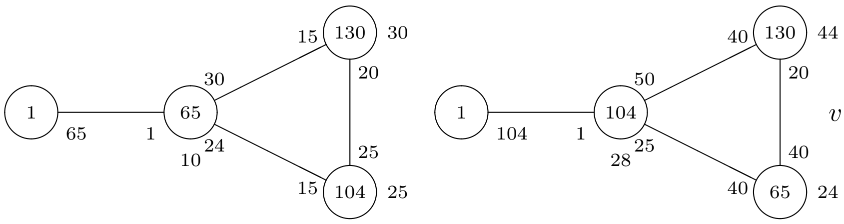
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$[2^{11}.3^4]$	165	9, 288	8	1
b	$\mathsf{U}_4(2):\mathsf{S}_3$	176	27, 270	10	3
c	$3^4:(2\times\mathsf{S}_5)$	1408	135, 162	22	15

These are the sets of vertices on a given isotropic point, in the perp of a given nonisotropic point (i.e., in a $\mathsf{U}_4(2)$ subspace), and in the perp of one point of a given orthogonal base.

10.64 $NO_5^{-\perp}(5)$ and $NO_5^{-}(5)$



$v = 300$

There are unique rank 4 strongly regular graphs Γ and Δ, with parameters $(v, k, \lambda, \mu) = (300, 65, 10, 15)$ and $(300, 104, 28, 40)$, and spectra $65^1\ 5^{195}\ (-10)^{104}$ and $104^1\ 4^{234}\ (-16)^{65}$, respectively. Their full group is $G = \mathsf{PGO}_5(5)$, with point stabilizer $2 \times \mathsf{P\Sigma L}(2, 25)$, giving a 4-class association scheme of which Γ and Δ are two relations. The graphs Γ and Δ are $NO_5^{-\perp}(5)$ and $NO_5^{-}(5)$, see §3.1.5 and §3.1.4, respectively.

The association scheme is that on the (nonisotropic) minus-points of $\mathsf{O}_5(5)$. The non-identity relations are being joined by a secant (Γ), being joined by a tangent (Δ), and being joined by an exterior line. Here the line xy is a secant precisely when x, y are orthogonal.

The local graph Σ of Γ is distance-regular with intersection array $\{10, 6, 4; 1, 2, 5\}$ and is locally Petersen.

Maximal cliques in Γ have size 4 (a single orbit), in Δ size 5 (two orbits). Maximal cocliques in Γ have sizes 11–28, 30 (a single orbit for sizes 27, 28, 30). Maximal cocliques in Δ have sizes 10–15, 17, 18, 20 (one orbit for sizes 18, 20).

The graph Δ satisfies the 4-vertex condition.

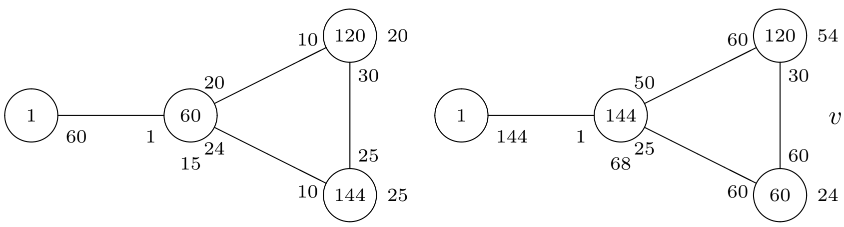
Regular sets

Examples of regular sets in Γ and Δ are obtained from subgroups H of G with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d_Γ	e_Γ	d_Δ	e_Δ	comment
a	$5^3 : (4 \times \mathsf{S}_5)$	156	50, 250	15	10	4	20	isotropic point
b		2340	100, 200	25	20	24	40	
c		3120	150, 150	35	30	44	60	

A t.i. line L determines a partition of the vertex set into 6 parts of size 50 as in case (a). A pair or triple on L determines splits as in (b) or (c).

10.65 $NO_5^{+\perp}(5)$ and $NO_5^{+}(5)$



$v = 325$

There are unique rank 4 strongly regular graphs Γ and Δ, with parameters $(v, k, \lambda, \mu) = (325, 60, 15, 10)$ and $(325, 144, 68, 60)$, and spectra $60^1\ 10^{104}\ (-5)^{220}$ and $144^1\ 14^{90}\ (-6)^{234}$, respectively. Their full group is $G = \mathsf{PGO}_5(5)$, with point stabilizer $2 \cdot (\mathsf{A}_5 \times \mathsf{A}_5).2^2$, giving a 4-class association scheme of which Γ and Δ are two relations. The graphs Γ and Δ are $NO_5^{+\perp}(5)$ and $NO_5^{+}(5)$, see §3.1.5 and §3.1.4, respectively.

The association scheme is that on the (nonisotropic) plus-points of $\mathsf{O}_5(5)$. The non-identity relations are being joined by a secant (Γ), being joined by a tangent (Δ), and being joined by an exterior line. Here the line xy is a secant precisely when x, y are orthogonal.

Maximal cliques in Γ have size 5 (a single orbit), in Δ size 7, 15, or 25 (one orbit each). Maximal cocliques in Γ have sizes 9 (1 orbit), 11–21, and 25 (reaching the Hoffman bound, 7 orbits). Maximal cocliques in Δ have sizes 7–10 and 13 (two orbits).

The graph Δ satisfies the 4-vertex condition.

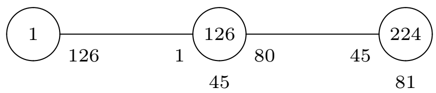
Regular sets

Examples of regular sets in Γ and Δ are obtained from subgroups H of G with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d_Γ	e_Γ	d_Δ	e_Δ	comment
a	$5_+^{1+2}:4\mathsf{S}_5$	156	25, 300	0	5	24	10	t.i. line
b	$5^3:(4 \times \mathsf{S}_5)$	156	75, 250	10	15	44	30	isotropic point

Case (a) are the 25-cliques $L^\perp \setminus L$ in Δ, where L is a t.i. line.

10.66 $NO_7^{-\perp}(3)$



$v = 351$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (351, 126, 45, 45)$. Its spectrum is $126^1\ 9^{168}\ (-9)^{182}$. The full group of automorphisms is $\mathsf{O}_7(3):2$ acting rank 3 with point stabilizer $2.\mathsf{U}_4(3):2^2$.

Construction

This is the graph $NO_7^{-\perp}(3)$, the graph on the 'minus' nonisotropic points in the $\mathsf{O}_7(3)$ geometry—the points that have perps that are elliptic hyperplanes—adjacent when orthogonal, cf. §3.1.4.

Other rank 3 representation

Γ is also the graph on the Hermitian spreads of the generalized hexagon $\mathsf{G}_2(3)$, adjacent when sharing four lines (necessarily the four lines of a regulus on the underlying quadric $\mathsf{O}_7(3)$, or equivalently, four lines at distance one from two give opposite points in $\mathsf{G}_2(3)$). This shows the rank 3 representation of Γ with automorphism group $\mathsf{G}_2(3)$ and point-stabilizer $\mathsf{U}_3(3):3 \simeq \mathsf{G}_2(2)$.

Maximal cliques and cocliques

The maximal cliques all have size 7 and form a single orbit. (They are the orthonormal bases.) The maximal cocliques have sizes 9 (two orbits) or 10, 13, 15 (a single orbit each).

Sub- and supergraphs

The local graph is strongly regular with parameters $(v, k, \lambda, \mu) = (126, 45, 12, 18)$. It is $NO_6^-(3)$ and rank 3, cf. §3.1.3.

This graph is the local graph of $NO_8^+(3)$, cf. §10.78, and that latter graph is the unique connected locally Γ graph (Pasechnik [599]).

This graph is a subgraph of the Fi_{22} graph, cf. §10.90.

This graph is the μ-graph of the Fi_{23} graph, cf. §10.96.

Regular sets

Below a few examples of regular sets in Γ obtained from subgroups H of $\mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

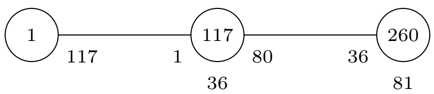
	H	index	orbitlengths	d	e
a	$\mathsf{L}_2(13)$	8398080	78, 273	21	30
b	$2 \times (\mathsf{L}_4(3):2)$	378	117, 234	36	45
c	$3_+^{1+6}:(2\mathsf{S}_4 \times \mathsf{S}_4)$	3640	27, 324	18	9
d	$\mathsf{O}_7(2)$	6318	63, 288	30	21
e	$3^5:(2 \times (\mathsf{O}_5(3):2))$	364	108, 243	45	36

Example (b) induces $NO_6^+(3)$ on the orbit of size 117. That orbit consists of the vertices in $y^\perp$, for a 'plus' nonisotropic point y.

Example (c) induces the 9-coclique extension of $NO_3^{-\perp}(3)$, that is $K_{9,9,9}$, on the orbit of size 27. That orbit consists of the vertices in $L^\perp$, for a t.i. line L.

Example (e) induces the 3-coclique extension of $NO_5^{-\perp}(3)$ on the orbit of size 108. That orbit consists of the vertices in $x^\perp$, for an isotropic point x.

10.67 $NO_7^{+\perp}(3)$



$v = 378$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (378, 117, 36, 36)$. Its spectrum is $117^1\ 9^{182}\ (-9)^{195}$. The full group of automorphisms is $\mathsf{O}_7(3):2$ acting rank 3 with point stabilizer $2 \times (\mathsf{L}_4(3):2)$.

This is the graph $NO_7^{+\perp}(3)$, the graph on the 'plus' nonisotropic points in the $\mathsf{O}_7(3)$ geometry—the points that have perps that are hyperbolic hyperplanes—adjacent when orthogonal, cf. §3.1.4. It is also the graph on the subhexagons of order $(1, 3)$ of the split Cayley hexagon $\mathsf{G}_2(3)$, adjacent if they share exactly 4 lines (and no points). The group $\mathsf{G}_2(3)$ acts rank 4.

The maximal cliques all have size 6 and form a single orbit. The maximal cocliques have sizes 9–15, 21 and 27 (reaching the Hoffman bound), with a single orbit for size 21 and two orbits for size 27.

The local graph is strongly regular with parameters $(v, k, \lambda, \mu) = (117, 36, 15, 9)$. It is $NO_6^+(3)$ and rank 3, cf. §10.35 and §3.1.3.

This graph is the local graph of $NO_8^-(3)$, cf. §10.79, and that latter graph is the unique connected locally Γ graph (Pasechnik [599]).

Regular sets

Below a few examples of regular sets in Γ obtained from subgroups H of $\mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$3^{3+3} : GL_3(3)$	1120	27, 351	0	9
b	$3_+^{1+6} : (2\mathsf{S}_4 \times \mathsf{S}_4)$	3640	54, 324	9	18
c	$3^5 : (2 \times (\mathsf{O}_5(3) : 2))$	364	135, 243	36	45
d	$\mathsf{L}_3(3) : 2$	816480	144, 234	39	48
e	$2^6 : \mathsf{S}_7$	28431	42, 336	21	12
f	$(2.\mathsf{U}_4(3)) : 2^2$	351	126, 252	45	36
g	S_8	227448	168, 210	57	48

Example (a) induces a 27-coclique on the orbit of size 27. That orbit consists of the vertices in $\pi^\perp$, for a t.i. plane π.

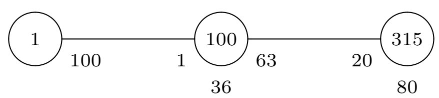
Example (b) induces the 9-coclique extension of $NO_3^{+\perp}(3)$, that is $3K_{9,9}$, on the orbit of size 54. That orbit consists of the vertices in $L^\perp$, for a t.i. line L.

Example (c) induces the 3-coclique extension of $NO_5^{+\perp}(3)$ on the orbit of size 135. That orbit consists of the vertices in $x^\perp$, for an isotropic point x.

Example (e) can be seen as the split between weight 2 and weight 5 vectors for the quadratic form $\sum_{i=1}^{7} x_i^2$.

Example (f) induces $NO_6^-(3)$ on the orbit of size 126. That orbit consists of the vertices in $y^\perp$, for a 'minus' nonisotropic point y.

10.68 The $\mathsf{G}_2(4)$ graph on 416 vertices



$v = 416$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (416, 100, 36, 20)$. Its spectrum is $100^1\ 20^{65}\ (-4)^{350}$. The full group of automorphisms is $\mathsf{G}_2(4).2$ acting rank 3 with point stabilizer $\mathsf{HJ}.2$.

This graph is a member of the Suzuki tower: it is the local graph of the Suzuki graph described below, and its local graph is the Hall-Janko graph.

Construction: $\mathsf{PG}(2, 16)$

Consider the projective plane $\mathsf{PG}(2, 16)$ provided with a nondegenerate Hermitian form. It has 273 points, 65 isotropic and 208 nonisotropic. There are $416 = 208 \cdot 12 \cdot 1/6$ orthogonal bases. These are the vertices of Γ. The group $\mathsf{U}_3(4) : 4$ of semilinear transformations preserving the form acts transitively on the 416 bases, with rank 5. The suborbit sizes are 1, 15, 100, 150, 150. The graph Γ is obtained by taking the suborbit of size 100 for adjacency.

These suborbits can be described geometrically as follows: Given one basis $\{a, b, c\}$, the suborbit of size 15 consists of the bases that have an element in common with $\{a, b, c\}$. The first suborbit of size 150 consists of the bases that are disjoint from $\{a, b, c\}$ but contain a point orthogonal to one of a, b, c. Associated with a basis $\{a, b, c\}$ is the triangle consisting of the 15 isotropic points on the three lines ab, ac, and bc. The suborbits of sizes 1, 15, 100, 150, 150 correspond to bases with triangles having 15, 5, 3, 2, 5 points in common, respectively. Thus, Γ can be described as the graph on the 416 triangles, adjacent when they have 3 points in common ([241]).

Cliques and cocliques

Maximal cliques have size 5 (since the local graph has maximal cliques of size 4). The smallest clique cover has size 84. Maximal cocliques have size 16, which is the Hoffman bound. The chromatic number is $\chi(\Gamma) = 26$.

Suzuki μ-graphs

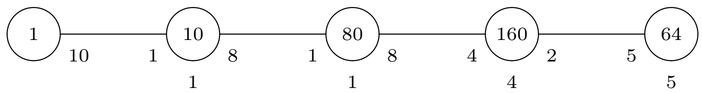
The group G acts imprimitively on the set of 65520 nonedges of Γ, it preserves a partition into 1365 sets of 48 nonedges. Each such set induces a subgraph of size 96 isomorphic to the disjoint union of three copies of the 2-coclique extension of the Clebsch graph. The stabilizer in G of such a subgraph is $2^{2+8} : (3 \times \mathsf{A}_5) : 2$.

If Γ is viewed as the neighborhood of a vertex x in the Suzuki graph Σ, then these 1365 subgraphs of size 96 are the sets of common neighbors of x and y, where y is a nonneighbor of x in Σ.

That these graphs are disconnected can be seen in the triple cover $\widetilde{\Sigma}$ of Σ. It plays a role in the construction of Jenrich's Borsuk example described below.

Jenrich [463] showed that the subgraph on the 320 vertices outside such a μ-graph can be extended by a 16-coclique to construct a strongly regular graph with parameters $(v, k, \lambda, \mu) = (336, 80, 28, 16)$.

Cohen-Tits near octagon



$v = 315$

There is a unique distance-regular graph Δ with intersection array $\{10, 8, 8, 2; 1, 1, 4, 5\}$. It was constructed in Cohen [202], and uniqueness (given the intersection array) was proved in Cohen & Tits [205]. (See also [123], §13.6 and [68].) It has spectrum $10^1\ 5^{36}\ 3^{90}\ (-2)^{160}\ (-5)^{28}$. This graph is the collinearity graph of a near polygon with lines of size 3 which we shall call the Cohen-Tits near octagon. The second subconstituent of Γ is the distance-2 graph of Δ. The 63-sets that are the intersection of $V\Delta$ with a vertex neighborhood in Γ, induce a generalized hexagon $\mathsf{GH}(2, 2)$ in Δ, isomorphic to the dual split Cayley hexagon $\mathsf{G}_2(2)$.

Construction: $\mathsf{G}_2(4)$

By [289], the Cohen-Tits near octagon admits an embedding in the dual, say Ω, of the split Cayley hexagon $\mathsf{G}_2(4)$. Now Γ is the graph on all copies of this near octagon embedded in Ω, adjacent when the intersection contains a copy of the dual split Cayley hexagon $\mathsf{G}_2(2)$. This shows the rank 3 action of $\mathsf{G}_2(4)$ on Γ.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$3.\mathsf{L}_3(4):2^2$	2080	56, 360	10	14	Gewirtz graph
b	$2^{2+8}:(3\times \mathsf{A}_5):2$	1365	96, 320	20	24	Suzuki μ-graph
c	$2^{4+6}:(\mathsf{A}_5\times 3):2$	1365	160, 256	36	40	

Objects of types (b) and (c), incident when the 96-set is contained in the 160-set, are the points and lines of the dual split Cayley hexagon $\mathsf{G}_2(4)$.

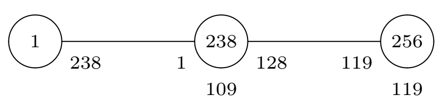
Borsuk conjecture

In 1933 Borsuk [91] asked whether each bounded set in $\mathbb{R}^n$ with nonzero diameter can be divided into $n+1$ parts, each of smaller diameter. In 1993 Kahn & Kalai [475] showed that this is false for $n = 1325$ and $n > 2014$. Various authors brought the smallest counterexample dimension down.

In 2013 Bondarenko [87] observed that the Euclidean representation of the present graph Γ in its θ-eigenspace for $\theta = 20$ yields 416 unit vectors in $\mathbb{R}^{65}$ with mutual inner products $\frac{1}{5}$ (for adjacent vertices) and $-\frac{1}{15}$ (for nonadjacent vertices). The diameter of this set is the distance between the images of two nonadjacent vertices, so that a partition into parts of smaller diameter must correspond to a partition of $V\Gamma$ into cliques. But Γ has clique number 5, so one needs at least (in fact: precisely) $\lceil \frac{416}{5} \rceil = 84$ parts. The argument is general: Given a strongly regular graph Γ with v vertices, where the 2nd largest eigenvalue has multiplicity f, one finds v unit vectors in $\mathbb{R}^f$ such that this set of vectors cannot be partitioned into fewer parts of smaller diameter than the clique covering number of Γ, that is, the chromatic number of its complement.

Today the counterexample with smallest dimension is that found by Jenrich [464] who observed that Bondarenko's 65-dimensional example contains a 64-dimensional example. It is a two-distance set of 352 points. Indeed, let M be a Suzuki μ-graph in Γ, with connected components M_i, $i = 1, 2, 3$. The vector u that is 1 on M_1, -1 on M_2, and 0 elsewhere, lies in the 65-dimensional θ-eigenspace, and is orthogonal to the vectors representing vertices outside $M_1 \cup M_2$. We find 352 unit vectors in the 64-space $u^\perp$, and a partition of this set into parts of smaller diameter needs at least 71 parts.

10.69 The $\mathsf{O}_{10}^-(2)$ graph on 495 vertices



$v = 495$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (495, 238, 109, 119)$. Its spectrum is $238^1\ 7^{340}\ (-17)^{154}$. The full group of automorphisms is $\mathsf{O}_{10}^-(2):2$ acting rank 3 with point stabilizer $2^8:\mathsf{O}_8^-(2):2$. This is the graph on the points of an $\mathsf{O}_{10}^-(2)$ geometry, adjacent when collinear. The graph induced on the nonneighbors of a point is $VO_8^-(2)$ (§10.59).

Construction

In $\mathbb{F}_2^{12}$, consider the quadratic form $Q(x) = \sum_{i<j} x_i x_j$. We have $Q(x) = \binom{\mathrm{wt}(x)}{2}$ and $B(x,y) = \mathrm{wt}(x)\mathrm{wt}(y) - \sum_i x_i y_i$. The space $\mathbf{1}^\top/\langle \mathbf{1} \rangle$ is a 10-dimensional elliptic orthogonal space. The isotropic points are the $495 = \binom{12}{4}$ cosets with a weight 4 representative. Two 4-sets are adjacent when they meet in 0 or 2 points. Thus, this is the distance 2-or-4 graph of the Johnson graph $J(12,4)$.

Cliques and cocliques

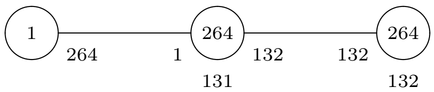
Maximal cliques have size 15 and form a single orbit. They are the maximal totally singular subspaces. Maximal cocliques have sizes 5 and 9, a single orbit each.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d, nexus e, and structure for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$2^{6+8}:(\mathsf{A}_8 \times \mathsf{S}_3)$	25245	15, 480	14	7	t.s. solid
b	$3^5:(2 \times 2^4{:}\mathsf{S}_5)$	53616640	90, 405	49	42	$\mathsf{O}_2^-(2)\,\mathrm{wr}\,5$
c	$\mathsf{S}_3 \times \mathsf{O}_8^+(2):2$	23936	135, 360	70	63	elliptic line
d	S_{11}	1253376	165, 330	84	77	$J(11,3)_2$
e	$\mathsf{L}_2(11):\mathsf{S}_3$	12634030080	165, 330	84	77	
f	$\mathsf{PSp}_8(2) \times 2$	528	240, 255	119	112	nonsg. pt

10.70 The rank 3 conference graphs on 529 vertices



$v = 529$

There are exactly three rank 3 strongly regular graphs with parameters $(v,k,\lambda,\mu) = (529, 264, 131, 132)$ namely the Paley graph $P(q)$, the Peisert graph $P^*(q)$ and the sporadic Peisert graph $P^{**}(q)$, where $q = 23^2$. Their spectrum is $264^1\ 11^{264} (-12)^{264}$. Each has $\mathrm{rk}_{23}(2A + I + bJ) = 144$ for all b.

Each of these graphs is self-complementary. Each has chromatic number 23, so that there are partitions into 23-cliques and partitions into 23-cocliques. The groups of automorphisms are $23^2:S$ where S is the point stabilizer given in the table below. This table also gives for each graph and each m the number of orbits of maximal m-cliques.

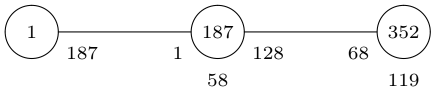
graph	name	S	6	7	8	9	10	11	12	13	23
Γ_1	Paley	$264:2$	-	85	108	80	7	9	-	4	1
Γ_2	Peisert	$11 \times (3:Q_8)$	1	222	442	186	22	1	1	-	1
Γ_3	sporadic Peisert	$11 \times \mathsf{SL}_2(3)$	3	362	448	87	2	1	1	-	1

Construction

Each of these three graphs is a Cayley graph for the additive group of $F = \mathbb{F}_{529}$. For Γ_1 the difference set D consists of the squares in F. For Γ_2, $D = K \cup \omega K$ where K is the subgroup (of size 132) of fourth powers and ω is a primitive element of F. For Γ_3, $D = \bigcup_{i \in I} \omega^i L$ where L is the subgroup (of size 66) of eighth powers and $I = \{0, 1, 3, 5\}$.

These graphs are self-complementary: for Γ_1 the map $d \mapsto \omega d$ interchanges D and its complement, for Γ_2 and Γ_3 the map $d \mapsto \omega^{-1} d^{23}$ works. For edge-transitivity: for Γ_1 one can multiply by a square, for Γ_2 one can multiply by a fourth power and apply $d \mapsto \omega d^{23}$. Finally, for Γ_3 one can multiply by an eighth power and apply the $\mathbb{F}_{23}$-linear transformation that maps 1 to ω and ω to -1.

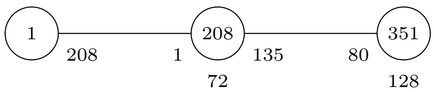
10.71 The $\mathsf{U}_4(2)$ graphs on 540 vertices



$v = 540$

Crnković, Rukavina & Švob [243] constructed two nonisomorphic strongly regular graphs with parameters $(v, k, \lambda, \mu) = (540, 187, 58, 68)$. The spectrum is $187^1\ 7^{374}\ (-17)^{165}$. The full groups of automorphisms are $2 \times (\mathsf{U}_4(2):2)$ and $2 \times \mathsf{U}_4(2)$. These groups act transitively, with ranks 13 and 17. In both cases, adjacency is defined by the union of 7 suborbits. In both cases, maximum cliques have size 12, meeting the Hoffman bound. In both cases, maximum cocliques have size 20.

10.72 The Aut($\mathsf{Sz}(8)$) graph on 560 vertices



$v = 560$

There is a strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (560, 208, 72, 80)$. Its spectrum is $208^1\ 8^{364}\ (-16)^{195}$. The full group of automorphisms is Aut(Sz(8)) acting rank 7 with point stabilizer 13 : 12 and suborbit sizes 1, 39, 52, 78^2, 156^2. The graph Γ is obtained by taking the union of the suborbit of size 52 and one of the two suborbits of size 156. This graph was discovered by Faradžev, Klin & Muzychuk [315].

Maximal cliques and cocliques

The maximal cliques have sizes 4–8. Those of sizes 4 and 8 form single orbits. The maximal cocliques have sizes 9–18 and 21. Those of size 21 form a single orbit.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$2^{3+3} : (7 : 3)$	65	112, 448	48	40

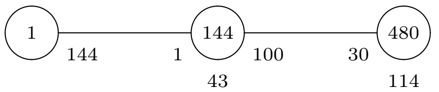
10.73 The rank 3 graphs on 625 vertices

There are precisely seven rank 3 strongly regular graphs on $v = 625$ vertices. The parameters are as follows.

	k	λ	μ	r^f	s^g	group	graph
a	48	23	2	23^{48}	$(-2)^{576}$	S_{25} wr 2	25×25
b	104	3	20	4^{520}	$(-21)^{104}$	$5^4 : 4.\mathsf{P\Gamma L}_2(25)$	$VO_4^-(5)$
c	144	43	30	19^{144}	$(-6)^{480}$	$5^4 : (\mathsf{GL}_2(5) \circ \mathsf{GL}_2(5)) : 2$	$VO_4^+(5)$
d	144	43	30	19^{144}	$(-6)^{480}$	$5^4 : 4.\mathsf{S}_6$	See A below
e	208	63	72	8^{416}	$(-17)^{208}$	$5^4 : 208 : 4$	Van Lint-Schrijver
f	240	95	90	15^{240}	$(-10)^{384}$	$5^4 : 4.(2^4.\mathsf{S}_6)$	See B below
g	312	155	156	12^{312}	$(-13)^{312}$	$5^4 : 312 : 4$	Paley

We discuss the cases of valency 144 or 240 more in detail below.

A. Valency 144



$v = 625$

There are precisely two rank 3 strongly regular graphs with parameters $(v, k, \lambda, \mu) = (625, 144, 43, 30)$. Their spectrum is $144^1\ 19^{144}\ (-6)^{480}$.

Both graphs are found on $\mathbb{F}_5^4$ by taking the union of the six lines of a dual BLT set at infinity: the first graph from the linear set, the second from the FTW set. See §10.47.

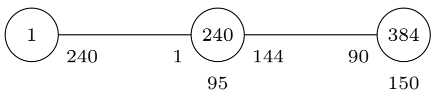
The first is $VO_4^+(5)$, also known as $H_5(2,2)$, also known as the graph found on $\mathbb{F}_{25}^2$ by taking a Baer subline at infinity. See §3.3.1, §3.4.1, §3.4.5. The full group is $5^4 : (\mathsf{GL}_2(5) \circ \mathsf{GL}_2(5)) : 2$.

Let us call the second graph Γ. Its full group of automorphisms is $5^4.4.\mathsf{S}_6$ with point stabilizer $4.\mathsf{S}_6$. It is due to Liebeck [517].

Cliques and cocliques

The maximal cliques in Γ have sizes 6 or 25, a single orbit of each. The maximal cliques of size 25 reach the Hoffman bound and are planes in the underlying vector space $\mathbb{F}_5^4$. Maximum cocliques have size 25 and reach the Hoffman bound. Both planes and non-planes occur.

B. Valency 240



$v = 625$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (625, 240, 95, 90)$. Its spectrum is $240^1\ 15^{240}\ (-10)^{384}$. The full group of automorphisms is $5^4.4.(2^4:\mathsf{S}_6)$ acting rank 3 with point stabilizer $4.(2^4:\mathsf{S}_6)$. Existence and uniqueness is due to LIEBECK [517] (using earlier work by Foulser). It is found on $\mathbb{F}_5^4$ by taking a regular set of $\Gamma(\mathsf{Sp}_4(5))$ of size 60 at infinity. See §10.47.

Construction

Let $V = \mathbb{F}_5^4$ and let H be the group generated by the four matrices

$$\begin{pmatrix} 1 & . & . & . \\ . & 4 & . & . \\ . & . & 4 & . \\ . & . & . & 1 \end{pmatrix}, \begin{pmatrix} . & . & 1 & . \\ . & . & . & 4 \\ 1 & . & . & . \\ . & 4 & . & . \end{pmatrix}, \begin{pmatrix} 4 & . & . & . \\ . & 4 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}, \begin{pmatrix} . & . & . & 4 \\ . & . & 4 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}.$$

Then H has order 32, and $H/\langle -I \rangle$ is elementary abelian of order 2^4. This group $H/\langle -I \rangle$ has 15 orbits of size 4 and 6 orbits of size 16 on PV (which is $\mathsf{PG}(3,5)$, with 156 points). The union X of the 15 orbits of size 4 is a two-character set: each plane of PV meets it in either 10 or 15 points. The graph Γ arises by joining two vectors $u, v \in V$ when $\langle v - u \rangle \in X$.

The set X is covered by 60 lines which, via the Klein correspondence, correspond in $\mathbb{F}_5^6$, provided with the quadratic form $Q(x) = x_1^2 + \cdots + x_6^2$, to the points of shape (000012), where $2^4:\mathsf{S}_6$ acts on the coordinates by permuting them and changing an even number of signs. The same set X is covered by the subset of 20 lines having a zero as last coordinate, and the corresponding group $2^4:\mathsf{S}_5$ acts transitively on X and is a subgroup of the symplectic group $\mathsf{Sp}_4(5)$ acting on the perp of (000001). Hence X is also a regular set (a 10-tight set) of $\mathsf{Sp}_4(5)$, see Case (e) in the second table of §10.47

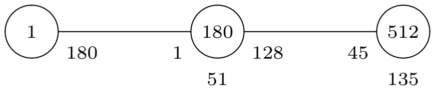
Cliques and cocliques

The maximal cliques in Γ have sizes 6, 8, 9, 17, 25 and those of sizes 9, 17, 25 each form a single orbit. The maximal cocliques in Γ have sizes 9–17 and 25 and those of sizes 16, 25 each form a single orbit. The 750 cliques and 2400 cocliques of size 25 reach the Hoffman bound and are planes in the underlying vector space $\mathbb{F}_5^4$; at infinity they have a line corresponding to a point of shape (000012) and (011111), respectively. Affine solids containing either type of affine planes are examples of regular sets of size 125 with degree 60 and 40, and nexus 45 and 50, respectively.

Cospectral graphs

There are many cospectral graphs, two of which are rank 4. One of these is $VNO_4^+(5)$ (see §3.3.2). The other is derived from the group generated by $x \mapsto x^{-1}$ and $x \mapsto a^3x$ on $\mathbb{F}_{25} \cup \{\infty\}$, where a is primitive in $\mathbb{F}_{25}^*$. It has orbits of sizes 2, 8, 16 on $\mathsf{PG}(1,25)$ and gives rise to a partition of the complete graph on $\mathbb{F}_{25}^2$ into three strongly regular graphs of valencies 48, 192, 384.

10.74 The $\mathsf{U}_6(2)$ graph on 693 vertices



$v = 693$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (693, 180, 51, 45)$. Its spectrum is $180^1\ 15^{252}\ (-9)^{440}$. The full group of automorphisms is $\mathsf{U}_6(2).\mathsf{S}_3$ acting rank 3 with point stabilizer $2_+^{1+8}:(\mathsf{U}_4(2)\times 3):2$. It is the collinearity graph of the $U_6(2)$ polar space (§2.7).

Maximal cliques have size 21, and are the maximal t.i. subspaces.

The smallest maximal cocliques have size 7 (a single orbit, with stabilizer $2 \times \mathsf{S}_7$).

These can be seen by viewing $\mathsf{U}_6(2)$ as $\mathbf{1}^\perp$ in $\mathsf{U}_7(2)$ defined by the form $\sum_{i=1}^7 x_i^3$ in $\mathbb{F}_4^7$. It contains the maximal coclique $\{\langle \mathbf{1} - e_i\rangle \mid 1 \le i \le 7\}$.

The largest maximal cocliques have size 27 (a single orbit, with stabilizer $[3^6]:2^2$). In particular, $U_6(2)$ does not contain an ovoid.

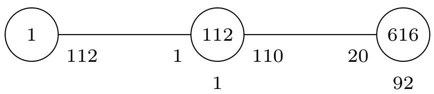
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e	comment
a	$(\mathsf{U}_5(2)\times 3):2$	672	165, 528	36	45	noniso. pt
b	$2^9:\mathsf{L}_3(4):\mathsf{S}_3$	891	21, 672	20	5	t.i. plane
c	$\mathsf{O}_7(2)\times 2$	19008	63, 630	30	15	$\mathsf{Sp}_6(2)$
d	$\mathsf{U}_4(3):2^2$	4224	126, 567	45	30	$NO_6^-(3)$
e	$\mathsf{M}_{22}:2$	62208	231, 462	70	55	

The action of $M_{22}:2$ is rank 4 on the 231 pairs of a 22-set. The graph of valency 70 is the union of the triangular graph $T(22)$ (valency 40) and the Cameron graph (valency 30). This situation arises in the Fi_{22} graph (§10.90), as the neighborhood of a point far from a 22-clique.

10.75 The Games graph



$v = 729$

There is a unique strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (729, 112, 1, 20)$. Its spectrum is $112^1\ 4^{616}\ (-23)^{112}$. The full group of automorphisms is $3^6.2.\mathsf{L}_3(4).2$ acting rank 4 with point stabilizer $2.\mathsf{L}_3(4).2$. Existence is due to Games [332]. Uniqueness to Bondarenko & Radchenko [90].

Construction

There is a unique 56-cap in $\mathsf{PG}(5,3)$ (that is, a set of 56 points, no three on a line) known as the *Hill cap*, see Hill [427, 429]. Take as vertices the points of $AG(6,3)$, adjacent when the connecting line hits the Hill cap at infinity.

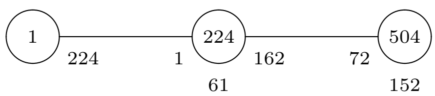
Calderbank & Kantor [169] give the following explicit construction. Let $e_\infty, e_0, e_1, e_2, e_3, e_4$ be a basis of $\mathbb{F}_3^6$. A group $2^5 : \mathsf{L}_2(5)$ acts: the elements of 2^5 are diagonal transformations $\mathrm{diag}(\pm 1, \ldots, \pm 1)$ of determinant 1, and $\mathsf{L}_2(5)$ acts by permuting the coordinates. Under this group the orbit of $\langle(111000)\rangle$ has size 40, and that of $\langle(111111)\rangle$ has size 16 in $\mathsf{PG}(5,3)$. The union of these two orbits is the Hill cap. It is contained in the elliptic quadric $\sum x_i^2 = 0$.

The stabilizer of a vertex x has orbit lengths $1 + 112 + 112 + 504$, where the second orbit of size 112 consists of the vertices z such that the line $\langle x, z\rangle$ hits the elliptic quadric outside the Hill cap.

Since $\lambda = 1$, this graph satisfies the 4-vertex condition.

The regular sets in Γ that arise as the smallest orbit of a subgroup of $\mathrm{Aut}\,\Gamma$ with two orbits on the vertex set are sets with size u, degree d, and nexus e, where $(u, d, e) = (81, 16, 12)$, $(243, 22, 45)$, $(243, 40, 36)$.

10.76 $VO_6^-(3)$



$v = 729$

The graph $VO_6^-(3)$ is the unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (729, 224, 61, 72)$. Its spectrum is $224^1\ 8^{504}\ (-19)^{224}$. The full group of automorphisms is $G = 3^6 : 2.\mathsf{U}_4(3) : \mathsf{D}_8$ acting rank 3 with point stabilizer $2.\mathsf{U}_4(3) : \mathsf{D}_8$. The group G has subgroups $3^6 : \mathsf{U}_3(3) : 4$ and $H = 3^6 : 2.\mathsf{L}_3(4).2$ that also act rank 3.

The group H is not isomorphic to the automorphism group of the Games graph, but has a subgroup $3^6 : 2.\mathsf{L}_3(4)$ for which the edges of Γ split into the edge-disjoint union of two copies of the Games graph.

The maximal cliques in Γ have size 9 (a single orbit, stabilized by $3^2 < 3^6$). The maximal cocliques in Γ have sizes 7 (a single orbit, with stabilizer S_7), 9–19, 22, and 27 (a single orbit, with stabilizer $GO_5(3)$).

In the representation of Γ on the affine hyperplane $\sum x_i = 1$ in $\mathbb{F}_3^7$, with $u \sim v$ when $Q(v - u) = 0$ for $Q(x) = \sum x_i^2$, the set of unit vectors is a maximal 7-coclique.

Among the regular sets in Γ that arise as an orbit of a subgroup of $\mathrm{Aut}\,\Gamma$ with two orbits on the vertex set are sets with size u, degree d, and nexus e, where $e = d + 19$ and $(u, d) = (81, 8)$, $(243, 62)$, $(324, 89)$, and where $e = d - 8$ and $(u, d) = (81, 32), (243, 80)$.

Let the *tensor product* of two graphs with adjacency matrices A and B be the graph with adjacency matrix $A \otimes B$. Then Γ contains regular sets with $(u, d, e) = (81, 32, 24)$ that induce $K_3 \otimes \Sigma$ where Σ is the Schläfli graph, and each 27-coclique is contained in a unique such regular set.

10.77 The rank 3 graphs on 961 vertices

There are precisely five rank 3 strongly regular graphs on $v = 961$ vertices. The parameters are as follows.

	v	k	λ	μ	r^f	s^g	group	graph
a	961	60	29	2	29^{60}	$(-2)^{900}$	S_{31} wr 2	31×31
b	961	240	71	56	23^{240}	$(-8)^{720}$	$31^2:30.\mathsf{S}_4$	
c	961	360	139	132	19^{360}	$(-12)^{600}$	$31^2:30.\mathsf{A}_5$	
d	961	480	239	240	15^{480}	$(-16)^{480}$	$31^2:240:2$	Peisert
e	961	480	239	240	15^{480}	$(-16)^{480}$	$31^2:480:2$	Paley

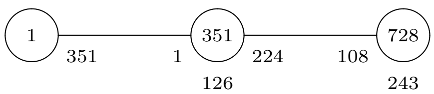
In cases (b)–(e) the group has shape $G = 31^2 : S$ with $Z(S) \simeq 30$.

Case (b) is from an action of S_4 on $\mathsf{PG}(1, 31)$ with orbits of sizes 8, 24.

Case (c) is from an action of A_5 on $\mathsf{PG}(1, 31)$ with orbits of sizes 12, 20. Cf. §7.5.

The graph on $\mathbb{F}_{961}$ where two elements are adjacent when they differ by a 4th power, is strongly regular with parameters $(v, k, \lambda, \mu) = (961, 240, 71, 56)$ and has a rank 4 group.

10.78 $NO_8^+(3)$



$v = 1080$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (1080, 351, 126, 108)$. Its spectrum is 351^1 27^{260} $(-9)^{819}$. The full group of automorphisms is $\mathsf{PGO}_8^+(3) = \mathsf{O}_8^+(3):2^2$ acting rank 3 with point stabilizer $2 \times (\mathsf{O}_7(3):2)$. The local graph is the $NO_7^{-\perp}(3)$ graph described in §10.66.

Construction: nonisotropic points in the $\mathsf{O}_8^+(3)$ geometry

This is the graph on one orbit of nonisotropic points in the $\mathsf{O}_8^+(3)$ geometry, adjacent when orthogonal, i.e., when joined by an elliptic line, cf. §3.1.3.

Construction: split Cayley hexagons on $\mathsf{O}_7(3)$

There are 2160 standard representations of $\mathsf{G}_2(3)$ on the $\mathsf{O}_7(3)$ polar space. The group $\mathsf{SO}_7(3)$ acts transitively on that set, but its index 2 subgroup $\mathsf{O}_7(3)$ acts with two orbits of length 1080. Then Γ is the graph on either of these orbits, two representations being adjacent when their line sets share exactly 28 lines (the 28 lines of a Hermitian spread in both), cf. §4.8.

Cliques, cocliques and chromatic number

The maximal cliques in Γ have size 8 and form a single orbit. They have stabilizer $(2^8 : \mathsf{A}_8) : 2^2$. For the quadratic form $q(x) = \sum_i x_i^2$, an 8-clique is given by $\{e_i \mid 1 \le i \le 8\}$.

Maximal cocliques have sizes 9–15, 18, 21 and 27. There are two orbits of 27-cocliques, reaching the Hoffman bound. One type is that of the sets $C(\pi)$ of vertices contained in $\pi^\perp$, where π is a totally isotropic plane. See §3.1.3.

Γ has chromatic number 40. A partition of the vertex set into 40 sets $C(\pi)$ is obtained by taking the 40 planes π in a totally isotropic solid.

Regular sets

Below a few examples of regular sets in Γ obtained from subgroups H of Aut Γ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$([3^6]:2):(3^3:\mathsf{GL}_3(3))$	44800	27, 1053	0	9
b	$[2^{10}.3^{12}]$	36400	108, 972	27	36
c	$[2^8.3^8]$	11793600	216, 864	63	72
d	$3^6:(2\times(\mathsf{PSL}_4(3):2))$	1120	351, 729	108	117
e	$2\times(\mathsf{O}_7(3):2)$	1080	378, 702	117	126
f	$[2^7.3^8]$	23587200	432, 648	135	144
g	$3:((3^4:\mathsf{S}_5)\times\mathsf{S}_3)$	113218560	540, 540	171	180
h	$\mathsf{O}_8^+(2):2$	56862	120, 960	63	36
i	S_9	54587520	240, 840	99	72
j	$(2.\mathsf{PSL}_3(4)):2^2$	122821920	240, 840	99	72
k	$(\mathsf{A}_6:\mathsf{S}_6):2^2$	19105632	360, 720	135	108
l	$3:(\mathsf{S}_6\times\mathsf{S}_3)$	1528450560	360, 720	135	108

Example (a) induces a 27-coclique on the orbit of size 27. That orbit consists of the vertices in $\pi^\perp$, for a t.s. plane π.

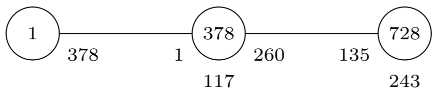
Example (b) induces $3K_{4\times 9}$ on the orbit of size 108. That orbit consists of the vertices in $L^\perp$, for a t.s. line L.

Example (d) induces the 3-coclique extension of $NO_6^+(3)$ on the orbit of size 351. That orbit consists of the vertices in $x^\perp$, for a singular point x.

Example (e) induces the rank 3 graph $NO_7^{+\perp}(3)$ (with parameters $(378, 117, 36, 36)$) on the orbit of size 378. That orbit consists of the vertices in $y^\perp$ for a nonsingular non-vertex point y.

Example (h) induces the rank 3 graph $NO_8^+(2)$ (see §10.39) on the orbit of size 120.

10.79 $NO_8^-(3)$



$v = 1107$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (1107, 378, 117, 135)$. Its spectrum is $378^1\ 9^{819}\ (-27)^{287}$. The full group of automorphisms is $\mathsf{PGO}_8^-(3) = O_8^-(3):2$ acting rank 3 with point stabilizer $2\times(\mathsf{O}_7(3):2)$. The local graph is the $NO_7^{+\perp}(3)$ graph described in §10.67.

Construction: nonisotropic points in the $\mathsf{O}_8^-(3)$ geometry

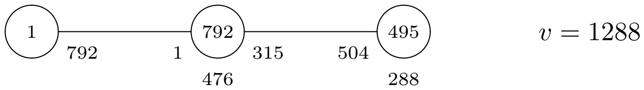
This is the graph on one orbit of nonisotropic points in the $\mathsf{O}_8^-(3)$ geometry, adjacent when orthogonal, i.e., when joined by an elliptic line, cf. §3.1.3.

Cliques and cocliques

The maximal cliques in Γ have size 7 and form a single orbit. They have stabilizer $2\times(2^6:\mathsf{S}_7)$. For the quadratic form $q(x) = \sum_{i=1}^{7} x_i^2 + 2x_8^2$, a 7-clique is given by $\{e_i \mid 1 \le i \le 7\}$.

Maximal cocliques have sizes 13–21, 23, 27, 30, 33 and 45 (with unique orbits for sizes 20, 21, 23, 30, 33, 45). A 45-coclique is found from a 5-coclique (ovoid) in $L^\perp/L$ where L is a t.s. line. The cocliques of sizes 27, 30, 33, 45 are found from cocliques of size 9, 10, 11, 15 in $x^\perp/\langle x\rangle$.

10.80 The dodecad graph



There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (1288, 792, 476, 504)$. Its spectrum is $792^1\ 8^{1035}\ (-36)^{252}$. Its complement has parameters $(v, k, \lambda, \mu) = (1288, 495, 206, 180)$ and spectrum $495^1\ 35^{252}(-9)^{1035}$. The full group of automorphisms is M_{24} acting rank 3 with point stabilizer $\mathsf{M}_{12} : 2$. This graph is the local graph of the $2^{11}.\mathsf{M}_{24}$ graph of valency 1288 on 2048 vertices.

Construction

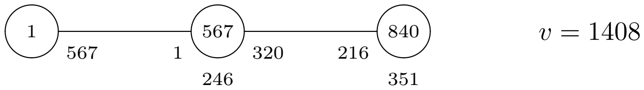
Let C be the extended binary Golay code. It has 2576 words of weight 12 (dodecads), so 1288 complementary pairs of dodecads. Given one dodecad, there are 1, 495, 1584, 495, 1 dodecads at distance 0, 8, 12, 16, 24, respectively. Given one complementary pair of dodecads, there are 1, 495, 792 such pairs at distance 0, 8, 12, respectively. The graph Γ is obtained if we call two dodecad pairs adjacent if they have distance 12.

Cliques and cocliques

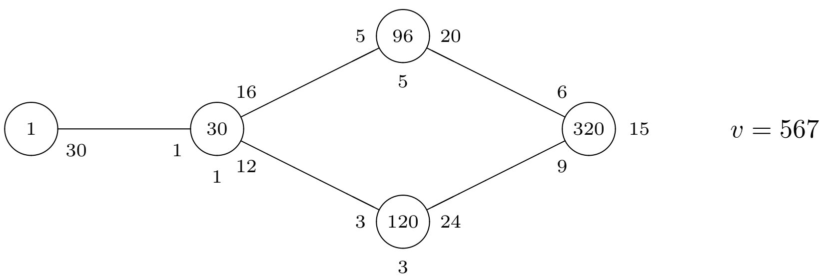
Maximum cliques have size 23 (since the $2^{11}.\mathsf{M}_{24}$ graph of valency 1288 on 2048 vertices has maximum cliques of size 24).

Maximal cocliques have sizes 9–14, 16, 24. There is a unique orbit of 24-cocliques (of size $26565 = 759 \cdot 35$). Given an octad B and a partition $\{S, T\}$ of B into two 4-sets, one finds a 24-coclique by taking all dodecad pairs that meet B precisely in $\{S, T\}$.

10.81 The Conway graph on 1408 vertices



There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (1408, 567, 246, 216)$. Its spectrum is $567^1\ 39^{252}\ (-9)^{1155}$. The full group of automorphisms is $\mathsf{U}_6(2).2$, acting rank 3 with point stabilizer $\mathsf{U}_4(3).2^2$. The local graph is the distance 1-or-2 graph of the Aschbacher near hexagon, cf. [14], [122].



Construction

This is the graph on the lines on a fixed point in the Fi_{22} Fischer space, adjacent when they span a dual affine plane. It follows that the complementary graph is the collinearity graph of a partial linear space with lines of size 4.

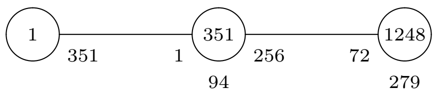
Cliques and cocliques

Maximal cliques have sizes 8, 10–14, 16, 17, 22, 28, 32. There is a unique orbit of 32-cliques. An example is obtained by taking a vertex x and a point neighborhood $p^{\perp}$ (of size $1 + 30$) in the near hexagon of which $\Gamma(x)$ is the distance 1-or-2 graph. Maximal cocliques have sizes 4, 8, 9, 11.

Supergraphs

This graph is the 2nd subconstituent of the Conway graph on 2300 vertices, (§10.88). It occurs as subgraph in the Fi_{22} graph on 14080 vertices (§10.94).

10.82 The Tits graph on 1600 vertices



$v = 1600$

There is a rank 4 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (1600, 351, 94, 72)$ that has as full group of automorphisms the simple Tits group $^2\mathsf{F}_4(2)'$ with point stabilizer $\mathsf{L}_3(3):2$ and suborbit lengths $1 + 351 + 312 + 936$. Its spectrum is $351^1\ 31^{351}\ (-9)^{1248}$. This graph was found by SAOUTER [635]. It is a subgraph of the Fi_{22} graph on 14080 vertices.

Maximal cliques in Γ have sizes 6 (2 orbits), 7, 8 (2 orbits), 10, 12, 16. The independence number $\alpha(\Gamma)$ satisfies $37 \le \alpha(\Gamma) \le 40$.

Regular sets

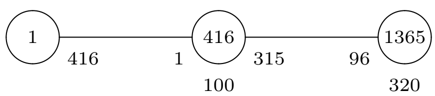
Examples of regular sets in Γ are obtained from subgroups H of $\mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$[2^9]:5:4$	1755	320, 1280	95	64

Generalized octagon

The 1755 subgraphs of size 320 and valency 95 found above are the points of a generalized octagon $\mathsf{GO}(2,4)$ (see p. 346). Two such 320-sets have distance 0, 1, 2, 3, 4 in this $\mathsf{GO}(2,4)$ when they have 320, 192, 96, 64, 60 vertices in common. The lines of $\mathsf{GO}(2,4)$ are triples of points on a common 192-set.

10.83 The Suzuki graph



$v = 1782$

There is a unique rank 3 strongly regular graph Σ with parameters $(v, k, \lambda, \mu) = (1782, 416, 100, 96)$. Its spectrum is $416^1\ 20^{780}\ (-16)^{1001}$. The full group of automorphisms is $\mathsf{Suz}.2$ acting rank 3 with point stabilizer $\mathsf{G}_2(4).2$.

This is the largest member of the *Suzuki tower*: the local graph is the $\mathsf{G}_2(4)$ graph on 416 vertices (§10.68), the local graph of that is the Hall-Janko graph on 100 vertices (§10.32), and the local graph of that is the $\mathsf{G}_2(2)$ graph on 36 vertices (§10.14). All are rank 3 strongly regular graphs.

For a combinatorial construction, see [133].

Cliques

Maximal cliques have size 6 (since the local graph has maximal cliques of size 5) and form a single orbit. The stabilizer is a nonmaximal $\mathsf{S}_6 \times \mathsf{S}_3$.

Each 6-clique K_6 determines a unique subgraph $3K_6$ stabilized by $3:(\mathsf{S}_6 \times \mathsf{S}_3)$. Each $3K_6$ determines a unique graph on 36 vertices of valency 20, union of two copies of $3K_6$, stabilized by $\mathsf{A}_6:((\mathsf{S}_3 \times \mathsf{S}_3):2)$, with a unique partition into six 6-cocliques. Each such graph determines a unique graph on 72 vertices of valency 26, union of two of the preceding, stabilized by a maximal subgroup of shape $((3^2:8) \times \mathsf{A}_6).2$ with vertex orbit sizes $72 + 270 + 1440$.

Cocliques

There is a single orbit of cocliques of size 66, reaching the Hoffman bound, see [133].

The stabilizer of one is $\mathsf{U}_3(4):4$ with vertex orbit sizes $1 + 65 + 416 + 1300$. The 1716 points outside a 66-coclique all have 16 neighbors inside, and we find a 3-(66,16,21) design. The smallest maximal cocliques have size 6 and form a single orbit stabilized by $2^{4+6}:3\mathsf{S}_6$ with vertex orbit sizes $6 + 240 + 1536$.

Nonedges and $K_{5\times 4}$ subgraphs

The stabilizer of a nonedge is $2^{2+8}:(\mathsf{S}_5 \times \mathsf{S}_3)$ with orbit sizes $2 + 20 + 96 + 640 + 1024$. The 96-orbit is the μ-graph. The graph induced on the 20-orbit is $K_{5\times 4}$, pointwise stabilized by a subgroup 2^2 of $\mathrm{Aut}\,\Sigma$. Each vertex of the 1024-orbit has five neighbors in the $K_{5\times 4}$, forming a clique. Each vertex of the 640-orbit has three neighbors in the $K_{5\times 4}$, forming a coclique.

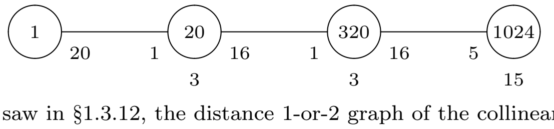
$K_{6,6}$ subgraphs

There is a single orbit of $K_{6,6}$ subgraphs. The stabilizer of one is $(\mathsf{A}_6:2_2 \times \mathsf{A}_5).2$ with vertex orbit sizes $12 + 150 + 720 + 900$.

Nonincidence graphs of $\mathsf{PG}(2, 4)$

The stabilizer of a 4-clique is $(\mathsf{L}_3(2):2) \times \mathsf{S}_4$. It is nonmaximal, contained in a maximal subgroup of shape $(\mathsf{A}_4 \times \mathsf{L}_3(4):2_3):2$ with orbit sizes $42 + 480 + 1260$. The graph induced on the 42-orbit A is the diameter 3 bipartite point-line nonincidence graph of $\mathsf{PG}(2, 4)$. Each vertex of the 480-orbit is adjacent to 14 vertices of A, and these induce the diameter 3 bipartite point-line nonincidence graph of $\mathsf{PG}(2, 2)$. The pointwise stabilizer of A is A_4 of order 12.

Second subconstituent



$v = 1365$

As we saw in §1.3.12, the distance 1-or-2 graph of the collinearity graph of a generalized hexagon of order s is strongly regular. For $s = 4$ this yields a strongly regular graph with parameters $(v, k, \lambda, \mu) = (1365, 340, 83, 85)$. Let Γ be the 2nd subconstituent of Σ. Then Γ

is the distance-2 graph of the $\mathsf{GH}(4,4)$ of which the distance 1-or-2 graph has automorphism group $\mathsf{G}_2(4).2$. The distance 1-or-2 graph of the dual $\mathsf{GH}(4,4)$ is the $\mathsf{Sp}_6(4)$ strongly regular graph.

Regular sets

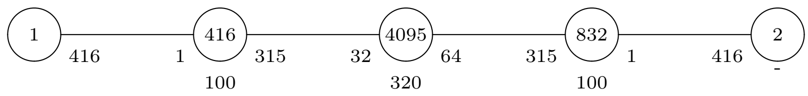
Examples of regular sets in Σ are obtained from subgroups H of $G = \mathrm{Aut}\,\Sigma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$\mathsf{U}_3(4){:}4$	3592512	$1+65$, $416+1300$	0	16
b	$\mathsf{M}_{12}.2 \times 2$	2358720	792, 990	176	192
c	$2_-^{1+6}.\mathsf{U}_4(2).2$	135135	54, 1728	32	12
d	$3_2.\mathsf{U}_4(3).(2^2)_{133}$	22880	162, 1620	56	36
e	$3^{2+4} : 2(\mathsf{S}_4 \times \mathsf{D}_8)$	3203200	324, 1458	92	72

In case (a) the group H has 4 orbits. In case (c) the subgraph induced on the short orbit is the 2-coclique extension of the Schläfli graph (§10.10). In case (d) the subgraph induced on the short orbit is the $\mathsf{U}_4(3)$ graph on 162 vertices (§10.48). Here H is the normalizer of a 3A element with 162 fixed points; this element permutes the nonfixed points in 540 triangles. In particular the long orbit is partitioned into triples of points with the same neighbors in the short orbit.

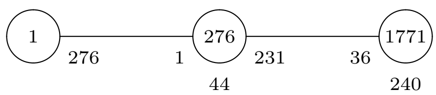
Triple cover

Soicher [664] showed that Σ has a distance-transitive triple cover $\widetilde{\Sigma}$ with diagram



on 5346 vertices. Pasechnik [601] showed that Σ and $\widetilde{\Sigma}$ are the only two locally Γ graphs, if Γ is the $\mathsf{G}_2(4)$ graph on 416 vertices.

10.84 $2^{11}.\mathsf{M}_{24}$ on 2048 vertices with valency 276



$v = 2048$

There is a unique rank 3 strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (2048, 276, 44, 36)$. Its spectrum is $276^1\ 20^{759}\ (-12)^{1288}$. The full group of automorphisms is $2^{11}.\mathsf{M}_{24}$ acting rank 3 with point stabilizer M_{24}.

Construction

Let C be the extended binary Golay code. Take the 2048 cosets of C in the 23-space of even weight vectors of length 24, and call two cosets adjacent when they have distance 2.

This is the Delsarte dual of the valency 759 graph below.

Cliques

The local graph of Γ is the triangular graph $T(24)$. The maximal cliques fall into two orbits, that of cliques of size 24, which is the Hoffman bound, and that of maximal cliques of size 4. For example, the 24 cosets C and $C + e_1 + e_i$ $(2 \le i \le 24)$ form a clique.

Cocliques

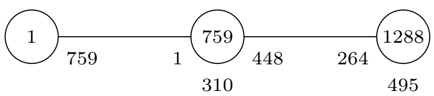
The independence number $\alpha(\Gamma)$ satisfies $72 \le \alpha(\Gamma) \le 84$.

Since Γ is the distance 1-or-2 graph of the coset graph of the perfect binary Golay code C, a coclique D is equivalent to a binary code of word length 23, size $|D| \cdot 2^{12}$ and minimum distance 3 that is a union of cosets of C. Four nonequivalent examples with $|D| = 72$ were found by MOGILNYKH [755], Krotov, JENRICH [752] and Brouwer.

Regular sets

Among the regular sets in Γ that arise as an orbit of a subgroup of Aut Γ with two orbits on the vertex set are sets with size u, degree d, and nexus e, where $e = d + 12$ and $(u, d) = (256, 24)$, $(512, 60)$, $(1024, 132)$, and where $e = d - 20$ and $(u, d) = (24, 23)$, $(128, 36)$, $(256, 52)$, $(512, 84)$, $(768, 116)$, $(1024, 148)$.

10.85 $2^{11}.\mathsf{M}_{24}$ on 2048 vertices with valency 759



$v = 2048$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (2048, 759, 310, 264)$. Its spectrum is $759^1\ 55^{276}\ (-9)^{1771}$. Its complement $\bar{\Gamma}$ has parameters $(2048, 1288, 792, 840)$ and spectrum $1288^1\ 8^{1771}\ (-56)^{276}$. The full group of automorphisms is $2^{11}.\mathsf{M}_{24}$ acting rank 3 with point stabilizer M_{24}. (Note that this $2^{11}.\mathsf{M}_{24}$ is not isomorphic to the $2^{11}.\mathsf{M}_{24}$ encountered for the valency 276 graph above.) This graph was found by GOETHALS & SEIDEL [355].

Construction

Let C be the extended binary Golay code. Take the 2048 cosets of $\{\mathbf{0}, \mathbf{1}\}$ in C, and call two cosets adjacent when they have distance 8.

This graph is the Delsarte dual of the valency 276 graph above.

Subconstituents

The local graph of Γ is the distance 1-or-2 graph of the near polygon on the blocks of $S(5, 8, 24)$ (cf. §6.2.3). See also CUYPERS [247].

The 2nd subconstituent of Γ is the complement of the dodecad graph (§10.80).

Cliques

The maximal cliques fall into 19 orbits, with maximal cliques having sizes 12, 13, 14, 16, 17, 22, 32. There are unique orbits of 32-cliques and 22-cliques. The stabilizer of a 32-clique is $2^{5+4}.\mathsf{A}_8$ and has orbit lengths 32, 896, 1120. The maximal 31-cliques in the local graph are the point neighborhoods in the local near polygon. The stabilizer of a 22-clique is $\mathsf{M}_{21}.\mathsf{S}_3$ and has orbit lengths 1, 21, 168, 210, 280, 360, 1008. The maximal 21-cliques in the local graph corresponding to the fixed point are the sets of octads on a given triple of symbols.

Cocliques and 5-designs

The maximum-size cocliques were determined in Horiguchi et al. [442]. These have size 24, which is the Hoffman bound, and fall into two orbits. (If we arbitrarily pick representatives from the 24 cosets in a 24-coclique containing the zero coset, and replace 0's by -1's, we get a Hadamard matrix of order 24 that spans the code C. Up to equivalence, there are two such Hadamard matrices.) The stabilizers of the two 24-cocliques in $2^{11}.\mathsf{M}_{24}$ are $\mathsf{PSL}_2(23)$ and $2 \times \mathsf{PGL}_2(11)$, respectively, both acting transitively on the coclique.

Since the Hoffman bound holds with equality, each point outside a 24-coclique X is adjacent to 9 points inside. It is shown in *loc. cit.* that in both cases the 2024 blocks obtained in this way form a 5-(24,9,6) design. These two 5-designs are the supports of the words of minimum weight of the ternary quadratic residue codes (see [18]) and Pless symmetry codes (see [620]), both with parameters $[24, 12, 9]_3$. In the former case the group acts transitively on the set of blocks. In the latter case there are three orbits, of sizes $264 + 440 + 1320$.

Construction: 24 + 2024

In both 5-(24,9,6) designs, any block meets n_i blocks in i points, with $(n_i)_{0 \le i \le 9} = (25, 0, 540, 480, 648, 270, 60, 0, 0, 1)$. If we start with one of these two designs $(X, \mathscr{B})$, and call two blocks adjacent when they meet in 3 or 5 points and a point adjacent to a block when it is in the block, we obtain the graph Γ again.

Regular sets

Among the regular sets in Γ that arise as an orbit of a subgroup of $\mathrm{Aut}\,\Gamma$ with two orbits on the vertex set are sets with size u, degree d, and nexus e, where $e = d + 9$ and $(u, d) = (24, 0)$, $(128, 39)$, $(256, 87)$, $(288, 99)$, $(512, 183)$, $(768, 279)$, $(1024, 375)$, and where $e = d - 55$ and $(u, d) = (256, 143)$, $(512, 231)$, $(1024, 407)$.

10.86 The rank 3 graphs on 2209 vertices

There are precisely five rank 3 strongly regular graphs on $v = 2209$ vertices. The parameters are as follows.

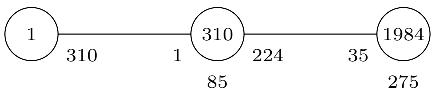
	v	k	λ	μ	r^f	s^g	group	graph
a	2209	92	45	2	45^{92}	$(-2)^{2116}$	$\mathsf{S}_{47}\,\mathrm{wr}\,2$	47×47
b	2209	736	255	240	31^{736}	$(-16)^{1472}$	$47^2:736:2$	cubes
c	2209	1104	551	552	23^{1104}	$(-24)^{1104}$	$47^2:46.\mathsf{S}_4$	
d	2209	1104	551	552	23^{1104}	$(-24)^{1104}$	$47^2:552:2$	Peisert
e	2209	1104	551	552	23^{1104}	$(-24)^{1104}$	$47^2:1104:2$	Paley

In cases (b)–(e) the group has shape $G = 47^2 : S$ with $Z(S) \simeq 46$.

Graph (c) has half-case parameters, but is not self-complementary.

Cases (c), (d), (e) correspond to an action of S_4, D_{24} and D_{48} on $\mathsf{PG}(1, 47)$ with two orbits of size 24.

10.87 $\mathsf{D}_{5,5}(2)$



$v = 2295$

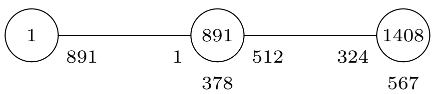
There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (2295, 310, 85, 35)$. Its spectrum is $310^1\ 55^{186}\ (-5)^{2108}$. The full group of automorphisms is $\mathsf{O}^+_{10}(2)$ acting rank 3 with point stabilizer $2^{10} : \mathsf{L}_5(2)$.

This graph is the graph $\mathsf{D}_{5,5}(2)$ of the t.s. 5-spaces of one kind in the polar geometry $\mathsf{O}^+_{10}(2)$, cf. §3.2.3. The local graph is the 2-clique extension of the Grassmann graph $\mathsf{A}_{4,2}(2)$ (a.k.a. $J_2(5, 2)$) of the lines in $\mathsf{PG}(4, 2)$. The μ-graphs are Grassmann graphs $\mathsf{A}_{3,2}(2)$ (a.k.a. $J_2(4, 2)$) of the lines in $\mathsf{PG}(3, 2)$, adjacent when intersecting, cf. §10.13.

The maximal cliques have sizes 15 or 31 (one orbit each) and are the shadows of objects of types 2 or 4. There are maximal cocliques of size 33. It is not known whether Γ contains larger cocliques. The Hoffman bound is 36.

The set of vertices containing any fixed point is a regular set of size 135, degree 70 and nexus 15 stabilized by $2^8 : \mathsf{O}^+_8(2)$. A regular set of size 945, degree 160 and nexus 105 is stabilized by a subgroup S_{10}.

10.88 The Conway graph on 2300 vertices



$v = 2300$

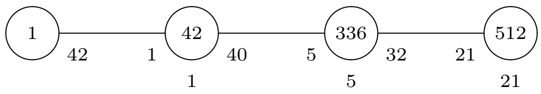
There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (2300, 891, 378, 324)$. Its spectrum is $891^1\ 63^{275}\ (-9)^{2024}$. The full group of automorphisms is Co_2 acting rank 3 with point stabilizer $\mathsf{U}_6(2) : 2$.

Construction in the Leech lattice

Let Λ be the Leech lattice as defined before. Fix $z = \frac{1}{\sqrt{8}}(4^2\, 0^{22})$ and look at all pairs x, y of lattice vectors of norm 4 with $x + y = z$. Omitting the $\frac{1}{\sqrt{8}}$, these are 44 pairs $(40\,(\pm 4)0^{21})$, $(04\,(\mp 4)0^{21})$, 1024 pairs $(31\,(\pm 1)^{22})$, $(13\,(\mp 1)^{22})$, and $77 \cdot 32\,/\,2 = 1232$ pairs $(22\,(\pm 2)^6 0^{16})$, $(22\,(\mp 2)^6 0^{16})$, altogether 2300 pairs. Call two pairs adjacent when the inner product of (arbitrarily chosen) representatives is even. This yields Γ.

1st subconstituent

The 1st subconstituent of Γ is the distance 1-or-2 graph of the near polygon that is the dual polar space for $\mathsf{U}_6(2)$.



$v = 891$

2nd subconstituent

The 2nd subconstituent of Γ is the Conway graph on 1408 vertices, cf. §10.81.

Cliques and cocliques

Maximal cliques have sizes 11, 14, 16, 23, 28, 44. There are unique orbits of m-cocliques for $m = 11, 28, 44$. A 44-clique can be seen in the Leech lattice description as the set of pairs containing the vectors $\frac{1}{\sqrt{8}}(40\,(\pm 4)0^{21})$. In the 1st subconstituent one sees a 43-clique as a point neighborhood $p^\perp$ in the $\mathsf{U}_6(2)$ polar graph.

Maximal cocliques have sizes 5, 9, 10, 12. Since the 2nd subconstituent is a rank 3 graph, G is transitive on 3-cocliques.

Line system and norm 3 vectors

The above construction in Λ was centered at $\frac{1}{2}z$. Shifting by $-\frac{1}{2}z$ yields a set Σ of 4600 vectors of norm 3 with mutual inner products $3, 1, 0, -1, -3$. The system is *tetrahedrally closed*: if $u, v, w \in \Sigma$ have mutual inner products -1, then $x = -u - v - w \in \Sigma$ and u, v, w, x have mutual inner products -1. The graph $\widetilde{\Gamma}$ on these 4600 vectors, adjacent when the inner product is -1, is a double cover of Γ. This graph is locally the above near polygon on 891 vertices, and in the local graph distances 0, 1, 2, 3 correspond to inner product 3, -1, 1, 0. That explains why $\widetilde{\Gamma}$ is locally a near polygon, and Γ is locally the distance 1-or-2 graph of this near polygon.

See also Cuypers [247].

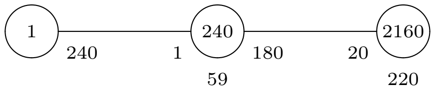
10.89 The rank 3 graphs on 2401 vertices

There are precisely ten rank 3 strongly regular graphs on $v = 2401$ vertices. The parameters are as follows.

	k	λ	μ	r^f	s^g	group	graph
a	96	47	2	47^{96}	$(-2)^{2304}$	$\mathsf{S}_{49}\,\mathrm{wr}\,2$	49×49
b	240	59	20	44^{240}	$(-5)^{2160}$	$7^4:6.\mathsf{O}_5(3)$	See A below
c	300	5	42	6^{2100}	$(-43)^{300}$	$7^4:6.\mathsf{P\Gamma L}_2(49)$	$VO_4^-(7)$
d	384	89	56	41^{384}	$(-8)^{2016}$	$7^4:(\mathsf{GL}_2(7)\circ\mathsf{GL}_2(7)):2$	$VO_4^+(7)$
e	480	119	90	39^{480}	$(-10)^{1920}$	$7^4:480:4$	Van Lint-Schrijver
f	480	119	90	39^{480}	$(-10)^{1920}$	$7^4:6.(2^4{:}\mathsf{S}_5)$	See B below
g	720	229	210	34^{720}	$(-15)^{1680}$	$7^4:6.\mathsf{S}_7$	See C below
h	960	389	380	29^{960}	$(-20)^{1440}$	$7^4:48.\mathsf{S}_5$	See D below
i	1200	599	600	24^{1200}	$(-25)^{1200}$	$7^4:1200:4$	Paley
j	1200	599	600	24^{1200}	$(-25)^{1200}$	$7^4:600:4$	Peisert

We discuss the cases b, f, g and h more in detail below.

A. Valency 240



$v = 2401$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (2401, 240, 59, 20)$. Its spectrum is $240^1\ 44^{240}\ (-5)^{2160}$. The full group of automorphisms is $G = 7^4 : S$ acting rank 3 with point stabilizer $S = 6.\mathsf{O}_5(3)$.

For a construction, see p. 144.

Cliques and cocliques

Maximal cliques in Γ have sizes 7–9, a single orbit for each size. Maximum cocliques have size 49, attaining the Hoffman bound. Among these are planes in the underlying $\mathsf{AG}(4, 7)$. It follows that $\chi(\Gamma) = 49$.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

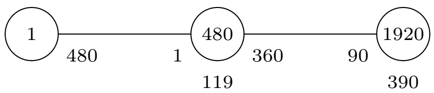
	H	index	orbitlengths	d	e
a	$7^2 : (3 \times 2.\mathsf{S}_4)$	52920	49, 2352	0	5
b	$7^2 : (3 \times \mathsf{SL}_2(3)) \times 7{:}6$	2520	343, 2058	30	35
c	$3 \times (7^3 : (2 \times (3^{2+1}{:}Q_8{:}3)))$	280	343, 2058	72	28

Case (a): these are the affine planes which are maximum cocliques.

Case (b): these are the affine solids with at infinity a plane that intersects the copolar space $HSp_4(3)$ in exactly five points (see p. 144).

Case (c): these are the affine solids with at infinity a plane that intersects the copolar space $HSp_4(3)$ in the twelve points of a dual affine plane $\mathsf{AG}(2, 3)^*$ (see p. 144).

B. Valency 480



$v = 2401$

There are precisely two rank 3 strongly regular graphs with parameters $(v, k, \lambda, \mu) = (2401, 480, 119, 90)$. Their spectrum is $480^1\ 39^{480}\ (-10)^{1920}$.

The first is the Van Lint-Schrijver graph on $\mathbb{F}_{2401}$ where two vertices are adjacent when their difference is a fifth power. See §7.3.1. The full group is $7^4 : 480 : 4$.

Let us call the second graph Γ. Its full group of automorphisms is $G = 7^4 : S$ with point stabilizer $S = 6.(2^4{:}\mathsf{S}_5)$.

Construction

The projective space $\mathsf{PG}(3, 7)$ has a unique spread with full automorphism group of order 1920 (namely $2^4{:}\mathsf{S}_5$). This group has two orbits on the spread, of sizes $10 + 40$, and on the space, of sizes $80 + 320$. Construct Γ by taking $\mathbb{F}_7^4$ as vertex set and joining two vertices when the connecting line meets the orbit of size 80 at infinity.

A classification of all spreads of $\mathsf{PG}(3,7)$ was given by MATHON & ROYLE [551] and by CHARNES & DEMPWOLFF [193]. The translation plane corresponding to our spread was found by MASON & OSTROM [541]. A nice description of the spread was given by MASON & SHULT [542]. Via the Klein correspondence it corresponds to the ovoid in $\mathbb{F}_7^6$ provided with the quadratic form $Q(x) = x_1^2 + \cdots + x_5^2 - x_6^2$ given by the $10 + 40$ points of shapes $(00001;1)$ or $(00022;1)$, where the $2^4 : \mathsf{S}_5$ acts on the first five coordinates by permuting them or changing an even number of signs.

Cliques and cocliques

Maximal cliques in Γ have sizes 6, 8, 9, 17, 49, a single orbit for the last three sizes. Maximum cocliques have size 49, attaining the Hoffman bound. Among these are planes in the underlying $\mathsf{AG}(4,7)$. It follows that $\chi(\Gamma) = \chi(\overline{\Gamma}) = 49$.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$2 \times (7^2 : (3 \times 2.\mathsf{S}_4))$	1960	49, 2352	0	10
b	$7^2 : (3 \times 2.\mathsf{S}_4)$	3920	49, 2352	0	10
c	$7^2 : (3 \times Q_{16})$	11760	49, 2352	0	10
	$7^2 : 6 : \mathsf{D}_{12}$	7840	49, 588, 1764	0	10
d	$7^3 : 6^2$	2240	343, 2058	60	70
e	$7^2 : (3 \times (Q_8 : 2.\mathsf{S}_4))$	490	49, 2352	48	9
f	$7^3 : (6 \times \mathsf{SL}_2(3))$	560	343, 2058	102	63

Cases (a), (b) and (c): These are the affine planes which are maximum cocliques; the lines at infinity of these planes correspond to the points of shapes $(0,0,0,2,2;1)$, $(0,3,3,3,3;1)$ and $(0,0,2,3,3;1)$, respectively.

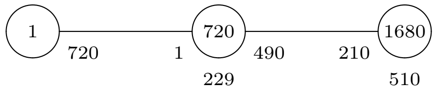
Case (d): These are the affine solids containing a maximum coclique.

Case (e): These are the affine planes which are the maximum cliques; they have a line at infinity corresponding to a point of shape $(0,0,0,0,1;1)$.

Case (f): These are the affine solids containing a maximum clique.

The fourth line of the table shows another type of maximum cocliques: affine planes with at infinity a line corresponding to a point of shape $(1,1,1,3,3;0)$. Their stabilizer has three orbits on the vertex set.

C. Valency 720



$v = 2401$

There is a unique rank 3 strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (2401, 720, 229, 210)$. Its spectrum is $720^1\ 34^{720}\ (-15)^{1680}$. The full group of automorphisms is $G = 7^4 : S$ acting rank 3 with point stabilizer $S = 6.\mathsf{S}_7$.

Construction

A construction is found by taking $\mathbf{1}^\perp/\langle\mathbf{1}\rangle$ in $\mathbb{F}_7^7$ provided with the quadratic form $q(x) = \sum x_i^2$. The isotropic vectors of shape $(1, 2, 4, 0, 0, 0, 0)$ define 70 points of the $\mathsf{O}_5(7)$ generalized quadrangle which, under (inverse) Klein correspondence, correspond to a set $\mathscr{L}$ of 70 lines of the $\mathsf{Sp}_4(7)$ generalized quadrangle with the property that every point of the quadrangle is contained in 0 or 2 such lines. Let $\mathscr{P}_i$ be the set of points contained in exactly i members of $\mathscr{L}$, $i = 0, 2$. With the point set $\mathscr{P}_0$ at infinity of $\mathbb{F}_7^4$, we find Γ.

The point set $\mathscr{P}_2$ is doubly covered by the members of $\mathscr{L}$. It is also 6-fold covered by the members of a set $\mathscr{L}'$ of 210 nonsingular lines, forming a single orbit under the action of S_7, and which can be found as follows. The eight points (1-spaces) of $\mathbb{F}_7^7$ obtained by applying the symmetric group S_4 on the first four coordinates of the vector $(1, 2, 4, 0, 0, 0, 0)$ correspond under inverse Klein correspondence to the eight lines of a regulus in $\mathsf{PG}(3, 7)$. The opposite regulus contains exactly two singular lines (corresponding to $(0, 0, 0, 0, 1, 2, 4)$ and $(0, 0, 0, 0, 1, 4, 2)$); the other six lines consist of points of $\mathscr{P}_2$ only and are nonsingular with respect to the $\mathsf{Sp}_4(7)$ geometry. Letting S_7 act, we obtain $6.\binom{7}{4} = 210$ nonsingular lines covering $\mathscr{P}_2$.

The sets $\mathscr{P}_0$ and $\mathscr{P}_2$ are complementary (exceptional) two-character sets of $\mathsf{PG}(3, 7)$; planes intersect $\mathscr{P}_0$ in either 15 (the perp of a point in $\mathscr{P}_2$) or 22 (the perp of a point in $\mathscr{P}_0$) points.

Cliques and cocliques

Maximal cliques in Γ have sizes 6–10, 12, 17, a single orbit for sizes 6, 12, 17. Maximum cocliques have size 49, attaining the Hoffman bound. There are two orbits. These cocliques are the planes in the underlying $\mathsf{AG}(4, 7)$ with a member of $\mathscr{L}$ (first orbit) or $\mathscr{L}'$ (second orbit) at infinity. It follows that $\chi(\Gamma) = 49$.

Regular sets

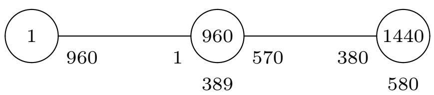
Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$3 \times (7^2 : (3 \times 2.\mathsf{S}_4))$	3430	49, 2352	0	15
b	$7^2 : (3 \times 2.\mathsf{S}_4)$	10290	49, 2352	0	15
c	$7^3 : (3 \times 6 \times \mathsf{S}_3)$	1960	343, 2058	90	105
d	$7^{2+2} : 6^2$	840	343, 2058	132	98

Cases (a), (b): these are the affine planes with a member of $\mathscr{L}$ (resp. $\mathscr{L}'$) at infinity.

Cases (c), (d): these are the affine solids with the perp of a point in $\mathscr{P}_2$ (resp. $\mathscr{P}_0$) at infinity.

D. Valency 960



$v = 2401$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (2401, 960, 389, 380)$. Its spectrum is $960^1\ 29^{960}\ (-20)^{1440}$. The full group of automorphisms is $G = 7^4 : S$ acting rank 3 with point stabilizer $S = 48.\mathsf{S}_5$.

Construction

The group $\mathsf{PSL}_2(49)$ has a maximal subgroup A_5 with orbit lengths $20 + 30$ on $\mathsf{PG}(1, 49)$ (cf. §7.5). In $\mathsf{P\Sigma L}_2(49)$ the stabilizer of this partition is S_5. Take $\mathbb{F}_{49}^2$ with the orbit of length 20 at infinity.

Cliques and cocliques

Maximal cliques in Γ have sizes 7–20, 22, 49, a single orbit for sizes 18–20, 22, and two orbits for size 49. Maximum cocliques have size 49. There are two orbits. The (co)cliques of size 49 are lines and Baer subplanes of $\mathsf{AG}(2, 49)$, and attain the Hoffman bound. It follows that $\chi(\Gamma) = \chi(\overline{\Gamma}) = 49$.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \operatorname{Aut}\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

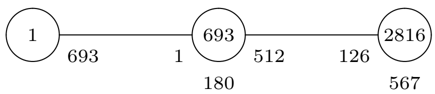
	H	index	orbitlengths	d	e
a	$2 \times (7^2 : (3 \times \mathsf{QD}_{32}))$	1470	49, 2352	0	20
b	$7^3 : (6 \times 2^2)$	1680	343, 2058	120	140
c	$3 \times (7^2 : (3 \times \mathsf{QD}_{32}))$	980	49, 2352	48	19
d	$7^2 : (3 \times 2.\mathsf{S}_4)$	1960	49, 2352	48	19
e	$7^3 : 6^2$	1120	343, 2058	162	133

Case (a): these are the maximum cocliques that are affine lines of $\mathsf{AG}(2, 49)$.

Cases (c), (d): these are the maximum cliques corresponding to the affine lines and the affine Baer subplanes of $\mathsf{AG}(2, 49)$, respectively.

For (b) and (e), view $\mathbb{F}_{49}^2$ as $\mathbb{F}_7^4$. The orbit of A_5 of length 20 becomes a set $\mathscr{S}$ of 20 lines (partial spread) at infinity of an $\mathsf{AG}(4, 7)$. Then (b) are the affine solids of $\mathsf{AG}(4, 7)$ with at infinity a plane that does not contain any member of $\mathscr{S}$, whereas (e) are the affine solids of $\mathsf{AG}(4, 7)$ with at infinity a plane that contains a unique member of $\mathscr{S}$.

10.90 The Fi_{22} graph



$v = 3510$

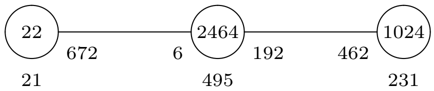
There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (3510, 693, 180, 126)$. Its spectrum is $693^1\ 63^{429}\ (-9)^{3080}$. The full group of automorphisms is $G = \mathsf{Fi}_{22}.2$ acting rank 3 with point stabilizer $2.\mathsf{U}_6(2).2$.

The local graph is the polar graph for $\mathsf{U}_6(2)$ (§10.74). It follows from Pasechnik [602] and De Bruyn [261] that Γ is the unique connected graph that is locally the polar graph for $\mathsf{U}_6(2)$.

The μ-graphs of Γ are $NO_6^-(3)$ graphs (see §10.41).

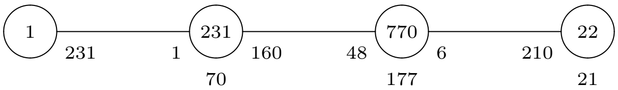
Cliques

The group G is transitive on triangles. Maximal cliques all have size 22, and form a single orbit. The stabilizer in G of a maximal clique M is $2^{10}{:}\mathsf{M}_{22}{:}2$ with three orbits of sizes $22, 2464, 1024$. Each vertex in the second orbit has 6 neighbors in M and each such 6-set occurs 32 times in this way. These 6-sets form the Steiner system $S(3, 6, 22)$. Diagram:



$v = 3510$

The subgraph induced on the 1024 vertices nonadjacent to M is distance-transitive with intersection array $\{231, 160, 6; 1, 48, 210\}$, the distance-2 graph of the coset graph of the truncated Golay code (cf. [123], §11.3F).



$v = 1024$

Cocliques

The smallest maximal cocliques have size 9 and come in two kinds. The first kind consists of the affine subplanes $AG(2, 3)$ of the Fischer space. Each such 9-coclique is contained in a unique $K_{9,9,9}$ stabilized by a maximal subgroup $3_+^{1+6} : 2^{3+4} : 3^2 : 2^2$ of G. The second kind consists of the 9-cocliques each contained in a unique subgraph $\overline{T(10)}$ stabilized by a maximal subgroup S_{10}.

Enright [307] gives a construction of Fi_{22} in terms of this S_{10}.

For the independence number of Γ we have $33 \le \alpha(\Gamma) \le 45$.

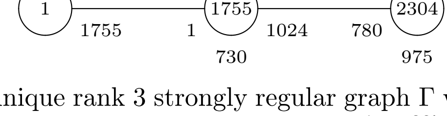
Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \operatorname{Aut}\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	$\mathsf{O}_8^+(2) : (\mathsf{S}_3 \times 2)$	61776	360, 3150	63	72	$3NO_8^+(2)$
b	$\mathsf{O}_7(3)$	28160	351, 3159	126	63	$NO_7^{-\perp}(3)$

Under (a), the graph induced on the orbit of size 360 is the disjoint union of three copies of $NO_8^+(2)$.

10.91 The Rudvalis graph



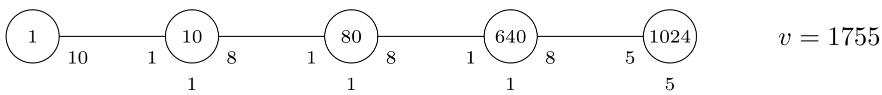
$v = 4060$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (4060, 1755, 730, 780)$. Its spectrum is $1755^1\ 15^{3276}\ (-65)^{783}$. The full group of automorphisms is Ru acting rank 3 with point stabilizer ${}^2\mathsf{F}_4(2)$. Construction of graph and group is due to CONWAY & WALES [218], after Rudvalis provided evidence for the existence of both.

Construction

COOLSAET [220] gave a construction starting with the Hoffman-Singleton graph Σ. Let Σ have adjacency matrix A. Then A has spectrum $7^1\ 2^{28}\ (-3)^{21}$, and $E = \frac{-1}{25}(A-7I)(A+3I) = \frac{1}{25}(5A+15I-J)$ is the projection on the 28-dimensional eigenspace of A. Let e_x be the unit vector corresponding to vertex x of Σ, and define $\overline{T} = E\sum_{x\in T} e_x$ for any set T of vertices of Σ. One has $(\bar{x},\bar{y}) = E_{xy}$, which is $\frac{14}{25}, \frac{4}{25}, \frac{-1}{25}$ when $x=y$, $x\sim y$, and $x \not\sim y$, respectively. This works over any field of characteristic different from 5. Look at this representation over $\mathbb{F}_2$. Then $(\bar{x},\bar{y}) = 0$ if $x=y$ or $x \sim y$, and $(\bar{x},\bar{y}) = 1$ if $x \not\sim y$. Let the $175+1260+2625 = 4060$ vertices of Γ be the 175 edges, the 1260 pentagons, and the 2625 hexads of Σ, where a hexad is the complement of a 4-coclique inside a Petersen subgraph. Two distinct vertices S,T of Γ are adjacent when $(\overline{S},\overline{T}) = 0$. This yields the Rudvalis graph.

Local graph



The ${}^2\mathsf{F}_4(2)$ generalized octagon has the above diagram. The local graph of the Rudvalis graph is the distance 1-or-2-or-3 graph of this generalized octagon.

A generalized octagon $\mathsf{GO}(s,t)$ (with lines of size $s+1$ and $t+1$ lines on each point), has eigenmatrix

$$P = \begin{pmatrix} 1 & s(t+1) & s^2t(t+1) & s^3t^2(t+1) & s^4t^3 \\ 1 & s-1+\sqrt{2st} & (s-1)\sqrt{2st}+st-s & -s\sqrt{2st}+s^2t-st & -s^2t \\ 1 & s-1 & -st-s & -s^2t+st & s^2t \\ 1 & s-1-\sqrt{2st} & -(s-1)\sqrt{2st}+st-s & s\sqrt{2st}+s^2t-st & -s^2t \\ 1 & -t-1 & t^2+t & -t^2-t^3 & t^3 \end{pmatrix}$$

so that its distance-4 matrix has fewer eigenvalues than the distance-1 matrix, and the latter cannot be a polynomial in the former. In particular, in our case $\mathsf{GO}(2,4)$ the distance 1-or-2-or-3 graph has spectrum $730^1\ 15^{1026}\ (-17)^{650}\ (-65)^{78}$. Does it determine the generalized octagon? The answer is yes, as one sees combinatorially (or from the group). There are 2925 27-cliques, corresponding to the 2925 lines of the generalized octagon.

Cliques

The largest cliques in Γ have size 28, and there are 424125 of them, forming a single orbit. The stabilizer of one is $2^{3+8}{:}\mathsf{L}_3(2)$ with vertex orbit sizes $28+448+3584$. An example is found by picking a vertex ∞, and in its local generalized octagon a line and the 24 points adjacent to it. The stabilizer of a 28-clique is transitive on the 28 vertices. Each vertex outside a 28-clique is adjacent to 12 vertices inside.

Cocliques

The largest cocliques in Γ have size 28, and there are 24128000 of them, forming a single orbit. The stabilizer of one is $\mathsf{U}_3(3)$ with vertex orbit sizes $28+63+$

189 + 756 + 1008 + 2016. The stabilizer of the orbit of size 2016 is a maximal subgroup $(2^6 : \mathsf{U}_3(3)) : 2$ of Ru with orbit sizes 252 + 1792 + 2016.

μ-graphs

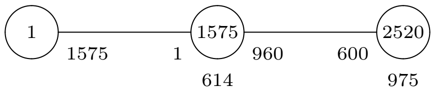
Let x, y be nonadjacent vertices and let M be the set of common neighbors. Then $|M| = \mu = 780$. Consider M as a subset of the generalized octagon on the neighbors of x. Every line meets M in 0 or 2 points, so M is a hyperplane complement, where the hyperplane has 975 points and 975 lines, 3 points/line and 3 lines/point. In the Rudvalis graph, these μ-graphs are disconnected, and have two connected components of size 390 each.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. Also the three orbits of subgroups $(2^6 : \mathsf{U}_3(3)) : 2$ are regular sets. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e
a	$(2^2 \times \mathsf{Sz}(8)) : 3$	417600	1820, 2240	795	780
b	$(2^6 : \mathsf{U}_3(3)) : 2$	188500	252, 1792+2016	123	108
			1792, 252+2016	783	768
			2016, 252+1792	879	864

10.92 $2^{12}.\mathsf{HJ}.\mathsf{S}_3$ on 4096 vertices



$v = 4096$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (4096, 1575, 614, 600)$. Its spectrum is $1575^1\ 39^{1575}\ (-25)^{2520}$. The full group of automorphisms is $2^{12} : (3 \times \mathsf{HJ}) : 2$ with point stabilizer $(3 \times \mathsf{HJ}) : 2$.

It arises because HJ, acting on $\mathsf{P}V$ for $V = \mathbb{F}_4^6$ via $\mathsf{HJ} < \mathsf{G}_2(4) < \mathsf{PSp}_6(4)$, has orbits of sizes 525 and 840 on the 1365 points ([517]), giving two-character sets.

Construction

The dual Ω of the Cohen-Tits near octagon is a geometry with 525 points and 315 lines. It is fully embedded in the generalized hexagon $\mathsf{G}_2(4)$, which has an embedding in the polar space $\mathsf{Sp}_6(4)$, which in turn is embedded in the projective space $\mathsf{PG}(5, 4)$. The automorphism group $\mathsf{HJ} : 2$ of Ω acts transitively on Ω and on its complement in $\mathsf{G}_2(4)$. Let V be a 6-dimensional vector space over $\mathbb{F}_4$ with $\mathsf{P}V = \mathsf{PG}(5, 4)$. Then Γ is the graph on the vectors of V, adjacent when their difference belongs to Ω.

As a set of 525 points, Ω is a two-character set of $\mathsf{PG}(5, 4)$. Taking perps with respect to the symplectic form associated to $\mathsf{Sp}_6(4)$, the perp of a point in Ω intersects Ω in 141 points and the perp of a point outside Ω has 125 points in common with Ω.

Maximal cliques and cocliques

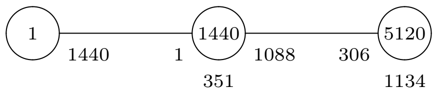
The maximal cliques in Γ have sizes 7–14, 16, 18, 20, 24, 25, 64 with unique orbits for sizes 20, 24, 25, 64. The Hoffman bound is 64, and 64-cliques have stabilizer $2^6 : \mathsf{G}_2(2)$. These arise as vector spaces over $\mathbb{F}_2$ in V whose 1-spaces (viewed as vector lines over $\mathbb{F}_4$) belong to a fixed subhexagon of Ω isomorphic to the split Cayley hexagon $\mathsf{G}_2(2)$ (Ω has 100 such subhexagons, which can be taken as the vertices of the Hall-Janko graph on 100 vertices, see §10.32).

Maximum cocliques have size 40 and fall into three orbits with stabilizers of orders 30, 50, and 150.

Regular sets

Among the regular sets in Γ that arise as an orbit of a subgroup of Aut Γ with two orbits on the vertex set are sets with size u, degree d, and nexus e, where $e = d+25$ and $(u, d) = (256, 75)$, $(512, 175)$, $(1024, 375)$, $(1536, 575)$, $(2048, 775)$, and where $e = d - 39$ and $(u, d) = (64, 63)$, $(256, 135)$, $(512, 231)$, $(1024, 423)$, $(1536, 615)$, $(1792, 711)$, $(2048, 807)$.

10.93 The $3^8.2^{1+6}.\mathsf{O}_6^-(2).2$ graph on 6561 vertices



$v = 6561$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (6561, 1440, 351, 306)$. Its spectrum is $1440^1\ 63^{1440}\ (-18)^{5120}$. The full group is $3^8 : 2^{1+6}.\mathsf{O}_6^-(2).2$, acting rank 3 with point stabilizer $2^{1+6}.\mathsf{O}_6^-(2).2$.

For a construction, see p. 144.

Maximum cliques have size 81 (a single orbit); they are subspaces $\mathsf{AG}(3, 4)$. Maximum cocliques have size 81.

Regular sets

Among the regular sets in Γ that arise as an orbit of a subgroup of Aut Γ with two orbits on the vertex set are sets with size u, degree d, and nexus e, where $e = d + 18$ and $(u, d) = (81, 0)$, $(729, 144)$, $(2187, 468)$, and where $e = d - 63$ and $(u, d) = (81, 80)$, $(729, 216)$, $(2187, 522)$.

The case $(81, 0)$ corresponds to affine spaces of dimension 4 with at infinity a solid disjoint from X (hence contained in X', see p. 144 for a construction).

The case $(729, 144)$ corresponds to affine spaces of dimension 6 with at infinity a 5-dimensional subspace containing a solid entirely contained in X, and intersecting X in the union of 18 lines in the orbit of the members of $\mathscr{S}$ (see $\langle L, S\rangle$ on p. 144 for a construction).

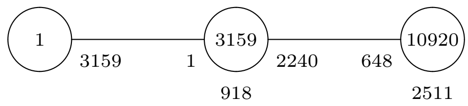
The case $(2187, 468)$ corresponds to affine spaces of dimension 7 with at infinity a hyperplane intersecting X in 234 points.

The case $(81, 80)$ corresponds to affine spaces of dimension 4 with at infinity a solid contained in X.

The case $(729, 216)$ corresponds to affine spaces of dimension 6 with at infinity a 5-dimensional subspace intersecting X in three solids not containing a common point (this can be realized by considering Σ' and an arbitrary member of $\mathscr{S}$).

The case $(2187, 522)$ corresponds to affine spaces of dimension 7 with at infinity a hyperplane intersecting X in 261 points.

10.94 The Fi_{22} graph on 14080 vertices



$v = 14080$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (14080, 3159, 918, 648)$. Its spectrum is $3159^1\ 279^{429}\ (-9)^{13650}$. The full group of automorphisms is $G = \mathsf{Fi}_{22}$ acting rank 3 with point stabilizer $\mathsf{O}_7(3)$. (The group G has two conjugacy classes of subgroups of index 14080 isomorphic to $\mathsf{O}_7(3)$, merged in $\mathsf{Fi}_{22}.2$. One of these has orbits of sizes 1, 3159, and 10920 on its own conjugacy class, and 364, 1080, and 12636 on the other.)

Construction

This is the graph on the lines on a fixed point in the Fi_{23} Fischer space, adjacent when they span a dual affine plane. It follows that the complementary graph is the collinearity graph of a partial linear space with lines of size 4.

The Rudvalis-Hunt design

Rudvalis and Hunt (cf. [183] or [621], p. 112) observed that the design of which the points are the subgroups in one conjugacy class of subgroups $\mathsf{O}_7(3)$ and the blocks the subgroups in the other conjugacy class, incident when one lies in an orbit of size 364 or 1080 of the other, is a square 2-(14080,1444,148) design. See also [280].

Maximal cliques and cocliques

The largest cliques have size 64. They form a single orbit, and the stabilizer of one is a maximal subgroup $2^6 : \mathsf{Sp}_6(2)$ with orbits of lengths 64, 5376, and 8640. Maximal cliques have sizes 10–14, 16, 18, 20, 22, 25, 28, 64.

The largest cocliques have size 40, reaching the Hoffman bound. There are several nonequivalent examples. One is invariant under a group $3^{3+3} : L_3(3)$.

1408-vertex subgraphs

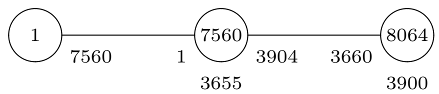
Γ has 3510 subgraphs isomorphic to the Conway graph on 1408 vertices (§10.81). Each is fixed by a Fischer transposition. The stabilizer of one in G is a nonsplit extension $2.\mathsf{U}_6(2)$. Each vertex outside such a subgraph has 288 neighbors inside. Distinct such subgraphs meet in 112 or 256 vertices. The graph with these 3510 subgraphs as vertices, adjacent when they have 256 vertices in common, is the Fi_{22} graph.

Regular sets

Examples of regular sets in Γ are obtained from subgroups H of $G = \mathrm{Aut}\,\Gamma$ with two orbits on the vertex set. We give degree d and nexus e for the smallest orbit.

	H	index	orbitlengths	d	e	graph
a	${}^2\mathsf{F}_4(2)'$	3592512	1600, 12480	351	360	Tits, §10.82
b	$\mathsf{O}_8^+(2):\mathsf{S}_3$	61776	2880, 11200	639	648	
c	$2.\mathsf{PSU}_6(2)$	3510	1408, 12672	567	288	Conway, §10.81
d	$2^{10}:\mathsf{M}_{22}$	142155	2816, 11264	855	576	

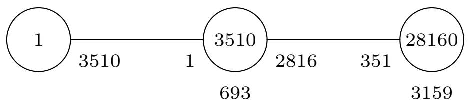
10.95 The $5^6.4.\mathsf{HJ}.2$ graph on 15625 vertices



There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (15625, 7560, 3655, 3660)$. Its spectrum is $7560^1\ 60^{8064}\ (-65)^{7560}$. The full group of automorphisms is $5^6:(2.\mathsf{HJ}):4$.

It arises because HJ, acting on $\mathsf{P}V$ for $V = \mathbb{F}_5^6$ via $2.\mathsf{HJ} < \mathsf{Sp}_6(5)$, has orbits of sizes 1890 and 2016 on the 3906 points ([517]), giving two-character sets.

10.96 The Fi_{23} graph

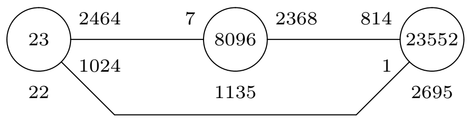


There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (31671, 3510, 693, 351)$. Its spectrum is $3510^1\ 351^{782}\ (-9)^{30888}$. The full group of automorphisms is $G = \mathsf{Fi}_{23}$ acting rank 3 with point stabilizer $2.\mathsf{Fi}_{22}$.

The local graph is the Fi_{22} graph, and Γ is the unique connected locally Fi_{22} graph (Pasechnik [602]). The μ-graphs are $NO_7^{-\perp}(3)$ graphs (see §10.66).

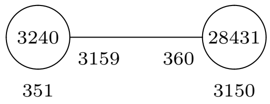
Cliques

The group G is transitive on triangles and 4-cliques. Maximal cliques all have size 23, and form a single orbit. The stabilizer in G of a maximal clique M is a nonsplit extension $2^{11}.\mathsf{M}_{23}$ with three orbits of sizes 23, 8096, 23552. Each vertex in the second orbit has 7 neighbors in M and each such 7-set occurs 32 times in this way. These 7-sets form the Steiner system $S(4, 7, 23)$. Each vertex in the third orbit has a unique neighbor in M. Diagram:



Regular sets

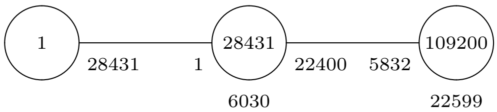
Given two nonadjacent vertices x, y in the Fi_{24} graph Δ, we see a split of $\Gamma = \Delta(y)$ into the μ-graph $\Delta(x) \cap \Delta(y)$, and the rest. This yields a regular partition fixed by $\mathsf{O}_8^+(3) : \mathsf{S}_3$:



$v = 31671$

The μ-graph is disconnected, with three components of size 1080. For a construction of Fi_{23} via this partition, see WILSON [735], p. 243.

10.97 The Fi_{23} graph on 137632 vertices



$v = 137632$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (137632, 28431, 6030, 5832)$. Its spectrum is $28431^1\ 279^{30888}\ (-81)^{106743}$. The full group of automorphisms is $G = \mathsf{Fi}_{23}$ acting rank 3 with point stabilizer $\mathsf{O}_8^+(3):\mathsf{S}_3$.

Construction

This is the graph on the lines on a fixed point in the Fi_{24} Fischer space, adjacent when they span a dual affine plane. It follows that the complementary graph is the collinearity graph of a partial linear space with lines of size 4.

Maximal cliques

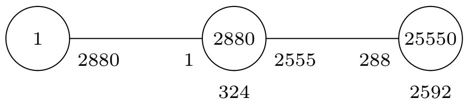
The largest cliques have size 136. They form a single orbit, and the stabilizer of one is a maximal subgroup $\mathsf{Sp}(8,2)$ with orbits of lengths 136, 45696, and 91800.

14080-vertex subgraphs

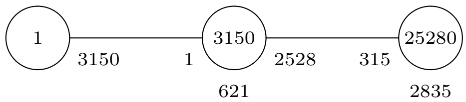
Γ has 31671 subgraphs isomorphic to the Fi_{22} graph on 14080 vertices (§10.94). Each is fixed by a Fischer transposition. The stabilizer of one in G is a nonsplit extension $2.\mathsf{Fi}_{22}$. Each vertex outside such a subgraph has 2880 neighbors inside. Distinct such subgraphs meet in 1408 or 1444 vertices. The graph with these 31671 subgraphs as vertices, adjacent when they have 1408 vertices in common, is the Fi_{23} graph.

The first subconstituent

The first subconstituent of Γ has full group of automorphisms $\mathsf{O}_8^+(3):\mathsf{S}_3$ acting rank 4 with suborbit sizes $1 + 2880 + 3150 + 22400$. The graphs induced by the suborbits of sizes 2880 and 3150 are strongly regular with parameters $(v, k, \lambda, \mu) = (28431, 2880, 324, 288)$ and $(28431, 3150, 621, 315)$, both with the same full group, as was found in [244].



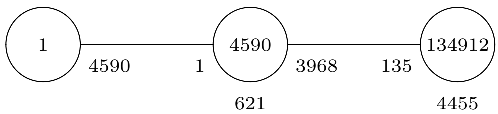
$v = 28431$



$v = 28431$

This latter graph is a subgraph of the Fi_{23} graph, see p. 350. The valency 2880 and 3150 graphs have clique numbers 9 and 21, respectively.

10.98 The $\mathsf{E}_6(2)$ graph



$v = 139503$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (139503, 4590, 621, 135)$. Its spectrum is $4590^1\ 495^{2482}\ (-9)^{137020}$. The full group of automorphisms is $\mathsf{E}_6(2)$ acting rank 3 with point stabilizer $2^{16} : \mathsf{O}_{10}^+(2)$.

The local graphs are 2-clique extensions of the strongly regular $\mathsf{D}_{5,5}(2)$ graph with parameters $(2295, 310, 85, 35)$ (see §2.2.12).

The μ-graphs are strongly regular $\mathsf{O}_8^+(2)$ graphs with parameters $(135, 70, 37, 35)$ (see §10.43).

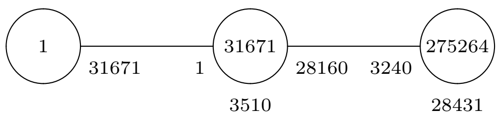
Maximal cliques have sizes 31 and 63. (If the vertices are the objects of type 1 in the E_6 geometry, then these maximal cliques are the objects of types 5 and 2, respectively.)

There are maximal cocliques of size 256 ($= 1+51+204$, union of three orbits of the normalizer of an element of order 17). It is not known whether Γ contains larger cocliques. The Hoffman bound is 273.

The maximal subgroup $\mathsf{F}_4(2)$ has two orbits, of lengths 69615 and 69888. The smallest orbit has degree $d = 2286$ and nexus $e = 2295$, see also §4.9.2.

The maximal subgroup $(7 \times {}^3\mathsf{D}_4(2)) : 3$ has two orbits, of lengths 17199 and 122304. The smallest orbit has degree $d = 558$ and nexus $e = 567$, see also §4.9.2.

10.99 The Fi_{24} graph



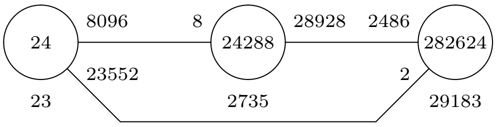
$v = 306936$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (306936, 31671, 3510, 3240)$. Its spectrum is $31671^1\ 351^{57477}\ (-81)^{249458}$. The full group of automorphisms is $G = \mathsf{Fi}_{24}$ acting rank 3 with point stabilizer $2 \times \mathsf{Fi}_{23}$.

The local graph is the Fi_{23} graph. The μ-graphs are disconnected, with three connected components, each carrying a copy of the $NO_8^+(3)$ graph with parameters $(v, k, \lambda, \mu) = (1080, 351, 126, 108)$ (§10.78).

Cliques

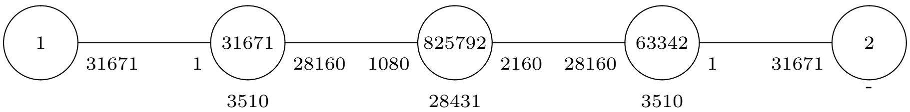
The group G is transitive on i-cliques for $i \le 5$. Maximal cliques all have size 24, and form a single orbit. The stabilizer in G of a maximal clique M is a nonsplit extension $2^{12}.\mathsf{M}_{24}$ with three orbits of sizes $24, 24288, 282624$. Each vertex in the second orbit has 8 neighbors in M and each such 8-set occurs 32 times in this way. These 8-sets form the Steiner system $S(5, 8, 24)$. Diagram:



$v = 306936$

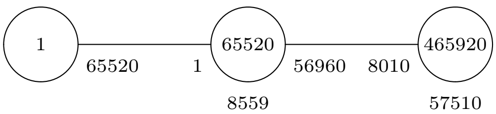
Triple cover

This graph Γ has a distance-transitive antipodal 3-cover 3Γ with diagram



on $v = 920808$ vertices, cf. NORTON [592]. The graphs Γ and 3Γ are the only connected locally Fi_{23} graphs (PASECHNIK [602]).

10.100 The Suz graph on 531441 vertices



$v = 531441 = 3^{12}$

There is a unique rank 3 strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (531441, 65520, 8559, 8010)$. Its spectrum is $65520^1\ 639^{65520}\ (-90)^{465920}$. The full group of automorphisms is $G = 3^{12}.2.\mathsf{Suz}.2$ acting rank 3 with point stabilizer $2.\mathsf{Suz}.2$. See also §6.3.3.

Cliques

The largest cliques have size 81 and form a single orbit. The stabilizer of one is a group of shape $3^4 : ((2^{2+6}.\mathsf{O}_5(3)) : 2)$ with orbit sizes $81+116640+414720$, acting 2-transitively on the 81-clique. Points outside have either 8 or 17 neighbors inside an 81-clique.

These 81-cliques are subspaces of the socle $V = \mathbb{F}_3^{12}$ of G, so that $\chi(\overline{\Gamma}) = 3^8$.

Chapter 11

Classification of rank 3 graphs

The classification of rank 3 graphs is due to Foulser, Kallaher, Kantor, Liebler, Liebeck, Saxl and others. The result is described in the following pages. We give all pairs (Γ, G), with Γ a strongly regular graph and G a group of automorphisms of Γ acting rank 3. Two such pairs (Γ, G) and (Γ', G') are called *equivalent* if there is an isomorphism $\alpha\colon \Gamma \to \Gamma'$ such that $G'\alpha = \alpha G$.

11.1 Primitive rank 3 permutation groups

The O'Nan-Scott theorem (cf. [518]) immediately implies

Theorem 11.1.1 *Let Γ be a primitive strongly regular graph with parameters (v, k, λ, μ), and G a primitive rank 3 permutation group acting as a group of automorphisms of Γ. Then we have one of the following cases.*

(i) $T \times T \lhd G \le T_0 \,\mathrm{wr}\, 2$, where T_0 is a 2-transitive group of degree v_0, the socle T of T_0 is simple and $v = v_0^2$.

(ii) The socle L of G is simple.

(iii) G is an affine group, that is, G has a regular elementary abelian normal subgroup and v is a power of a prime.

Hence, the classification must handle these three cases.

For (i), see the classification of doubly transitive groups (Theorem 11.2.1 below). The graphs here are the lattice graphs.

For (ii), if L is alternating, see Bannai [45] (Theorem 11.3.1 below). If L is classical, see Kantor & Liebler [481] (Theorems 11.3.2, 11.3.3 below). If L is exceptional, see Liebeck & Saxl [520] (Theorem 11.3.4 below). For sporadic L the list was determined by Brouwer, Soicher and Wilson and given in [520] (Theorem 11.3.5 below).

For (iii), see Liebeck [517] (Theorem 11.4.1 below).

11.2 Wreath product

This case depends on the classification of doubly transitive groups. We follow Cohen & Zantema [206].

Theorem 11.2.1 *Let G be a doubly transitive permutation group on a finite set Ω. Then we have one of the cases in Table 11.1.*

G	$\lvert\Omega\rvert$	Ω	restrictions
S_n, A_n	n	n symbols	$n \ge 2$, $n \ge 4$
$\mathsf{PSL}_n(q) \le G \le \mathsf{P\Gamma L}_n(q)$	$\frac{q^n-1}{q-1}$	points of $\mathsf{PG}(n-1, q)$	$n \ge 2$
$\mathsf{PSU}_3(q) \le G \le \mathsf{P\Gamma U}_3(q)$	q^3+1	points of a Hermitian unital	
${}^2\mathsf{G}_2(q) \le G \le \mathrm{Aut}({}^2\mathsf{G}_2(q))$	q^3+1	points of a Ree unital	$q = 3^{2m+1}$, $m \ge 0$
${}^2\mathsf{B}_2(q) \le G \le \mathrm{Aut}({}^2\mathsf{B}_2(q))$	q^2+1	points of a Suzuki ovoid	$q = 2^{2m+1}$, $m \ge 1$
$\mathsf{Sp}_{2m}(2)$	$2^{m-1}(2^m \pm 1)$	nondegenerate quadrics	$m \ge 3$
$\mathsf{PSL}_2(11)$	11	2-$(11, 5, 2)$	
A_7	15	points of $\mathsf{PG}(3, 2)$	
M_{11}	11	$S(4, 5, 11)$	
M_{11}	12	3-$(12, 6, 2)$	
M_{12}	12	$S(5, 6, 12)$	
M_{22}, $\mathrm{Aut}(\mathsf{M}_{22})$	22	$S(3, 6, 22)$	
M_{23}	23	$S(4, 7, 23)$	
M_{24}	24	$S(5, 8, 24)$	
HS	176	2-$(176, 50, 14)$	
Co_3	276	2-$(276, 100, 2 \cdot 3^6)$	
$\mathsf{SL}_d(q) \le G_0 \le \Gamma L_d(q)$	q^d		$d \ge 1$
$\mathsf{Sp}_{2d}(q) \trianglelefteq G_0$	q^{2d}		$d \ge 2$
$\mathsf{G}_2(q)' \trianglelefteq G_0$	q^6		$q = 2^a$
$2^{1+2} \trianglelefteq G_0$	q^2		$q = 3, 5, 7, 11, 23$
$2^{1+4} \trianglelefteq G_0$	3^4		
$G_0^{(\infty)} \simeq \mathsf{SL}_2(5)$	q^2	$\mathsf{SL}_2(5) < \mathsf{SL}_2(q)$	$q = 9, 11, 19, 29, 59$
$G_0 \simeq \mathsf{A}_6$	2^4	$\mathsf{A}_6 \simeq \mathsf{Sp}_4(2)'$	
$G_0 \simeq \mathsf{A}_7$	2^4	$\mathsf{A}_7 < \mathsf{A}_8 \simeq \mathsf{SL}_4(2)$	
$G_0 \simeq \mathsf{SL}_2(13)$	3^6	$\mathsf{SL}_2(13) < \mathsf{Sp}_6(3)$	

Table 11.1: The doubly transitive permutation groups G acting on a set Ω. In the second part of the table, Ω is elementary abelian, and $0 \in \Omega \le G$. Here $G_0^{(\infty)}$ is the last term of the commutator series of G_0.

For the first part of this table (G without regular normal subgroup), see [173], [476]. For the second part, see [417], [418]. For the application to rank 3 groups, only the first part is used.[1] The only rank 3 graphs this describes are the grids (lattice graphs) $n \times n$ (with full automorphism group $\mathsf{S}_n \,\mathrm{wr}\, 2$). For a given $n \times n$ lattice graph Γ and doubly transitive group T_0 of degree n with nonabelian simple socle T, a group G with $T \times T \lhd G \le T_0 \,\mathrm{wr}\, 2$ acts as a rank 3 automorphsm group on Γ if and only if G contains an element (not necessarily of order 2) that interchanges the two partitions of $V\Gamma$ in maximal cliques (the two 'directions' of the lattice).

11.3 Simple socle

11.3.1 Alternating socle

Theorem 11.3.1 (Bannai [45])

Let G be either S_n or A_n, and let H be a maximal subgroup of G, such that the permutation representation of G on the cosets of H is rank 3. Then we have one of

(i) H is the stabilizer of a pair of symbols (of index $\binom{n}{2}$). The corresponding graph is the triangular graph $T(n)$ (or its complement). Parameters are v =

[1] The lattice graphs with doubly transitive groups of affine type acting are contained in (iii), see (2) of Theorem 11.4.1.

$n(n-1)/2$, $k = 2(n-2)$, $\lambda = n-2$, $\mu = 4$, $r = n-4$, $s = -2$, $f = n-1$, $g = n(n-3)/2$.

(ii) $n = 6$ and H is the stabilizer of a partition of the 6 symbols into three pairs. The graph is $T(6)$, with $(v, k, \lambda, \mu) = (15, 6, 1, 3)$.

(iii) $n = 8$ and H is the stabilizer of a partition of the 8 symbols into two 4-sets. Parameters are $(v, k, \lambda, \mu) = (35, 16, 6, 8)$.

(iv) $n = 10$ and H is the stabilizer of a partition of the 10 symbols into two 5-sets. Parameters are $(v, k, \lambda, \mu) = (126, 25, 8, 4)$.

(v) $n = 4$ and $G = \mathsf{A}_4$ and $H = 2^2$. The graph is K_3.

(vi) $n = 9$ and $G = \mathsf{A}_9$ and H is $\mathsf{P\Gamma L}_2(8)$ (two classes). Parameters are $(v, k, \lambda, \mu) = (120, 56, 28, 24)$.

The graph from case (vi) is the graph $NO_8^+(2)$.

For the triangular graphs, see §1.1.7. Case (ii) is equivalent to Case (i) for $n = 6$. For the graphs from cases (iii), (iv), (vi), see §10.13, §10.40, §10.39.

11.3.2 Classical simple socle

Theorem 11.3.2 (Kantor & Liebler [481])

Let M be one of the groups $\mathsf{Sp}_{2m-2}(q)$, $\Omega^{\pm}_{2m}(q)$, $\Omega_{2m-1}(q)$ or $\mathsf{SU}_m(q)$ for $m \geq 3$ and let q be a prime power. Let $M \trianglelefteq G$ with $G/Z(M) \leq \mathrm{Aut}(M/Z(M))$. Assume that G acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of G. Then at least one of the following holds up to conjugacy under $\mathrm{Aut}(M/Z(M))$.

(i) X is an M-orbit of singular (or isotropic) points.

(ii) X is an M-orbit of maximal totally singular (or isotropic) subspaces and M is one of $\mathsf{Sp}_4(q)$, $\mathsf{SU}_4(q)$, $\mathsf{SU}_5(q)$, $\Omega^-_6(q)$, $\Omega^+_8(q)$ or $\Omega^+_{10}(q)$.

(iii) X is any M-orbit of nonsingular points and M is one of $\mathsf{SU}_m(2)$, $\Omega^{\pm}_{2m}(2)$, $\Omega^{\pm}_{2m}(3)$ or $\Omega_{2m-1}(3)$.

(iv) X is either orbit of nonsingular hyperplanes for $M = \Omega_{2m-1}(4)$ or $M = \Omega_{2m-1}(8)$, where in the latter case $G = \Omega_{2m-1}(8).3$.

(v) $M = \mathsf{SU}_3(3)$ and $K \cap M = \mathsf{PSL}_3(2)$.

(vi) $M = \mathsf{SU}_3(5)$ and $K \cap M = 3.\mathsf{A}_7$.

(vii) $M = \mathsf{SU}_4(3)$ and $K \cap M = 4.\mathsf{PSL}_3(4)$.

(viii) $M = \mathsf{Sp}_6(2)$ and $K = \mathsf{G}_2(2)$.

(ix) $M = \Omega_7(3)$ and $K \cap M = \mathsf{G}_2(3)$.

(x) $M = \mathsf{SU}_6(2)$ and $K \cap M = 3.\mathsf{PSU}_4(3).2$.

The graphs here are in case (i) the polar graphs (with parameters given in Theorem 2.2.12 (in terms of the order (q, t), and orders given in Theorem 2.3.6), in case (ii) given in Theorem 2.2.19 and Theorem 2.2.20, in case (iii) in §3.1.6, §3.1.2, §3.1.3, §3.1.4, in case (iv) in §3.1.4. In cases (v)–(x) the graphs have parameters $(v, k, \lambda, \mu) = (36, 14, 4, 6)$ (§10.14), $(50, 7, 0, 1)$ (§10.19), $(162, 56, 10, 24)$ (§10.48), $(120, 56, 28, 24)$ (§10.39), $(1080, 351, 126, 108)$ (§10.78) and $(1408, 567, 246, 216)$ (§10.81). Hence the graphs of (viii) and (ix) are already contained in (iii).

Theorem 11.3.3 (Kantor & Liebler [481])

Let $M = \mathsf{PSL}_n(q) \leq G \leq \mathrm{Aut}\, M$. Assume that G acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of G. Then at least one of the following occurs up to conjugacy under $\mathrm{Aut}\, M$.

(i) X *is the set of lines for* M*,* $n \geq 4$*.*
(ii) $M = \mathsf{PSL}_2(4) \simeq \mathsf{PSL}_2(5)$*,* $|X| = \binom{5}{2}$*, or*
$M = \mathsf{PSL}_2(9) \simeq \mathsf{A}_6$*,* $|X| = \binom{6}{2}$*, or*
$M = \mathsf{PSL}_4(2) \simeq \mathsf{A}_8$*,* $|X| = \binom{8}{2}$*, or*
$G = \mathsf{P\Gamma L}_2(8)$*,* $|X| = \binom{9}{2}$*.*
(iii) $M = \mathsf{PSL}_3(4)$*,* $M \cap K \simeq \mathsf{A}_6$*.*
(iv) $M = \mathsf{PSL}_4(3)$*,* $M \cap K = \mathsf{PSp}_4(3)$*.*

The graphs here are (i) those of §3.5.1, (ii) the triangular graphs $T(m)$, $m = 6, 8, 9$, (iii) the Gewirtz graph (§10.20), (iv) the $NO_6^+(3)$ graph (§10.35).

11.3.3 Exceptional simple socle

Theorem 11.3.4 (Liebeck & Saxl [520])
Let G *be a finite primitive rank* 3 *permutation group of degree* v*. Assume that the socle* L *of* G *is a simple group of exceptional Lie type, and let* H *be the stabilizer in* L *of a point. Then either* $L = \mathsf{E}_6(q)$*,* H *is a parabolic* $\mathsf{D}_5(q)$*, and* $v = \frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)}$*,* $k = q(q^3+1)\frac{q^8-1}{q-1}$*,* $f = q(q^4+1)(q^6+q^3+1)$*,* $g = q^2(q^6+1)(q^4+1)\frac{q^5-1}{q-1}$ *(see Proposition 4.9.1; two classes), or* L*,* H *and the parameters of* Γ *are as in Table 11.2 below. The comment 'two classes' means that there are two classes of such rank* 3 *representations of* L*, interchanged by a graph automorphism.*

v	k, l	L	H	graph
351	$126, 224$	$\mathsf{G}_2(3)$	$\mathsf{U}_3(3).2$ two classes	$NO_7^{-\perp}(3)$, §10.66
416	$100, 315$	$\mathsf{G}_2(4)$	HJ	§10.68
2016	$975, 1040$	$\mathsf{G}_2(4)$	$\mathsf{U}_3(4).2$	$NO_7^-(4)$, §3.1.4
130816	$32319, 98496$	$\mathsf{G}_2(8)$ $G = \mathsf{G}_2(8).3$	$\mathsf{SU}_3(8).2$	$NO_7^-(8)$, §3.1.4

Table 11.2: Rank 3 graphs with exceptional simple socle. Here $l = v - k - 1$.

In the cases of $\mathsf{G}_2(q)$ of degree $q^3(q^3-1)/2$ the full automorphism group of the graph contains $\mathsf{O}_7(q)$. (See also [519].) Hence the corresponding strongly regular graphs are contained in the classes (iii) and (iv) of Theorem 11.3.2. See also §3.1.4 for a construction of these graphs exhibiting the rank 3 action of $\mathsf{G}_2(3)$, $\mathsf{G}_2(4)$ and $\mathsf{G}_2(8).3$.

11.3.4 Sporadic simple socle

The list of rank 3 representations of the sporadic simple groups other than BM had been determined by Brouwer, and Soicher and R. A. Wilson checked that there are no further examples. The proof is by inspection of the Atlas [215].

Theorem 11.3.5 *Let* G *be a finite primitive rank* 3 *permutation group of degree* v*. Assume that the socle* L *of* G *is a sporadic simple group, and let* H *be the stabilizer in* L *of a point. Then we have one of the cases in Table 11.3 below.*

We see two graphs for M_{23} on 253 vertices with the same permutation character but nonisomorphic permutation representations.

v	k, l	L	H	ref	comment
55	18,36	M_{11}	$M_9.2$	§11.3.5	$T(11)$
66	20,45	M_{12}	$M_{10}.2$ two classes	§11.3.5	$T(12)$
77	16,60	M_{22}	$2^4.A_6$	§10.27	$S(3,6,22)$
100	22,77	HS	M_{22}	§10.31	
100	36,63	HJ	$U_3(3)$	§10.32	
176	70,105	M_{22}	A_7 two classes	§10.51	$S(4,7,23) \setminus S(3,6,22)$
253	42,210	M_{23}	$M_{21}.2$	§11.3.5	$T(23)$
253	112,140	M_{23}	$2^4.A_7$	§10.56	$S(4,7,23)$
275	112,162	McL	$U_4(3)$	§10.61	
276	44,231	M_{24}	$M_{22}.2$	§11.3.5	$T(24)$
1288	495,792	M_{24}	$M_{12}.2$	§10.80	
1782	416,1365	Suz	$G_2(4)$	§10.83	
2300	891,1408	Co_2	$U_6(2).2$	§10.88	
3510	693,2816	Fi_{22}	$2.U_6(2)$	§10.90	
4060	1755,2304	Ru	${}^2F_4(2)$	§10.91	
14080	3159,10920	Fi_{22}	$\Omega_7(3)$ two classes	§10.94	
31671	3510,28160	Fi_{23}	$2.Fi_{22}$	§10.96	
137632	28431,109200	Fi_{23}	$P\Omega_8^+(3).S_3$	§10.97	
306936	31671,275264	Fi'_{24}	Fi_{23}	§10.99	

Table 11.3: Rank 3 graphs with sporadic simple socle

11.3.5 Triangular graphs

If G acts 4-transitively on a set Ω, then G will act as a rank 3 group on the set $\binom{\Omega}{2}$ of unordered pairs from Ω. The corresponding graphs are triangular graphs $T(m)$, where $m = |\Omega|$. These graphs have $\binom{m}{2}$ vertices, and full automorphism group S_m (if $m > 4$).

For sporadic G this happens with M_{11}, M_{12}, M_{23} and M_{24} acting on 11, 12, 23 and 24 points, respectively. The corresponding graphs have 55, 66, 253 and 276 vertices, respectively.

11.4 The affine case

The affine case was finished by Liebeck after substantial earlier work by Foulser and Kallaher.

Theorem 11.4.1 (LIEBECK [517])

Let G be a finite primitive affine permutation group of rank 3 and of degree $v = p^d$, with socle V, where $V \simeq (Z_p)^d$ for some prime p, and let G_0 be the stabilizer of the zero vector in V. Then G_0 belongs to one of the following classes (and, conversely, each of the possibilities listed below does give rise to a rank 3 affine group).

(A) Infinite classes. These are:

(1) $G_0 \le \Gamma L_1(p^d)$. This case is handled in FOULSER & KALLAHER [330], §3. *(See Theorem 11.4.2 below.)*

(2) G_0 imprimitive: G_0 stabilizes a pair $\{V_1, V_2\}$ of subspaces of V, where $V = V_1 \oplus V_2$ and $\dim V_1 = \dim V_2$; moreover, $(G_0)_{V_i}$ is transitive on $V_i \setminus 0$ for $i = 1, 2$ (and hence G_0 is determined by HERING [418]*; see Table 11.1 above).*

(3) Tensor product case: for some a, q *with* $q^a = p^d$, *consider* V *as a vector space* $V_a(q)$ *of dimension* a *over* $\mathbb{F}_q$; *then* G_0 *stabilizes a decomposition of* $V_a(q)$ *as a tensor product* $V_1 \otimes V_2$ *where* $\dim_{\mathbb{F}_q} V_1 = 2$; *moreover,* $G_0^{V_2} \rhd \mathsf{SL}(V_2)$, *or* $G_0^{V_2} = \mathsf{A}_7 < \mathsf{SL}_4(2)$ *(and* $p = q = 2$, $d = a = 8$*), or* $\dim_{\mathbb{F}_q} V_2 \leq 3$.

(4) $G_0 \rhd \mathsf{SL}_a(q)$ *and* $p^d = q^{2a}$.

(5) $G_0 \rhd \mathsf{SL}_2(q)$ *and* $p^d = q^6$.

(6) $G_0 \rhd \mathsf{SU}_a(q)$ *and* $p^d = q^{2a}$.

(7) $G_0 \rhd \Omega_{2a}^{\pm}(q)$ *and* $p^d = q^{2a}$ *(and if* q *is odd,* G_0 *contains an automorphism interchanging the two orbits of* $\Omega_{2a}^{\pm}(q)$ *on nonsingular 1-spaces).*

(8) $G_0 \rhd \mathsf{SL}_5(q)$ *and* $p^d = q^{10}$ *(from the action of* $\mathsf{SL}_5(q)$ *on the exterior square of* $V_5(q)$*).*

(9) $G_0/Z(G_0) \rhd \Omega_7(q).Z_{(2,q-1)}$ *and* $p^d = q^8$ *(from the action of* $B_3(q)$ *on a spin module).*

(10) $G_0/Z(G_0) \rhd P\Omega_{10}^{+}(q)$ *and* $p^d = q^{16}$ *(from the action of* $\mathsf{D}_5(q)$ *on a spin module).*

(11) $G_0 \rhd \mathsf{Sz}(q)$, $q = 2^{2m+1}$, *and* $p^d = q^4$ *(from the embedding* $\mathsf{Sz}(q) < \mathsf{Sp}_4(q)$*).*

(B) 'Extraspecial' classes. Here $G_0 \leq N_{\mathsf{GL}_d(p)}(R)$ *where* R *is an* r*-group, irreducible on* V. *Either* $r = 3$ *and* $R \simeq 3^{1+2}$ *(extraspecial of order* 27*) or* $r = 2$ *and* $|R/Z(R)| = 2^{2m}$ *with* $m = 1$ *or* 2. *If* $r = 2$, *then either* $|Z(R)| = 2$ *and* R *is one of the two extraspecial groups* R_1^m, R_2^m *of order* 2^{1+2m}, *or* $|Z(R)| = 4$, *when we write* $R = R_3^m$. *The possibilities are listed in Table 11.5. (Note that this includes all the soluble rank* 3 *groups from* FOULSER [329]*.)*

(C) 'Exceptional classes'. Here the socle L *of* $G_0/Z(G_0)$ *is simple, and the possibilities are listed in Table 11.6.*

In part (B) the groups are $R_1^m = 2_+^{1+2m}$, $R_2^m = 2_-^{1+2m}$ and $R_3^m = Z_4 \circ 2^{1+2m}$.

case	$v = p^d$	k, l	ref
(A2)	p^{2m}	$2(p^m-1),\ (p^m-1)^2$	$L_2(p^m)$
(A3)	q^{2m}	$(q+1)(q^m-1),\ q(q^m-1)(q^{m-1}-1)$	$H_q(2,m)$, §3.4.1
(A4)	q^{2a}	$(q+1)(q^a-1),\ q(q^a-1)(q^{a-1}-1)$	Baer subspace, §3.4.5
(A5)	q^6	$(q+1)(q^3-1),\ q(q^3-1)(q^2-1)$	cube root subspace, §3.4.5
(A6)	q^{2a}	$(q^a-\varepsilon)(q^{a-1}+\varepsilon),\ q^{a-1}(q-1)(q^a-\varepsilon)$	$\varepsilon = (-1)^a$, §3.3.1
(A7)	q^{2a}	$(q^a-\varepsilon)(q^{a-1}+\varepsilon),\ q^{a-1}(q-1)(q^a-\varepsilon)$	$VO_{2a}^{\varepsilon}(q)$, §3.3.1
(A8)	q^{10}	$(q^5-1)(q^2+1),\ q^2(q^5-1)(q^3-1)$	§3.4.2
(A9)	q^8	$(q^4-1)(q^3+1),\ q^3(q^4-1)(q-1)$	$VO_8^+(q)$, §3.3.1
(A10)	q^{16}	$(q^8-1)(q^3+1),\ q^3(q^8-1)(q^5-1)$	$VD_{5,5}(q)$, §3.3.3
(A11)	q^4	$(q^2+1)(q-1),\ q(q^2+1)(q-1)$	$VSz(q)$, §8.7.1(iv)

Table 11.4: Infinite classes

r	$v = p^d$	k, l	R	ref
3	$64 = 2^6$	27,36	3^{1+2}	§10.25
2	m^2 $(m = 7, 13, 17, 19, 23, 29, 31, 47)$, or 3^4 or 3^6	see Thm. 11.4.4	R_1^1 or R_2^1 (i.e., D_8 or Q_8)	§7.5.2
2	$81 = 3^4$	32,48	R_1^2 or R_2^2	$VO_4^+(3)$, §3.3.1
	$81 = 3^4$	16,64	R_2^2	9×9
	$625 = 5^4$	240,384	R_2^2 or R_3^2	§10.73B
	$2401 = 7^4$	480,1920	R_2^2	§10.89B
	$6561 = 3^8$	1440,5120	R_2^3	§10.93

Table 11.5: Extraspecial classes

Case (A1) above is in more detail:

L	$v = p^d$	k, l	embedding of L	ref
A_5	3^4 or 7^4 or m^2 ($m = 31, 41, 71, 79, 89$)	see Thm. 11.4.3	$\mathsf{A}_5 < \mathsf{PSL}_2(p^{d/2})$	§10.30, §10.89D §7.5
A_6	$64 = 2^6$	18,45	$\mathsf{A}_6 < \mathsf{PSL}_3(4)$	§10.24
M_{11}	$243 = 3^5$	22,220	$\mathsf{M}_{11} < \mathsf{PSL}_5(3)$	§10.55
M_{11}	$243 = 3^5$	110,132	$\mathsf{M}_{11} < \mathsf{PSL}_5(3)$	§10.55
A_7	$256 = 2^8$	45,210	$\mathsf{A}_7 < \mathsf{PSL}_4(4)$	§10.57
A_{10}	$256 = 2^8$	45,210	$\mathsf{A}_{10} < \mathsf{Sp}_8(2)$	§10.57
$\mathsf{PSL}_2(17)$	$256 = 2^8$	102,153	$\mathsf{PSL}_2(17) < \mathsf{Sp}_8(2)$	§10.58
A_9	$256 = 2^8$	120,135	$\mathsf{A}_9 < \Omega_8^+(2)$	§10.60
A_6	$625 = 5^4$	144,480	$\mathsf{A}_6 < \mathsf{PSp}_4(5)$	§10.73A
$\mathsf{PSL}_3(4)$	$729 = 3^6$	224,504	$\mathsf{PSL}_3(4) < P\Omega_6^-(3)$	§10.76
M_{24}	$2048 = 2^{11}$	276,1771	$\mathsf{M}_{24} < \mathsf{PSL}_{11}(2)$	§10.84
M_{24}	$2048 = 2^{11}$	759,1288	$\mathsf{M}_{24} < \mathsf{PSL}_{11}(2)$	§10.85
$\mathsf{PSU}_4(2)$	$2401 = 7^4$	240,2160	$\mathsf{PSU}_4(2) < \mathsf{PSL}_4(7)$	§10.89A
A_7	$2401 = 7^4$	720,1680	$\mathsf{A}_7 < \mathsf{PSp}_4(7)$	§10.89C
HJ	$4096 = 2^{12}$	1575,2520	$\mathsf{HJ} < \mathsf{G}_2(4) < \mathsf{Sp}_6(4)$	§10.92
HJ	$15625 = 5^6$	7560,8064	$\mathsf{HJ} < \mathsf{PSp}_6(5)$	§10.95
$\mathsf{G}_2(4)$, Suz	$531441 = 3^{12}$	65520,465920	$\mathsf{G}_2(4) < \mathsf{Suz} < \mathsf{PSp}_{12}(3)$	§6.3.3, §10.100

Table 11.6: Exceptional classes

Theorem 11.4.2 (FOULSER & KALLAHER [330], §3) *Let $q = p^r$ be a prime power. Let $G = \mathsf{A\Gamma L}(1, q)$, the group consisting of the semilinear maps $x \mapsto ax^\sigma + b$ on $\mathbb{F}_q$. Let T be the subgroup of size q consisting of the translations $x \mapsto x+b$. Let $G_0 = \mathsf{\Gamma L}_1(q)$, so that $G = G_0T$. Let H be a subgroup of G_0. Then HT acts as a rank 3 group on $\mathbb{F}_q$ precisely when H has two orbits on $\mathbb{F}_q^*$. The possible H are found in Theorem 7.4.5 (the case where $H < \mathsf{GL}_1(q)$), Theorem 7.4.6 (the case where H has two orbits of different sizes), and Theorem 7.4.7 (the case where H has two orbits of equal size).*

The graphs here were determined by MUZYCHUK [581], and are the Paley, Peisert and Van Lint-Schrijver graphs.

The case of A_5 in Case (C) above was described in [330], Theorem 5.3.

Theorem 11.4.3 (FOULSER & KALLAHER [330]; see also §7.5) *Let $q = p^r$ be a prime power. Let TH be a rank 3 collineation group of the Desarguesian affine plane $\mathsf{AG}(2, q)$, where T is the translation group of order q^2 and $\bar{H} \cap \mathsf{PSL}(2, q) \simeq \mathsf{A}_5$, where $\bar{H}$ is the image of H under the homomorphism that maps $\mathsf{\Gamma L}_2(q)$ onto $\mathsf{P\Gamma L}_2(q)$. Then either $p^r = 3^2$, or $\bar{H}$ has two orbits on l_∞ of lengths a and b, and (p^r, a, b) is one of $(2^4, 5, 12)$, $(5^2, 6, 20)$, $(31, 12, 20)$, $(41, 12, 30)$, $(7^2, 20, 30)$, $(2^6, 5, 60)$, $(71, 12, 60)$, $(79, 20, 60)$, $(89, 30, 60)$, $(5^3, 6, 120)$. If $p^r = 3^2$, then $\bar{H}$ is transitive on (the 10 points of) l_∞, but H has two orbits of size 40 on the nonzero vectors.*

The graphs here have $v = q^2$ and $k = (q-1)a$, $l = (q-1)b$.

In [330] also the possibility $(p^r, a, b) = (119, 60, 60)$ was listed, but, as Liebeck noted, the authors overlooked there that 119 is not a prime power. The cases $(p^r, a, b) = (16, 5, 12)$, (25,6,20), (64,5,60), and (125,6,120) are not listed in Table 11.6 because $\mathsf{A}_5 \simeq \mathsf{L}_2(4) \simeq \mathsf{L}_2(5)$, so that these occur under the cases (A4) and (A5) of Theorem 11.4.1.

Table 14 of [517] gives subdegrees 105, 150 for the case $L = \mathsf{A}_9$ in Case (C), but Table 12 of [159] corrects that to 120, 135.

The solvable primitive permutation groups of low rank were determined in FOULSER [329]. In particular:

Theorem 11.4.4 (FOULSER [329])

Let G be a maximal solvable primitive permutation group of degree v. Then G_0 is a semilinear group on a vector space V over a field $\mathbb{F}_{p^m}$. Suppose G has rank 3, and let k, l be the lengths of the nontrivial orbits of G_0. Then we have one of the following cases.
(i) G is a collineation group of affine lines.

(ii) G is vector space primitive and has an irreducible minimal normal non-abelian subgroup N which is a q-group for some prime q, such that $|N/Z(N)| = q^{2a}$ for some a, and one of the following cases applies.

(a) $q^a = 3$, $p^m = 4$, $v = 2^6$, $|G_0| = 2^4 \cdot 3^4$, and $(k,l) = (27,36)$.

(b) $q^a = 2$, $v = p^{2m}$, $|G_0| = 24m(p^m - 1)$, and (p^m, k, l) occurs among $(3^2, 32, 48)$, $(13, 72, 96)$, $(17, 96, 192)$, $(19, 144, 216)$, $(3^3, 104, 624)$, $(29, 168, 672)$, $(31, 240, 720)$, $(47, 1104, 1104)$.

(c) $q^a = 4$, $p^m = 3$, $v = 3^4$, $|G_0| = 2^8 \cdot 3^2$, and $(k,l) = (32,48)$.

(d) $q^a = 4$, $p^m = 7$, $v = 7^4$, $|G_0| = 2^7 \cdot 3 \cdot 5$, and $(k,l) = (480,1920)$.

(iii) G is imprimitive and there exists a decomposition $V = V_1 \oplus V_2$ of V into minimal imprimitivity subspaces for G_0, and $G_0|V_i$ is transitive on the nonzero elements of V_i $(i = 1,2)$ (hence $G_0|V_i$ is determined by Huppert's theorem). Moreover, the nontrivial orbits of G_0 are $V_1 \cup V_2 \setminus \{0\}$ and $V \setminus (V_1 \cup V_2)$.

Some of the groups mentioned contain proper rank 3 subgroups. Moreover, there exist two cases in which exceptional doubly transitive groups have proper rank 3 subgroups. Here G is as in (ii), $q^a = 2$, $v = p^{2m} = 2k+1$, and $(p^m, |G_0|)$ is either $(7, 2^3 \cdot 3^2)$ or $(23, 2^3 \cdot 3 \cdot 11)$.

For the cases in (ii)(b), and those of the last sentence, see §7.5.2.

11.5 Rank 3 parameter index

Below we index the parameters of the rank 3 graphs found above not as part of an infinite family, and refer to the theorem or table where they occur.

v	k	λ	μ	r^f	s^g	ref
15	6	1	3	1^9	$(-3)^5$	Thm. 11.3.1
	8	4	4	2^5	$(-2)^9$	
35	16	6	8	2^{20}	$(-4)^{14}$	Thm. 11.3.1
	18	9	9	3^{14}	$(-3)^{20}$	
36	14	4	6	2^{21}	$(-4)^{14}$	Thm. 11.3.2 (v)
	21	12	12	3^{14}	$(-3)^{21}$	
49	24	11	12	3^{24}	$(-4)^{24}$	Thm. 11.4.4
50	7	0	1	2^{28}	$(-3)^{21}$	Thm. 11.3.2 (vi)
	42	35	36	2^{21}	$(-3)^{28}$	
55	18	9	4	7^{10}	$(-2)^{44}$	Table 11.3
	36	21	28	1^{44}	$(-8)^{10}$	
56	10	0	2	2^{35}	$(-4)^{20}$	Thm. 11.3.3 (iii)
	45	36	36	3^{20}	$(-3)^{35}$	
64	18	2	6	2^{45}	$(-6)^{18}$	Table 11.6
	45	32	30	5^{18}	$(-3)^{45}$	
64	27	10	12	3^{36}	$(-5)^{27}$	Thm. 11.4.4, Table 11.5
	36	20	20	4^{27}	$(-4)^{36}$	

continued...

v	k	λ	μ	r^f	s^g	ref
66	20	10	4	8^{11}	$(-2)^{54}$	Table 11.3
	45	28	36	1^{54}	$(-9)^{11}$	
77	16	0	4	2^{55}	$(-6)^{21}$	Table 11.3
	60	47	45	5^{21}	$(-3)^{55}$	
81	16	7	2	7^{16}	$(-2)^{64}$	Table 11.5
	64	49	56	1^{64}	$(-8)^{16}$	
81	32	13	12	5^{32}	$(-4)^{48}$	Thm. 11.4.4, Table 11.5
	48	27	30	3^{48}	$(-6)^{32}$	
81	40	19	20	4^{40}	$(-5)^{40}$	Thm. 11.4.3
100	22	0	6	2^{77}	$(-8)^{22}$	Table 11.3
	77	60	56	7^{22}	$(-3)^{77}$	
100	36	14	12	6^{36}	$(-4)^{63}$	Table 11.3
	63	38	42	3^{63}	$(-7)^{36}$	
120	56	28	24	8^{35}	$(-4)^{84}$	Thm. 11.3.1
	63	30	36	3^{84}	$(-9)^{35}$	
126	25	8	4	7^{35}	$(-3)^{90}$	Thm. 11.3.1
	100	78	84	2^{90}	$(-8)^{35}$	
162	56	10	24	2^{140}	$(-16)^{21}$	Thm. 11.3.2 (vii)
	105	72	60	15^{21}	$(-3)^{140}$	
169	72	31	30	7^{72}	$(-6)^{96}$	Thm. 11.4.4
	96	53	56	5^{96}	$(-8)^{72}$	
176	70	18	34	2^{154}	$(-18)^{21}$	Table 11.3
	105	68	54	17^{21}	$(-3)^{154}$	
243	22	1	2	4^{132}	$(-5)^{110}$	Table 11.6
	220	199	200	4^{110}	$(-5)^{132}$	
243	110	37	60	2^{220}	$(-25)^{22}$	Table 11.6
	132	81	60	24^{22}	$(-3)^{220}$	
253	42	21	4	19^{22}	$(-2)^{230}$	Table 11.3
	210	171	190	1^{230}	$(-20)^{22}$	
253	112	36	60	2^{230}	$(-26)^{22}$	Table 11.3
	140	87	65	25^{22}	$(-3)^{230}$	
256	45	16	6	13^{45}	$(-3)^{210}$	Table 11.6 (twice)
	210	170	182	2^{210}	$(-14)^{45}$	
256	75	26	20	11^{75}	$(-5)^{180}$	Thm. 11.4.3
	180	124	132	4^{180}	$(-12)^{75}$	
256	102	38	42	6^{153}	$(-10)^{102}$	Table 11.6
	153	92	90	9^{102}	$(-7)^{153}$	
256	120	56	56	8^{120}	$(-8)^{135}$	Table 11.6
	135	70	72	7^{135}	$(-9)^{120}$	
275	112	30	56	2^{252}	$(-28)^{22}$	Table 11.3
	162	105	81	27^{22}	$(-3)^{252}$	
276	44	22	4	20^{23}	$(-2)^{252}$	Table 11.3
	231	190	210	1^{252}	$(-21)^{23}$	
289	96	35	30	11^{96}	$(-6)^{192}$	Thm. 11.4.4
	192	125	132	5^{192}	$(-12)^{96}$	
351	126	45	45	9^{168}	$(-9)^{182}$	Table 11.2
	224	142	144	8^{182}	$(-10)^{168}$	
361	144	59	56	11^{144}	$(-8)^{216}$	Thm. 11.4.4
	216	127	132	7^{216}	$(-12)^{144}$	
416	100	36	20	20^{65}	$(-4)^{350}$	Table 11.2
	315	234	252	3^{350}	$(-21)^{65}$	
529	264	131	132	11^{264}	$(-12)^{264}$	Thm. 11.4.4
625	144	43	30	19^{144}	$(-6)^{480}$	Table 11.6, Thm. 11.4.3
	480	365	380	5^{480}	$(-20)^{144}$	
625	240	95	90	15^{240}	$(-10)^{90}$	Table 11.5
	384	233	240	9^{384}	$(-16)^{240}$	
729	104	31	12	23^{104}	$(-4)^{624}$	Thm. 11.4.4
	624	531	552	3^{624}	$(-24)^{104}$	
729	224	61	72	8^{504}	$(-19)^{224}$	Table 11.6
	504	351	342	18^{224}	$(-9)^{504}$	
841	168	47	30	23^{168}	$(-6)^{672}$	Thm. 11.4.4

continued...

v	k	λ	μ	r^f	s^g	ref
	672	533	552	5^{672}	$(-24)^{168}$	
961	240	71	56	23^{240}	$(-8)^{720}$	Thm. 11.4.4
	720	535	552	7^{720}	$(-24)^{240}$	
961	360	139	132	19^{360}	$(-12)^{600}$	Thm. 11.4.3
	600	371	380	11^{600}	$(-20)^{360}$	
1288	495	206	180	35^{252}	$(-9)^{1035}$	Table 11.3
	792	476	504	8^{1035}	$(-36)^{252}$	
1408	567	246	216	39^{252}	$(-9)^{1155}$	Thm. 11.3.2 (x)
	840	488	520	8^{1155}	$(-40)^{252}$	
1681	480	149	132	29^{480}	$(-12)^{1200}$	Thm. 11.4.3
	1200	851	870	11^{1200}	$(-30)^{480}$	
1782	416	100	96	20^{780}	$(-16)^{1001}$	Table 11.3
	1365	1044	1050	15^{1001}	$(-21)^{780}$	
2016	975	462	480	15^{1365}	$(-33)^{650}$	Table 11.2
	1040	544	528	32^{650}	$(-16)^{1365}$	
2048	276	44	36	20^{759}	$(-12)^{1288}$	Table 11.6
	1771	1530	1540	11^{1288}	$(-21)^{759}$	
2048	759	310	264	55^{276}	$(-9)^{1771}$	Table 11.6
	1288	792	840	8^{1771}	$(-56)^{276}$	
2209	1104	551	552	23^{1104}	$(-24)^{1104}$	Thm. 11.4.4
2300	891	378	324	63^{275}	$(-9)^{2024}$	Table 11.3
	1408	840	896	8^{2024}	$(-64)^{275}$	
2401	240	59	20	44^{240}	$(-5)^{2160}$	Table 11.6
	2160	1939	1980	4^{2160}	$(-45)^{240}$	
2401	480	119	90	39^{480}	$(-10)^{1920}$	Thm. 11.4.4, Table 11.5
	1920	1529	1560	9^{1920}	$(-40)^{480}$	
2401	720	229	210	34^{720}	$(-15)^{1680}$	Table 11.6
	1680	1169	1190	14^{1680}	$(-35)^{720}$	
2401	960	389	380	29^{960}	$(-20)^{1440}$	Thm. 11.4.3
	1440	859	870	19^{1440}	$(-30)^{960}$	
3510	693	180	126	63^{429}	$(-9)^{3080}$	Table 11.3
	2816	2248	2304	8^{3080}	$(-64)^{429}$	
4060	1755	730	780	15^{3276}	$(-65)^{783}$	Table 11.3
	2304	1328	1280	64^{783}	$(-16)^{3276}$	
4096	315	74	20	59^{315}	$(-5)^{3780}$	Thm. 11.4.3
	3780	3484	3540	4^{3780}	$(-60)^{315}$	
4096	1575	614	600	39^{1575}	$(-25)^{2520}$	Table 11.6
	2520	1544	1560	24^{2520}	$(-40)^{1575}$	
5041	840	179	132	59^{840}	$(-12)^{4200}$	Thm. 11.4.3
	4200	3491	3540	11^{4200}	$(-60)^{840}$	
6241	1560	419	380	59^{1560}	$(-20)^{4680}$	Thm. 11.4.3
	4680	3499	3540	19^{4680}	$(-60)^{1560}$	
6561	1440	351	306	63^{1440}	$(-18)^{5120}$	Table 11.5
	5120	3985	4032	17^{5120}	$(-64)^{1440}$	
7921	2640	899	870	59^{2640}	$(-30)^{5280}$	Thm. 11.4.3
	5280	3509	3540	29^{5280}	$(-60)^{2640}$	
14080	3159	918	648	279^{429}	$(-9)^{13650}$	Table 11.3
	10920	8408	8680	8^{13650}	$(-280)^{429}$	
15625	744	143	30	119^{744}	$(-6)^{14880}$	Thm. 11.4.3
	14880	14165	14280	5^{14880}	$(-120)^{744}$	
15625	7560	3655	3660	60^{8064}	$(-65)^{7560}$	Table 11.6
	8064	4163	4160	64^{7560}	$(-61)^{8064}$	
31671	3510	693	351	351^{782}	$(-9)^{30888}$	Table 11.3
	28160	25000	25344	8^{30888}	$(-352)^{782}$	
130816	32319	7742	8064	63^{112347}	$(-385)^{18468}$	Table 11.2
	98496	74240	73920	384^{18468}	$(-64)^{112347}$	
137632	28431	6030	5832	279^{30888}	$(-81)^{106743}$	Table 11.3
	109200	86600	86800	80^{106743}	$(-280)^{30888}$	
306936	31671	3510	3240	351^{57477}	$(-81)^{249458}$	Table 11.3
	275264	246832	247104	80^{249458}	$(-352)^{57477}$	
531441	65520	8559	8010	639^{65520}	$(-90)^{465920}$	Table 11.6
	465920	408409	408960	89^{465920}	$(-640)^{65520}$	

Table 11.7: Parameters of rank 3 graphs

11.6 Small rank 3 graphs

Below we give the parameters of the primitive rank 3 graphs with $v \leq 1024$. The full group of automorphisms is G, the point stabilizer S. Of a complementary pair of graphs only the one with smallest k is given.

v	k	λ	μ	G	S	ref	graph
5	2	1	1	D_{10}	2	§10.1	Paley
9	4	1	2	$3^2:\mathsf{D}_8$	D_8	§10.2	Paley, 3×3
10	3	0	1	S_5	D_{12}	§10.3	$\overline{T(5)}$, Petersen
13	6	2	3	$13:6$	6	§10.4	Paley
15	6	1	3	S_6	$2 \times \mathsf{S}_4$	§10.5	$\overline{T(6)}$, $\mathsf{GQ}(2,2)$
16	5	0	2	$2^4:\mathsf{S}_5$	S_5	§10.7	$VO_4^-(2)$, $\overline{\text{Clebsch}}$, cubes
16	6	2	2	$(\mathsf{S}_4 \times \mathsf{S}_4):2$	$(\mathsf{S}_3 \times \mathsf{S}_3):2$		$\overline{VO_4^+(2)}$, 4×4
17	8	3	4	$17:8$	8	§10.8	Paley
21	10	5	4	S_7	$2 \times \mathsf{S}_5$		$T(7)$
25	8	3	2	$(\mathsf{S}_5 \times \mathsf{S}_5):2$	$(\mathsf{S}_4 \times \mathsf{S}_4):2$		5×5, cubes
25	12	5	6	$5^2:(4 \times \mathsf{S}_3)$	$4 \times \mathsf{S}_3$		Paley
27	10	1	5	$\mathsf{O}_5(3):2$	$2^4:\mathsf{S}_5$	§10.10	$\mathsf{O}_6^-(2)$, $\mathsf{GQ}(2,4)$, $\overline{\text{Schläfli}}$
28	12	6	4	S_8	$2 \times \mathsf{S}_6$	§10.11	$T(8)$
29	14	6	7	$29:14$	14	§10.12	Paley
35	16	6	8	S_8	$(\mathsf{S}_4 \times \mathsf{S}_4):2$	§10.13	$\overline{J}(8,4)$
36	10	4	2	$(\mathsf{S}_6 \times \mathsf{S}_6):2$	$(\mathsf{S}_5 \times \mathsf{S}_5):2$		6×6
36	14	4	6	$\mathsf{U}_3(3):2$	$\mathsf{L}_3(2):2$	§10.14	
36	14	7	4	S_9	$2 \times \mathsf{S}_7$		$T(9)$
36	15	6	6	$\mathsf{O}_5(3):2$	$2 \times \mathsf{S}_6$	§10.15	$NO_6^-(2)$
37	18	8	9	$37:18$	18		Paley
40	12	2	4	$\mathsf{O}_5(3):2$	$3^3:(\mathsf{S}_4 \times 2)$	§10.16	$\mathsf{O}_5(3)$, $\mathsf{GQ}(3,3)$
40	12	2	4	$\mathsf{O}_5(3):2$	$3_+^{1+2}:2\mathsf{S}_4$	§10.16	$\mathsf{Sp}_4(3)$, $\mathsf{GQ}(3,3)$
41	20	9	10	$41:20$	20		Paley
45	12	3	3	$\mathsf{O}_5(3):2$	$((2^{3+2}:3^2):2):2$	§10.17	$\mathsf{U}_4(2)$, $\mathsf{GQ}(4,2)$
45	16	8	4	S_{10}	$2 \times \mathsf{S}_8$		$T(10)$
49	12	5	2	$(\mathsf{S}_7 \times \mathsf{S}_7):2$	$(\mathsf{S}_6 \times \mathsf{S}_6):2$		7×7
49	24	11	12	$7^2:S$	$3 \times \mathsf{D}_{16}$	§10.18	Paley
49	24	11	12	$7^2:S$	$3 \times \mathsf{SL}_2(3)$	§10.18	Peisert
50	7	0	1	$\mathsf{U}_3(5):2$	S_7	§10.19	Hoffman-Singleton
53	26	12	13	$53:26$	26		Paley
55	18	9	4	S_{11}	$2 \times \mathsf{S}_9$		$T(11)$
56	10	0	2	$\mathsf{L}_3(4):2^2$	$\mathsf{A}_6:2^2$	§10.20	Gewirtz
61	30	14	15	$61:30$	30		Paley
63	30	13	15	$\mathsf{O}_7(2)$	$2^5:\mathsf{S}_6$	§10.21	$\mathsf{Sp}_6(2)$
64	14	6	2	$(\mathsf{S}_8 \times \mathsf{S}_8):2$	$(\mathsf{S}_7 \times \mathsf{S}_7):2$		8×8
64	18	2	6	$2^6:S$	$3\mathsf{S}_6$	§10.24	$\mathsf{GQ}(3,5)$
64	21	8	6	$2^6:S$	$\mathsf{L}_3(2) \times \mathsf{S}_3$	§3.4.1	$H_2(2,3)$, cubes
64	27	10	12	$2^6:S$	$\mathsf{O}_5(3):2$	§10.25	$VO_6^-(2)$
64	28	12	12	$2^6:S$	S_8	§10.26	$\overline{VO_6^+(2)}$
66	20	10	4	S_{12}	$2 \times \mathsf{S}_{10}$		$T(12)$
73	36	17	18	$73:36$	36		Paley
77	16	0	4	$\mathsf{M}_{22}:2$	$2^4:\mathsf{S}_6$	§10.27	
78	22	11	4	S_{13}	$2 \times \mathsf{S}_{11}$		$T(13)$
81	16	7	2	$(\mathsf{S}_9 \times \mathsf{S}_9):2$	$(\mathsf{S}_8 \times \mathsf{S}_8):2$		9×9
81	20	1	6	$3^4:S$	$(2 \times \mathsf{S}_6):2$	§10.28	$VO_4^-(3)$
81	32	13	12	$3^4:S$	$(2^{3+2}:3^2):\mathsf{D}_8$		$VO_4^+(3)$
81	40	19	20	$3^4:S$	$40:4$	§10.30	Paley
81	40	19	20	$3^4:S$	$\mathsf{SL}_2(5):2^2$	§10.30	Peisert
85	20	3	5	$\mathsf{O}_5(4):2$	$2^6:(\mathsf{A}_5:\mathsf{S}_3)$		$\mathsf{Sp}_4(4)$
89	44	21	22	$89:44$	44		Paley
91	24	12	4	S_{14}	$2 \times \mathsf{S}_{12}$		$T(14)$
97	48	23	24	$97:48$	48		Paley
100	18	8	2	S_{10} wr 2	S_9 wr 2		10×10
100	22	0	6	$\mathsf{HS}:2$	$\mathsf{M}_{22}:2$	§10.31	Higman-Sims
100	36	14	12	$\mathsf{HJ}:2$	$\mathsf{U}_3(3):2$	§10.32	Hall-Janko

continued...

v	k	λ	μ	G	S	ref	graph
101	50	24	25	$101:50$	50		Paley
105	26	13	4	S_{15}	$2\times\mathsf{S}_{13}$		$T(15)$
109	54	26	27	$109:54$	54		Paley
112	30	2	10	$\mathsf{U}_4(3):\mathsf{D}_8$	$3^4:((2\times\mathsf{A}_6).2^2)$	§10.34	$\mathsf{GQ}(3,9)$
113	56	27	28	$113:56$	56		Paley
117	36	15	9	$\mathsf{L}_4(3):2$	$2\times(\mathsf{O}_5(3):2)$	§10.35	$NO_6^+(3)$
119	54	21	27	$\mathsf{O}_8^-(2):2$	$2^6:(\mathsf{O}_5(3):2)$	§10.36	$O_8^-(2)$
120	28	14	4	S_{16}	$2\times\mathsf{S}_{14}$		$T(16)$
120	51	18	24	$\mathsf{O}_5(4):2$	$\mathsf{L}_2(16):4$	§10.38	$NO_5^-(4)$
120	56	28	24	$\mathsf{O}_8^+(2):2$	$2\times\mathsf{O}_7(2)$	§10.39	$\overline{NO_8^+(2)}$
121	20	9	2	S_{11} wr 2	S_{10} wr 2		11×11
121	40	15	12	$11^2:S$	$40:2$	§7.4.5	cubes
121	60	29	30	$11^2:S$	$5\times\mathsf{D}_{24}$		Paley
121	60	29	30	$11^2:S$	$5\times(3:4)$		Peisert
125	62	30	31	$5^3:S$	$2\times(31:3)$		Paley
126	25	8	4	S_{10}	S_5 wr 2	§10.40	
126	45	12	18	$\mathsf{U}_4(3):2^2$	$2\times(\mathsf{O}_5(3):2)$	§10.41	$NO_6^-(3)$
130	48	20	16	$\mathsf{O}_6^+(3):2^2$	$[2^8.3^6]$		$\mathsf{O}_6^+(3)$
135	64	28	32	$\mathsf{O}_8^+(2):2$	$2^6:\mathsf{S}_8$	§10.43	
136	30	15	4	S_{17}	$2\times\mathsf{S}_{15}$		$T(17)$
136	60	24	28	$\mathsf{O}_5(4):2$	$(\mathsf{A}_5\times\mathsf{A}_5):2^2$		$\overline{NO_5^+(4)}$
136	63	30	28	$\mathsf{O}_8^-(2):2$	$2\times\mathsf{O}_7(2)$	§10.44	$NO_8^-(2)$
137	68	33	34	$137:68$	68		Paley
144	22	10	2	S_{12} wr 2	S_{11} wr 2		12×12
149	74	36	37	$149:74$	74		Paley
153	32	16	4	S_{18}	$2\times\mathsf{S}_{16}$		$T(18)$
155	42	17	9	$\mathsf{L}_5(2)$	$2^6:(\mathsf{L}_3(2)\times\mathsf{S}_3)$	§3.5	$J_2(5,2)$
156	30	4	6	$\mathsf{O}_5(5):2$	$5^3:(4\times\mathsf{S}_5)$	§10.47	$\mathsf{O}_5(5)$, $\mathsf{GQ}(5,5)$
156	30	4	6	$\mathsf{O}_5(5):2$	$5_+^{1+2}:4\mathsf{S}_5$	§10.47	$\mathsf{Sp}_4(5)$, $\mathsf{GQ}(5,5)$
157	78	38	39	$157:78$	78		Paley
162	56	10	24	$\mathsf{U}_4(3):2^2$	$\mathsf{L}_3(4):2^2$	§10.48	
165	36	3	9	$\mathsf{P\Gamma U}_5(2)$	$2_-^{1+6}:3_+^{1+2}:2\mathsf{S}_4$		$\mathsf{U}_5(2)$, $\mathsf{GQ}(4,8)$
169	24	11	2	S_{13} wr 2	S_{12} wr 2		13×13
169	72	31	30	$13^2:S$	$3\times(\mathsf{SL}_2(3):4)$	§7.5.2	
169	84	41	42	$13^2:S$	$84:2$		Paley
171	34	17	4	S_{19}	$2\times\mathsf{S}_{17}$		$T(19)$
173	86	42	43	$173:86$	86		Paley
176	40	12	8	$\mathsf{U}_5(2):2$	$\mathsf{U}_4(2):\mathsf{S}_3$	§10.49	
176	70	18	34	M_{22}	A_7	§10.51	
181	90	44	45	$181:90$	90		Paley
190	36	18	4	S_{20}	$2\times\mathsf{S}_{18}$		$T(20)$
193	96	47	48	$193:96$	96		Paley
196	26	12	2	S_{14} wr 2	S_{13} wr 2		14×14
197	98	48	49	$197:98$	98		Paley
210	38	19	4	S_{21}	$2\times\mathsf{S}_{19}$		$T(21)$
225	28	13	2	S_{15} wr 2	S_{14} wr 2		15×15
229	114	56	57	$229:114$	114		Paley
231	40	20	4	S_{22}	$2\times\mathsf{S}_{20}$		$T(22)$
233	116	57	58	$233:116$	116		Paley
241	120	59	60	$241:120$	120		Paley
243	22	1	2	$3^5:S$	$2\times\mathsf{M}_{11}$	§10.55	Berlekamp-Van Lint-Seidel
243	110	37	60	$3^5:S$	$2\times\mathsf{M}_{11}$	§10.55	Delsarte dual of BvLS
253	42	21	4	S_{23}	$2\times\mathsf{S}_{21}$		$T(23)$
253	112	36	60	M_{23}	$2^4:\mathsf{A}_7$	§10.56	$S(4,7,23)$
255	126	61	63	$\mathsf{O}_9(2)$	$2^7:\mathsf{O}_7(2)$		$\mathsf{Sp}_8(2)$
256	30	14	2	S_{16} wr 2	S_{15} wr 2		16×16
256	45	16	6	$2^8:S$	$\mathsf{A}_8\times\mathsf{S}_3$	§10.57	
256	45	16	6	$2^8:S$	S_{10}	§10.57	
256	51	2	12	$2^8:S$	$(3\times\mathsf{SL}_2(16)):4$		$VO_4^-(4)$
256	75	26	20	$2^8:S$	$(\mathsf{A}_5\times\mathsf{A}_5):\mathsf{D}_{12}$		$VO_4^+(4)$
256	85	24	30	$2^8:S$	$85:8$	§7.4.5	cubes
256	102	38	42	$2^8:S$	$\mathsf{L}_2(17)$	§10.58	

continued...

v	k	λ	μ	G	S	ref	graph
256	119	54	56	$2^8:S$	$\mathsf{O}_8^-(2):2$	§10.59	$VO_8^-(2)$
256	120	56	56	$2^8:S$	$\mathsf{O}_8^+(2):2$	§10.60	$VO_8^+(2)$
257	128	63	64	$257:128$	128		Paley
269	134	66	67	$269:134$	134		Paley
275	112	30	56	$\mathsf{McL}:2$	$\mathsf{U}_4(3):2$	§10.61	McLaughlin
276	44	22	4	S_{24}	$2\times\mathsf{S}_{22}$		$T(24)$
277	138	68	69	$277:138$	138		Paley
280	36	8	4	$\mathsf{U}_4(3):\mathsf{D}_8$	$[2^7.3^6]$		$\mathsf{GQ}(9,3)$
281	140	69	70	$281:140$	140		Paley
289	32	15	2	$\mathsf{S}_{17}\,\mathrm{wr}\,2$	$\mathsf{S}_{16}\,\mathrm{wr}\,2$		17×17
289	96	35	30	$17^2:S$	$96:2$	§7.4.5	cubes
289	96	35	30	$17^2:S$	$8.\mathsf{S}_4:2$	§7.5.2	
289	144	71	72	$17^2:S$	$144:2$		Paley
293	146	72	73	$293:146$	146		Paley
297	40	7	5	$\mathsf{P\Gamma U}_5(2)$	$[2^8]:(\mathsf{A}_5:\mathsf{S}_3)$	§10.63	$\mathsf{U}_5(2)$, $\mathsf{GQ}(8,4)$
300	46	23	4	S_{25}	$2\times\mathsf{S}_{23}$		$T(25)$
313	156	77	78	$313:156$	156		Paley
317	158	78	79	$317:158$	158		Paley
324	34	16	2	$\mathsf{S}_{18}\,\mathrm{wr}\,2$	$\mathsf{S}_{17}\,\mathrm{wr}\,2$		18×18
325	48	24	4	S_{26}	$2\times\mathsf{S}_{24}$		$T(26)$
325	68	3	17	$U_4(4):4$	$2^8:(\mathsf{L}_2(16):(3:4))$		$\mathsf{O}_6^-(4)$, $\mathsf{GQ}(4,16)$
337	168	83	84	$337:168$	168		Paley
349	174	86	87	$349:174$	174		Paley
351	50	25	4	S_{27}	$2\times\mathsf{S}_{25}$		$T(27)$
351	126	45	45	$\mathsf{O}_7(3):2$	$(2.\mathsf{U}_4(3)):2^2$	§10.66	$NO_7^{-\perp}(3)$
353	176	87	88	$353:176$	176		Paley
357	100	35	25	$L_4(4):2^2$	$2^8:(\mathsf{A}_5\times\mathsf{A}_4):D_{12})$		$\mathsf{O}_6^+(4)$
361	36	17	2	$\mathsf{S}_{19}\,\mathrm{wr}\,2$	$\mathsf{S}_{18}\,\mathrm{wr}\,2$		19×19
361	144	59	56	$19^2:S$	$9\times\mathsf{GL}_2(3)$	§7.5.2	
361	180	89	90	$19^2:S$	$180:2$		Paley
361	180	89	90	$19^2:S$	$9\times(5:4)$		Peisert
364	120	38	40	$\mathsf{O}_7(3):2$	$3^5:(2\times(\mathsf{O}_5(3):2))$		$\mathsf{O}_7(3)$
364	120	38	40	$\mathsf{PSp}_6(3):2$	$[3^5]:(2.\mathsf{O}_5(3):2)$		$\mathsf{Sp}_6(3)$
373	186	92	93	$373:186$	186		Paley
378	52	26	4	S_{28}	$2\times\mathsf{S}_{26}$		$T(28)$
378	117	36	36	$\mathsf{O}_7(3):2$	$2\times(\mathsf{L}_4(3):2)$	§10.67	$NO_7^{+\perp}(3)$
389	194	96	97	$389:194$	194		Paley
397	198	98	99	$397:198$	198		Paley
400	38	18	2	$\mathsf{S}_{20}\,\mathrm{wr}\,2$	$\mathsf{S}_{19}\,\mathrm{wr}\,2$		20×20
400	56	6	8	$\mathsf{O}_5(7):2$	$7^3:(6\times(\mathsf{L}_3(2):2))$		$\mathsf{O}_5(7)$, $\mathsf{GQ}(7,7)$
400	56	6	8	$\mathsf{O}_5(7):2$	$7_+^{1+2}:\mathsf{GL}_2(7)$		$\mathsf{Sp}_4(7)$, $\mathsf{GQ}(7,7)$
401	200	99	100	$401:200$	200		Paley
406	54	27	4	S_{29}	$2\times\mathsf{S}_{27}$		$T(29)$
409	204	101	102	$409:204$	204		Paley
416	100	36	20	$\mathsf{G}_2(4):2$	$\mathsf{HJ}:2$	§10.68	
421	210	104	105	$421:210$	210		Paley
433	216	107	108	$433:216$	216		Paley
435	56	28	4	S_{30}	$2\times\mathsf{S}_{28}$		$T(30)$
441	40	19	2	$\mathsf{S}_{21}\,\mathrm{wr}\,2$	$\mathsf{S}_{20}\,\mathrm{wr}\,2$		21×21
449	224	111	112	$433:449$	224		Paley
457	228	113	114	$457:228$	228		Paley
461	230	114	115	$461:230$	230		Paley
465	58	29	4	S_{31}	$2\times\mathsf{S}_{29}$		$T(31)$
484	42	20	2	$\mathsf{S}_{22}\,\mathrm{wr}\,2$	$\mathsf{S}_{21}\,\mathrm{wr}\,2$		22×22
495	238	109	119	$\mathsf{O}_{10}^-(2):2$	$2^8:(\mathsf{O}_8^-(2):2)$	§10.69	$\mathsf{O}_{10}^-(2)$
496	60	30	4	S_{32}	$2\times\mathsf{S}_{30}$		$T(32)$
496	240	120	112	$\mathsf{O}_{10}^+(2):2$	$2\times\mathsf{O}_9(2)$	§3.1.2	$\overline{NO_{10}^+(2)}$
509	254	126	127	$509:254$	254		Paley
521	260	129	130	$521:260$	260		Paley
527	256	120	128	$\mathsf{O}_{10}^+(2):2$	$2^8:\mathsf{O}_8^+(2):2$	§2.6.1	$\overline{\Gamma(\mathsf{O}_{10}^+(2))}$
528	62	31	4	S_{33}	$2\times\mathsf{S}_{31}$		$T(33)$
528	255	126	120	$\mathsf{O}_{10}^-(2):2$	$2\times\mathsf{O}_9(2)$	§3.1.2	$NO_{10}^-(2)$

continued...

v	k	λ	μ	G	S	ref	graph
529	44	21	2	S_{23} wr 2	S_{22} wr 2		23 × 23
529	176	63	56	$23^2:S$	$176:2$	§7.4.5	cubes
529	264	131	132	$23^2:S$	$264:2$	§10.70	Paley
529	264	131	132	$23^2:S$	$11\times(3:Q_8)$	§10.70	Peisert
529	264	131	132	$23^2:S$	$11\times SL_2(3)$	§10.70	sporadic Peisert
541	270	134	135	$541:270$	270		Paley
557	278	138	139	$557:278$	278		Paley
561	64	32	4	S_{34}	$2\times S_{32}$		$T(34)$
569	284	141	142	$569:284$	284		Paley
576	46	22	2	S_{24} wr 2	S_{23} wr 2		24 × 24
577	288	143	144	$577:288$	288		Paley
585	72	7	9	$P\Gamma O_5(8)$	$2^9:\Gamma L_2(8)$		$GQ(8,8)$
593	296	147	148	$593:296$	296		Paley
595	66	33	4	S_{35}	$2\times S_{33}$		$T(35)$
601	300	149	150	$601:300$	300		Paley
613	306	152	153	$613:306$	306		Paley
617	308	153	154	$617:308$	308		Paley
625	48	23	2	S_{25} wr 2	S_{24} wr 2		25 × 25
625	104	3	20	$5^4:S$	$L_2(25):(8:2)$	§3.3.1	$VO_4^-(5)$
625	144	43	30	$5^4:S$	$4.(S_5 \text{ wr } 2)$	§3.4.1	$VO_4^+(5)$
625	144	43	30	$5^4:S$	$4.S_6$	§10.73A	
625	208	63	72	$5^4:S$	$208:4$	§7.4.5	cubes
625	240	95	90	$5^4:S$	$4.(2^4:S_6)$	§10.73B	
625	312	155	156	$5^4:S$	$312:4$		Paley
630	68	34	4	S_{36}	$2\times S_{34}$		$T(36)$
641	320	159	160	$641:320$	320		Paley
651	90	33	9	$L_6(2)$	$2^8:(A_8\times S_3)$	§3.5	$J_2(6,2)$
653	326	162	163	$653:326$	326		Paley
661	330	164	165	$661:330$	330		Paley
666	70	35	4	S_{37}	$2\times S_{35}$		$T(37)$
672	176	40	48	$U_6(2):S_3$	$U_5(2):S_3$		$NU_6(2)$
673	336	167	168	$673:336$	336		Paley
676	50	24	2	S_{26} wr 2	S_{25} wr 2		26 × 26
677	338	168	169	$677:338$	338		Paley
693	180	51	45	$U_6(2)\times S_3$	$2_+^{1+8}:(O_5(3):S_3)$	§10.74	$U_6(2)$
701	350	174	175	$701:350$	350		Paley
703	72	36	4	S_{38}	$2\times S_{36}$		$T(38)$
709	354	176	177	$709:354$	354		Paley
729	52	25	2	S_{27} wr 2	S_{26} wr 2		27 × 27
729	104	31	12	$3^6:S$	$L_3(3)\times GL_2(3)$	§3.4.1	$H_3(2,3)$
729	224	61	72	$3^6:S$	$2.U_4(3):D_8$	§3.3.1	$VO_6^-(3)$
729	260	97	90	$3^6:S$	$2.L_4(3):2^2$	§3.3.1	$VO_6^+(3)$
729	364	181	182	$3^6:S$	$364:6$		Paley
729	364	181	182	$3^6:S$	$182:6$		Peisert
733	366	182	183	$733:366$	366		Paley
741	74	37	4	S_{39}	$2\times S_{37}$		$T(39)$
756	130	4	26	$U_4(5):2^2$	$5^4:(L_2(25):(8:2))$		$GQ(5,25)$
757	378	188	189	$757:378$	378		Paley
761	380	189	190	$761:380$	380		Paley
769	384	191	192	$769:384$	384		Paley
773	386	192	193	$773:386$	386		Paley
780	76	38	4	S_{40}	$2\times S_{38}$		$T(40)$
784	54	26	2	S_{28} wr 2	S_{27} wr 2		28 × 28
797	398	198	199	$797:398$	398		Paley
806	180	54	36	$L_4(5):D_8$	$5^4:2.(A_5\times A_5).2.2.4$		$O_6^+(5)$
809	404	201	202	$809:404$	404		Paley
820	78	39	4	S_{41}	$2\times S_{39}$		$T(41)$
820	90	8	10	$O_5(9):2^2$	$3^6:(A_6.2:QD_{16})$		$GQ(9,9)$
820	90	8	10	$O_5(9):2^2$	$[3^6]:SL(2,9):QD_{16}$		$GQ(9,9)$
821	410	204	205	$821:410$	410		Paley
829	414	206	207	$829:414$	414		Paley
841	56	27	2	S_{29} wr 2	S_{28} wr 2		29 × 29
841	168	47	30	$29^2:S$	$7\times(SL_2(3):4)$	§7.5.2	
841	280	99	90	$29^2:S$	$280:2$	§7.4.5	cubes

continued...

v	k	λ	μ	G	S	ref	graph
841	420	209	210	$29^2:S$	$420:2$		Paley
853	426	212	213	$853:426$	426		Paley
857	428	213	214	$857:428$	428		Paley
861	80	40	4	S_{42}	$2\times\mathsf{S}_{40}$		$T(42)$
877	438	218	219	$877:438$	438		Paley
881	440	219	220	$881:440$	440		Paley
900	58	28	2	S_{30} wr 2	S_{29} wr 2		30×30
903	82	41	4	S_{43}	$2\times\mathsf{S}_{42}$		$T(43)$
929	464	231	232	$929:464$	464		Paley
937	468	233	234	$937:468$	468		Paley
941	470	234	235	$941:470$	470		Paley
946	84	42	4	S_{44}	$2\times\mathsf{S}_{42}$		$T(44)$
953	476	237	238	$953:476$	476		Paley
961	60	29	2	S_{31} wr 2	S_{30} wr 2		31×31
961	240	71	56	$31^2:S$	$15\times2.\mathsf{S}_4$	§10.77	
961	360	139	132	$31^2:S$	$15\times\mathsf{SL}_2(5)$	§10.77	
961	480	239	240	$31^2:S$	$480:2$		Paley
961	480	239	240	$31^2:S$	$240:2$		Peisert
977	488	243	244	$977:488$	488		Paley
990	86	43	4	S_{45}	$2\times\mathsf{S}_{43}$		$T(45)$
997	498	248	249	$997:498$	498		Paley
1009	504	251	252	$1009:504$	504		Paley
1013	506	252	253	$1013:506$	506		Paley
1021	510	254	255	$1021:510$	510		Paley
1023	510	253	255	$\mathsf{O}_{11}(2)$	$2^9:\mathsf{O}_9(2)$		$\mathsf{Sp}_{10}(2)$
1024	62	30	2	S_{32} wr 2	S_{31} wr 2		32×32
1024	93	32	6	$2^{10}:S$	$\mathsf{L}_5(2)\times\mathsf{S}_3$		$H_2(2,5)$
1024	155	42	20	$2^{10}:S$	$\mathsf{L}_5(2)$	§3.4.2	
1024	341	120	110	$2^{10}:S$	$341:10$	§7.4.5	cubes
1024	495	238	240	$2^{10}:S$	$\mathsf{O}^-_{10}(2):2$		$\overline{VO^-_{10}(2)}$
1024	496	240	240	$2^{10}:S$	$\mathsf{O}^+_{10}(2):2$		$\overline{VO^+_{10}(2)}$

Table 11.8: Small rank 3 graphs

11.7 Small rank 4–10 strongly regular graphs

Below we give the parameters of the strongly regular with $v \le 1024$ with a full automorphism group acting primitively of rank 4–10.

For rank 3 graphs the group action is imprimitive if and only if the graph is imprimitive (aK_m or its complement $K_{a\times m}$). For $r \ge 4$, a primitive strongly regular graph can have an automorphism group that acts imprimitively with rank r. For example, the graph on the lines of $\mathsf{AG}(3,q)$ has a rank 4 group with imprimitive action, preserving parallelism.

Since there are very many graphs with Latin square parameters $\mathrm{LS}_n(q)$ (that is, $v = q^2$, $k = (q-1)n$, $\lambda = q + n(n-3)$, $\mu = n(n-1)$) where q is a prime power, we omit those.

The full group of automorphisms is G, the rank is 'rk', and '#' gives the number of nonisomorphic such graphs. Of a complementary pair of graphs only the one with smallest k is given.

v	k	λ	μ	#	rk	G	suborbit sizes	ref
63	30	13	15	1	4	$\mathsf{PSU}_3(3).2$	$1,6,24,32$	§10.22
81	30	9	12	1	4	$3^4:(2\times\mathsf{S}_6)$	$1,20,30,30$	§10.29
105	32	4	12	1	4	$\mathsf{L}_3(4).D_{12}$	$1,8,32,64$	§10.33
120	42	8	18	1	4	$\mathsf{L}_3(4).2^2$	$1,21,42,56$	§10.37
120	56	28	24	1	4	S_{10}	$1,21,35,63$	p. 299
120	56	28	24	1	7	S_7	$1,7,14,14,21,21,42$	
144	39	6	12	1	6	$\mathsf{L}_3(3).2$	$1,13,26,26,39,39$	§10.45
144	55	22	20	1	4	$\mathsf{M}_{12}.2$	$1,22,55,66$	§10.46
144	66	30	30	2	4	$\mathsf{M}_{12}.2$	$1,22,55,66$	§10.46
175	72	20	36	1	4	$\mathsf{P\Sigma U}_3(5)$	$1,12,72,90$	p. 269
208	75	30	25	1	4	$\mathsf{P\Gamma U}_3(4)$	$1,12,75,120$	$NU_3(4)$
231	30	9	3	1	4	$M_{22}.2$	$1,30,40,160$	§10.54

continued...

v	k	λ	μ	#	rk	G	suborbit sizes	ref
256	68	12	20	1	4	$\mathsf{A\Sigma L}_2(16)$	$1, 51, 68, 136$	
256	102	38	42	1	4	$2^8:(3\times(17:4))$	$1, 51, 102, 102$	§10.58
280	36	8	4	1	4	$\mathsf{HJ}.2$	$1, 36, 108, 135$	p. 287
280	117	44	52	1	5	S_9	$1, 27, 36, 54, 162$	§10.62
280	135	70	60	1	4	$\mathsf{HJ}.2$	$1, 36, 108, 135$	p. 287
300	65	10	15	1	4	$\mathsf{PGO}_5(5)$	$1, 65, 104, 130$	$NO_5^{-\perp}(5)$
300	104	28	40	1	4	$\mathsf{PGO}_5(5)$	$1, 65, 104, 130$	$NO_5^-(5)$
325	60	15	10	1	4	$\mathsf{PGO}_5(5)$	$1, 60, 120, 144$	$NO_5^{+\perp}(5)$
325	144	68	60	1	4	$\mathsf{PGO}_5(5)$	$1, 60, 120, 144$	$NO_5^+(5)$
330	63	24	9	1	5	S_{11}	$1, 28, 35, 126, 140$	p. 26
525	144	48	36	1	6	$\mathsf{P\Gamma U}_3(5)$	$1, 20, 120, 120, 120, 144$	$NU_3(5)$
540	224	88	96	1	4	$\mathsf{PSU}_4(3).D_8$	$1, 63, 224, 252$	$NU_4(3)$
560	208	72	80	1	7	$\mathsf{PSz}(8).3$	$1, 39, 52, 78, 78, 156, 156$	§10.72
625	208	63	72	1	4	$5^4:(13:(16:2))$	$1, 208, 208, 208$	
625	208	63	72	1	5	$5^4:(13:(8:4))$	$1, 104, 104, 208, 208$	
625	208	63	72	1	7	$5^4:(4.(4\times 4).6)$	$1, 16, 64, 96, 128, 128, 192$	
625	260	105	110	1	4	$5^4:4.\mathsf{PGO}_4^-(5)$	$1, 104, 260, 260$	$VNO_4^-(5)$
729	112	1	20	1	4	$3^6:2.\mathsf{L}_3(4).2$	$1, 112, 112, 504$	§10.75
729	168	27	42	1	8	$3^6:2.\mathsf{S}_5$	$1, 40, 40, 48, 120, 120, 120, 240$	
729	224	61	72	1	7	$3^6:2.\mathsf{P\Gamma L}_2(9)$	$1, 80, 90, 90, 144, 144, 180$	
729	252	81	90	1	4	$3^6:2.\mathsf{PGO}_6^-(3)$	$1, 224, 252, 252$	$VNO_6^-(3)$
729	252	81	90	1	7	$3^6:2.\mathsf{P\Gamma L}_2(9)$	$1, 72, 72, 80, 144, 180, 180$	
729	252	81	90	2	10	$3^6:2.\mathsf{PGL}_2(9)$	$1, 72, 72, 72, 72, 80, 90, 90, 90, 90$	
729	280	103	110	2	5	G	$1, 24, 192, 256, 256$	
729	280	103	110	2	8	$3^6:2.\mathsf{S}_5$	$1, 40, 40, 48, 120, 120, 120, 240$	
729	336	153	156	2	8	G	$1, 24, 48, 48, 96, 128, 192, 192$	
729	336	153	156	1	9	G	$1, 24, 48, 48, 64, 64, 96, 192, 192$	
729	336	153	156	1	10	G	$1, 24, 32, 48, 48, 96, 96, 96, 96, 192$	
775	150	45	25	1	8	$\mathsf{L}_3(5).2$	$1, 30, 48, 96, 120, 120, 120, 240$	
784	243	82	72	1	4	$\mathsf{L}_2(8)^2.6$	$1, 54, 243, 486$	§8.11
784	297	116	110	1	4	$\mathsf{L}_2(8)^2.6$	$1, 54, 243, 486$	§8.11
1024	363	122	132	1	9	G	$1, 22, 55, 55, 66, 110, 165, 220, 330$	
1024	495	238	240	1	9	G	$1, 22, 55, 55, 66, 110, 165, 220, 330$	

Table 11.9: Small primitive rank 4–10 non-LS strongly regular graphs

Chapter 12

Parameter table

In this chapter we give a table with the feasible parameter sets of arbitrary strongly regular graphs on at most 512 vertices, and add comments about the known examples.

The columns are:

(i) Existence: A number indicates the precise number of nonisomorphic examples. '!' when there is a unique such graph, '+' when there is a known example, '−' when no example exists (the reason is indicated after '†'), and '?' otherwise.

(ii) The parameters v, k, λ, μ: the number of vertices, the valency, the number of common neighbors of two adjacent vertices, and the number of common neighbors of two nonadjacent vertices, respectively.

(iii) The spectrum of the adjacency matrix: k (with multiplicity 1), r (with multiplicity f), and s (with multiplicity g). Eigenvalues are integral, except when $(v, k, \lambda, \mu) = (4t+1, 2t, t-1, t)$ for some t, in which case $r, s = (-1 \pm \sqrt{v})/2$, and we give an approximation.

(iv) Comments.

The symbol '$\downarrow$' labels a descendant of a regular 2-graph on $v+1$ vertices.
The symbol '$\uparrow$' labels a graph in the switching class of a regular 2-graph.
'$[n,k]_q$ (wts w_1, w_2)' indicates a projective two-weight code.
The parameters of a partial geometry are written $pg(K,R,T)$ (not $pg(s,t,\alpha)$).
The label $OA(2m+1,m)^*$ refers to the construction of p. 194.
The labels 'ConfMat$(2m+2)^2$' and 'ConfMat$(2m+2)^{2*}$'refer to the construction of p. 190.
Labels are postfixed '?' when the corresponding object is unknown.

ex	v	k	λ	μ	r^f	s^g	comment
!	5	2	0	1	0.62^2	-1.62^2	§10.1; pentagon; Paley(5); $\downarrow$
!	9	4	1	2	1^4	-2^4	§10.2; Paley(9); 3×3; $\downarrow$
!	10	3	0	1	1^5	-2^4	§10.3; Petersen graph; $NO_4^-(2)$; $NO_3^{-\perp}(5)$; OA$(3,2)^*$; $\uparrow$
		6	3	4	1^4	-2^5	$T(5)$; $\uparrow$
!	13	6	2	3	1.30^6	-2.30^6	§10.4; Paley(13); $\downarrow$
!	15	6	1	3	1^9	-3^5	§10.5; $\mathsf{O}_5(2)$; $\mathsf{Sp}_4(2)$; $NO_4^-(3)$; $\mathsf{GQ}(2,2)$; $\downarrow$
		8	4	4	2^5	-2^9	$T(6)$; $\downarrow$
!	16	5	0	2	1^{10}	-3^5	$q_{22}^2 = 0$; vanLint-Schrijver, §7.3.1; $VO_4^-(2)$; $[5,4]_2$ (wts $2,4$); RSHCD$^-$; $\uparrow$
		10	6	6	2^5	-2^{10}	§10.7; Clebsch graph; $q_{11}^1 = 0$; vanLint-Schrijver, §7.3.1; $\uparrow$
2!	16	6	2	2	2^6	-2^9	§10.6; Shrikhande graph; 4×4; vanLint-Schrijver, §7.3.1; Wallis [718]; $[6,4]_2$ (wts $2,4$); RSHCD$^+$; $\uparrow$
		9	4	6	1^9	-3^6	OA$(4,3)$; $H_2(2,2)$; vanLint-Schrijver, §7.3.1; Wallis [718]; Goethals-Seidel [355]; $VO_4^+(2)$; $\uparrow$
!	17	8	3	4	1.56^8	-2.56^8	§10.8; Paley(17); $\downarrow$
!	21	10	3	6	1^{14}	-4^6	
		10	5	4	3^6	-2^{14}	$T(7)$
−	21	10	4	5	1.79^{10}	-2.79^{10}	† $v \neq a^2 + b^2$

continued...

ex	v	k	λ	μ	r^f	s^g	comment
!	25	8	3	2	3^{8}	-2^{16}	5×5; vanLint-Schrijver, §7.3.1
		16	9	12	1^{16}	-4^{8}	OA(5, 4); vanLint-Schrijver, §7.3.1
15!	25	12	5	6	2^{12}	-3^{12}	§10.9; Paulus-Rozenfel'd; Paley(25); OA(5, 3); ↓
10!	26	10	3	4	2^{13}	-3^{12}	§10.9; Paulus-Rozenfel'd; OA(5, 3)*; ↑
		15	8	9	2^{12}	-3^{13}	S(2,3,13); ↑
!	27	10	1	5	1^{20}	-5^{6}	$q^2_{22} = 0$; $\mathsf{O}_6^-(2)$; Godsil [345]; $\mathsf{GQ}(2, 4)$; ↓
		16	10	8	4^{6}	-2^{20}	§10.10; Schläfli graph; $q^1_{11} = 0$; ↓
−	28	9	0	4	1^{21}	-5^{6}	† $q^2_{22} < 0$; † Absolute bound
		18	12	10	4^{6}	-2^{21}	† $q^1_{11} < 0$; † Absolute bound
4!	28	12	6	4	4^{7}	-2^{20}	§10.11; Chang graphs; $T(8)$; Wallis [718]; ↑
		15	6	10	1^{20}	-5^{7}	$NO_6^+(2)$; Goethals-Seidel [355]; no pg(4,5,2) (De Clerck); Taylor ↑
41!	29	14	6	7	2.19^{14}	-3.19^{14}	§10.12; Enumerated by Bussemaker and by Spence; Paley(29); ↓
−	33	16	7	8	2.37^{16}	-3.37^{16}	† $v \neq a^2 + b^2$
3854!	35	16	6	8	2^{20}	-4^{14}	§10.13; Enumerated by McKay & Spence [556]; no pg(5,4,2) (De Clerck); ↓
		18	9	9	3^{14}	-3^{20}	S(2,3,15); lines in $\mathsf{PG}(3, 2)$; $\mathsf{O}_6^+(2)$; ↓
!	36	10	4	2	4^{10}	-2^{25}	6×6
		25	16	20	1^{25}	-5^{10}	OA(6,5) does not exist (Tarry)
180!	36	14	4	6	2^{21}	-4^{14}	§10.14; $\mathsf{U}_3(3).2/\mathsf{L}_2(7).2$ - subconstituent of Hall-Janko graph; Enumerated by McKay & Spence [556]; RSHCD$^-$; ↑
		21	12	12	3^{14}	-3^{21}	↑
!	36	14	7	4	5^{8}	-2^{27}	$T(9)$
		21	10	15	1^{27}	-6^{8}	
32548!	36	15	6	6	3^{15}	-3^{20}	§10.15; Enumerated by McKay & Spence [556]; OA(6, 3); $NO_6^-(2)$; RSHCD$^+$; ↑
		20	10	12	2^{20}	-4^{15}	$NO_5^-(3)$; OA(6,4) does not exist (Tarry); ↑
+	37	18	8	9	2.54^{18}	-3.54^{18}	see McKay & Spence [556]; Crnković & Maksimović [240]; Maksimović & Rukavina [754]; Paley(37); ↓
28!	40	12	2	4	2^{24}	-4^{15}	§10.16; Enumerated by Spence [670]; $\mathsf{O}_5(3)$; $\mathsf{Sp}_4(3)$; $\mathsf{GQ}(3, 3)$
		27	18	18	3^{15}	-3^{24}	$NU_4(2)$
+	41	20	9	10	2.70^{20}	-3.70^{20}	Maksimović & Rukavina [754]; Paley(41); ↓
78!	45	12	3	3	3^{20}	-3^{24}	§10.17; Enumerated by Coolsaet, Degraer & Spence [223]; $\mathsf{U}_4(2)$; Wallis [718]; $\mathsf{GQ}(4, 2)$
		32	22	24	2^{24}	-4^{20}	$NO_5^+(3)$
!	45	16	8	4	6^{9}	-2^{35}	$T(10)$
		28	15	21	1^{35}	-7^{9}	pg(5,7,3)
+	45	22	10	11	2.85^{22}	-3.85^{22}	Mathon [544]; ↓
!	49	12	5	2	5^{12}	-2^{36}	7×7
		36	25	30	1^{36}	-6^{12}	OA(7, 6)
−	49	16	3	6	2^{32}	-5^{16}	† Bussemaker-Haemers-Mathon-Wilbrink [162]
		32	21	20	4^{16}	-3^{32}	
+	49	18	7	6	4^{18}	-3^{30}	Behbahani-Lam [55]; Crnkovic-Maksimovic [240]; OA(7, 3); Pasechnik (§8.12)
		30	17	20	2^{30}	-5^{18}	OA(7, 5)
+	49	24	11	12	3^{24}	-4^{24}	§10.18; Paley(49); OA(7, 4); ↓
!	50	7	0	1	2^{28}	-3^{21}	§10.19; Hoffman-Singleton graph
		42	35	36	2^{21}	-3^{28}	
−	50	21	4	12	1^{42}	-9^{7}	† Absolute bound
		28	18	12	8^{7}	-2^{42}	† Absolute bound
+	50	21	8	9	3^{25}	-4^{24}	OA(7, 4)*; ConfMat(8)^{2*}; ↑
		28	15	16	3^{24}	-4^{25}	S(2,4,25); ↑
+	53	26	12	13	3.14^{26}	-4.14^{26}	Paley(53); ↓
!	55	18	9	4	7^{10}	-2^{44}	$T(11)$
		36	21	28	1^{44}	-8^{10}	
!	56	10	0	2	2^{35}	-4^{20}	§10.20; Gewirtz graph; Cossidente-Penttila [233]

continued...

ex	v	k	λ	μ	r^f	s^g	comment
		45	36	36	3^{20}	-3^{35}	qs 2-(21,6,4)
−	56	22	3	12	1^{48}	-10^{7}	† $q^2_{22} < 0$; † Absolute bound
		33	22	15	9^{7}	-2^{48}	† $q^1_{11} < 0$; † Absolute bound
−	57	14	1	4	2^{38}	-5^{18}	† Wilbrink-Brouwer [732]
		42	31	30	4^{18}	-3^{38}	
+	57	24	11	9	5^{18}	-3^{38}	S(2,3,19)
		32	16	20	2^{38}	-6^{18}	
−	57	28	13	14	3.27^{28}	-4.27^{28}	† $v \neq a^2 + b^2$
+	61	30	14	15	3.41^{30}	-4.41^{30}	Paley(61); ↓
−	63	22	1	11	1^{55}	-11^{7}	† $q^2_{22} < 0$; † Absolute bound
		40	28	20	10^{7}	-2^{55}	† $q^1_{11} < 0$; † Absolute bound
+	63	30	13	15	3^{35}	-5^{27}	§10.21; §10.22; §10.23; qs 2-(36,16,12); $O_7(2)$; $Sp_6(2)$; pg(7,5,3); ↓
		32	16	16	4^{27}	-4^{35}	S(2,4,28); qs 2-(28,12,11); $NU_3(3)$; ↓
!	64	14	6	2	6^{14}	-2^{49}	8×8; $[14,6]_2$ (wts 4, 8)
		49	36	42	1^{49}	-7^{14}	OA(8, 7)
167!	64	18	2	6	2^{45}	-6^{18}	§10.24; Enumerated by Haemers & Spence [384]; $GQ(3,5)$; $[6,3]_4$ (wts 4, 6); $[18,6]_2$ (wts 8, 12)
		45	32	30	5^{18}	-3^{45}	
−	64	21	0	10	1^{56}	-11^{7}	† $q^2_{22} < 0$; † Absolute bound
		42	30	22	10^{7}	-2^{56}	† $q^1_{11} < 0$; † Absolute bound
+	64	21	8	6	5^{21}	-3^{42}	OA(8, 3); $H_2(2,3)$; vanLint-Schrijver, §7.3.1; $[7,3]_4$ (wts 4, 6); Brouwer [112]; $[21,6]_2$ (wts 8, 12)
		42	26	30	2^{42}	-6^{21}	OA(8, 6); vanLint-Schrijver, §7.3.1
+	64	27	10	12	3^{36}	-5^{27}	§10.25; Mesner; $[9,3]_4$ (wts 6, 8); $VO^-_6(2)$; RSHCD$^-$; ↑
		36	20	20	4^{27}	-4^{36}	↑
+	64	28	12	12	4^{28}	-4^{35}	§10.26; OA(8, 4); Wallis [718]; $[28,6]_2$ (wts 12, 16); RSHCD$^+$; ↑
		35	18	20	3^{35}	-5^{28}	OA(8, 5); Wallis [718]; Goethals-Seidel [355]; $VO^+_6(2)$; ↑
−	64	30	18	10	10^{8}	-2^{55}	† Absolute bound
		33	12	22	1^{55}	-11^{8}	† Absolute bound
+	65	32	15	16	3.53^{32}	-4.53^{32}	Gritsenko [366]; ↓
!	66	20	10	4	8^{11}	-2^{54}	$T(12)$
		45	28	36	1^{54}	-9^{11}	no pg(6,9,4) (Lam et al.)
?	69	20	7	5	5^{23}	-3^{45}	
		48	32	36	2^{45}	-6^{23}	no S(2,6,46) [443]
−	69	34	16	17	3.65^{34}	-4.65^{34}	† $v \neq a^2 + b^2$
+	70	27	12	9	6^{20}	-3^{49}	S(2,3,21)
		42	23	28	2^{49}	-7^{20}	pg(7,7,4)?
+	73	36	17	18	3.77^{36}	-4.77^{36}	Paley(73); ↓
−	75	32	10	16	2^{56}	-8^{18}	† Azarija-Marc [20]
		42	25	21	7^{18}	-3^{56}	
−	76	21	2	7	2^{56}	-7^{19}	† Haemers [378]
		54	39	36	6^{19}	-3^{56}	
−	76	30	8	14	2^{57}	-8^{18}	† Bondarenko, Prymak & Radchenko [89]
		45	28	24	7^{18}	-3^{57}	
−	76	35	18	14	7^{19}	-3^{56}	
		40	18	24	2^{56}	-8^{19}	† no ↑
!	77	16	0	4	2^{55}	-6^{21}	§10.27; S(3,6,22); Mesner [560]; unique by [111]; qs 2-(56,16,6)
		60	47	45	5^{21}	-3^{55}	Witt: qs 2-(22,6,5)
−	77	38	18	19	3.89^{38}	-4.89^{38}	† $v \neq a^2 + b^2$
!	78	22	11	4	9^{12}	-2^{65}	$T(13)$
		55	36	45	1^{65}	-10^{12}	
!	81	16	7	2	7^{16}	-2^{64}	9×9; $[8,4]_3$ (wts 3, 6)
		64	49	56	1^{64}	-8^{16}	OA(9, 8); vanLint-Schrijver, §7.3.1
!	81	20	1	6	2^{60}	-7^{20}	§10.28; Mesner [560]; Brouwer-Haemers; $VO^-_4(3)$; $[10,4]_3$ (wts 6, 9)

continued...

ex	v	k	λ	μ	r^f	s^g	comment
		60	45	42	6^{20}	-3^{60}	
+	81	24	9	6	6^{24}	-3^{56}	OA$(9,3)$; Wallis [718]; $VNO_4^+(3)$; $[12,4]_3$ (wts $6,9$)
		56	37	42	2^{56}	-7^{24}	OA$(9,7)$
+	81	30	9	12	3^{50}	-6^{30}	§10.29; Mesner [560]; Van Lint & Schrijver pg(6,6,2); $VNO_4^-(3)$; $[15,4]_3$ (wts $9,12$)
		50	31	30	5^{30}	-4^{50}	
+	81	32	13	12	5^{32}	-4^{48}	OA$(9,4)$; $H_3(2,2)$; vanLint-Schrijver, §7.3.1; Wallis [718]; $VO_4^+(3)$; $[16,4]_3$ (wts $9,12$)
		48	27	30	3^{48}	-6^{32}	OA$(9,6)$; vanLint-Schrijver, §7.3.1
−	81	40	13	26	1^{72}	-14^{8}	† Absolute bound
		40	25	14	13^{8}	-2^{72}	† Absolute bound
+	81	40	19	20	4^{40}	-5^{40}	§10.30; Paley(81); OA$(9,5)$; ↓
+	82	36	15	16	4^{41}	-5^{40}	OA$(9,5)^*$; ↑
		45	24	25	4^{40}	-5^{41}	S(2,5,41); ↑
?	85	14	3	2	4^{34}	-3^{50}	
		70	57	60	2^{50}	-5^{34}	
+	85	20	3	5	3^{50}	-5^{34}	$\mathsf{O}_5(4)$; $\mathsf{Sp}_4(4)$; $\mathsf{GQ}(4,4)$
		64	48	48	4^{34}	-4^{50}	
?	85	30	11	10	5^{34}	-4^{50}	
		54	33	36	3^{50}	-6^{34}	S(2,6,51)?
?	85	42	20	21	4.11^{42}	-5.11^{42}	↓?
?	88	27	6	9	3^{55}	-6^{32}	
		60	41	40	5^{32}	-4^{55}	
+	89	44	21	22	4.22^{44}	-5.22^{44}	Paley(89); ↓
!	91	24	12	4	10^{13}	-2^{77}	$T(14)$
		66	45	55	1^{77}	-11^{13}	pg(7,11,5)?
−	93	46	22	23	4.32^{46}	-5.32^{46}	† $v \neq a^2+b^2$
−	95	40	12	20	2^{75}	-10^{19}	† Azarija-Marc [21]
		54	33	27	9^{19}	-3^{75}	
+	96	19	2	4	3^{57}	-5^{38}	Haemers [376, (6.2.3), $q=4$]; Muzychuk [580]; Brouwer-Koolen-Klin [135]; Golemac-Mandić-Vučičić [358]
		76	60	60	4^{38}	-4^{57}	
+	96	20	4	4	4^{45}	-4^{50}	Wallis [718]; $\mathsf{GQ}(5,3)$; Brouwer-Koolen-Klin [135]; Golemac-Mandić-Vučičić [358]
		75	58	60	3^{50}	-5^{45}	
?	96	35	10	14	3^{63}	-7^{32}	pg(6,7,2)?
		60	38	36	6^{32}	-4^{63}	
−	96	38	10	18	2^{76}	-10^{19}	† Degraer [273]
		57	36	30	9^{19}	-3^{76}	
−	96	45	24	18	9^{20}	-3^{75}	
		50	22	30	2^{75}	-10^{20}	† no ↑
+	97	48	23	24	4.42^{48}	-5.42^{48}	Paley(97); ↓
?	99	14	1	2	3^{54}	-4^{44}	
		84	71	72	3^{44}	-4^{54}	
?	99	42	21	15	9^{21}	-3^{77}	
		56	28	36	2^{77}	-10^{21}	
+	99	48	22	24	4^{54}	-6^{44}	no pg(9,6,4) (Lam et al.); ↓
		50	25	25	5^{44}	-5^{54}	S(2,5,45); ↓
!	100	18	8	2	8^{18}	-2^{81}	10×10
		81	64	72	1^{81}	-9^{18}	
!	100	22	0	6	2^{77}	-8^{22}	§10.31; Higman-Sims graph; $q_{22}^2 = 0$
		77	60	56	7^{22}	-3^{77}	$q_{11}^1 = 0$
+	100	27	10	6	7^{27}	-3^{72}	OA$(10,3)$
		72	50	56	2^{72}	-8^{27}	OA$(10,8)$?
?	100	33	8	12	3^{66}	-7^{33}	
		66	44	42	6^{33}	-4^{66}	
+	100	33	14	9	8^{24}	-3^{75}	S(2,3,25)
		66	41	48	2^{75}	-9^{24}	
−	100	33	18	7	13^{11}	-2^{88}	† Absolute bound
		66	39	52	1^{88}	-14^{11}	† Absolute bound

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ex	v	k	λ	μ	r^f	s^g	comment
+	100	36	14	12	6^{36}	-4^{63}	§10.32; Hall-Janko graph; OA$(10,4)$
		63	38	42	3^{63}	-7^{36}	OA$(10,7)$?
+	100	44	18	20	4^{55}	-6^{44}	Jørgensen-Klin graph [471]; RSHCD$^-$; ↑
		55	30	30	5^{44}	-5^{55}	↑
+	100	45	20	20	5^{45}	-5^{54}	OA$(10,5)$?; RSHCD$^+$; ↑
		54	28	30	4^{54}	-6^{45}	OA$(10,6)$?; ↑
+	101	50	24	25	4.52^{50}	-5.52^{50}	Paley(101); ↓
!	105	26	13	4	11^{14}	-2^{90}	$T(15)$
		78	55	66	1^{90}	-12^{14}	
!	105	32	4	12	2^{84}	-10^{20}	§10.33; flags of $\mathsf{PG}(2,4)$, unique by [221]
		72	51	45	9^{20}	-3^{84}	
?	105	40	15	15	5^{48}	-5^{56}	
		64	38	40	4^{56}	-6^{48}	
?	105	52	21	30	2^{84}	-11^{20}	
		52	29	22	10^{20}	-3^{84}	
−	105	52	25	26	4.62^{52}	-5.62^{52}	† $v \neq a^2 + b^2$
+	109	54	26	27	4.72^{54}	-5.72^{54}	Paley(109); ↓
?	111	30	5	9	3^{74}	-7^{36}	
		80	58	56	6^{36}	-4^{74}	
+	111	44	19	16	7^{36}	-4^{74}	S(2,4,37)
		66	37	42	3^{74}	-8^{36}	
!	112	30	2	10	2^{90}	-10^{21}	§10.34; unique by [178]; subconstituent of McLaughlin graph; $q^2_{22} = 0$; $\mathsf{O}^-_6(3)$; $\mathsf{GQ}(3,9)$
		81	60	54	9^{21}	-3^{90}	$q^1_{11} = 0$
?	112	36	10	12	4^{63}	-6^{48}	pg(7,6,2)?
		75	50	50	5^{48}	-5^{63}	
+	113	56	27	28	4.82^{56}	-5.82^{56}	Paley(113); ↓
?	115	18	1	3	3^{69}	-5^{45}	
		96	80	80	4^{45}	-4^{69}	
+	117	36	15	9	9^{26}	-3^{90}	§10.35; S(2,3,27); $NO^+_6(3)$; lines in $AG(3,3)$ (rk 4); Wallis [718]
		80	52	60	2^{90}	-10^{26}	pg(9,10,6)?
?	117	58	28	29	4.91^{58}	-5.91^{58}	↓?
+	119	54	21	27	3^{84}	-9^{34}	§10.36; $\mathsf{O}^-_8(2)$; pg(7,9,3)?; ↓
		64	36	32	8^{34}	-4^{84}	↓
!	120	28	14	4	12^{15}	-2^{104}	$T(16)$
		91	66	78	1^{104}	-13^{15}	pg(8,13,6)?
?	120	34	8	10	4^{68}	-6^{51}	
		85	60	60	5^{51}	-5^{68}	
?	120	35	10	10	5^{56}	-5^{63}	pg(8,5,2) does not exist (no dual)
		84	58	60	4^{63}	-6^{56}	
!	120	42	8	18	2^{99}	-12^{20}	§10.37; Baer subplanes of $\mathsf{PG}(2,4)$, unique by [274]
		77	52	44	11^{20}	-3^{99}	qs 2-(21,7,12)
+	120	51	18	24	3^{85}	-9^{34}	§10.38; $NO^-_5(4)$; ↑
		68	40	36	8^{34}	-4^{85}	Fickus et al. [324]; ↑
+	120	56	28	24	8^{35}	-4^{84}	§10.39; Wallis [718]; ↑
		63	30	36	3^{84}	-9^{35}	dist. 2 in J(10,3); $NO^+_8(2)$; Goethals-Seidel [355]; Cohen pg(8,9,4); see also [266]; ↑
!	121	20	9	2	9^{20}	-2^{100}	11×11
		100	81	90	1^{100}	-10^{20}	OA$(11,10)$
+	121	30	11	6	8^{30}	-3^{90}	OA$(11,3)$
		90	65	72	2^{90}	-9^{30}	OA$(11,9)$
?	121	36	7	12	3^{84}	-8^{36}	
		84	59	56	7^{36}	-4^{84}	
+	121	40	15	12	7^{40}	-4^{80}	OA$(11,4)$; vanLint-Schrijver, §7.3.1
		80	51	56	3^{80}	-8^{40}	OA$(11,8)$; vanLint-Schrijver, §7.3.1
?	121	48	17	20	4^{72}	-7^{48}	
		72	43	42	6^{48}	-5^{72}	
+	121	50	21	20	6^{50}	-5^{70}	OA$(11,5)$; Pasechnik (§8.12)
		70	39	42	4^{70}	-7^{50}	OA$(11,7)$
−	121	56	15	35	1^{112}	-21^{8}	† Absolute bound

continued...

ex	v	k	λ	μ	r^f	s^g	comment
		64	42	24	20^{8}	-2^{112}	† Absolute bound
+	121	60	29	30	5^{60}	-6^{60}	Paley(121); OA(11, 6); ↓
+	122	55	24	25	5^{61}	-6^{60}	OA(11, 6)*; ConfMat(12)^{2*}; ↑
		66	35	36	5^{60}	-6^{61}	S(2,6,61)?; ↑
+	125	28	3	7	3^{84}	-7^{40}	Godsil [345]; GQ(4, 6)
		96	74	72	6^{40}	-4^{84}	
−	125	48	28	12	18^{10}	-2^{114}	† Absolute bound
		76	39	57	1^{114}	-19^{10}	† Absolute bound
+	125	52	15	26	2^{104}	-13^{20}	Godsil [345]; pg(5,13,2)?; ↓
		72	45	36	12^{20}	-3^{104}	↓
+	125	62	30	31	5.09^{62}	-6.09^{62}	Paley(125); ↓
+	126	25	8	4	7^{35}	-3^{90}	§10.40; dist. 1 or 4 in J(9,4)
		100	78	84	2^{90}	-8^{35}	
+	126	45	12	18	3^{90}	-9^{35}	§10.41; $NO_6^-(3)$; pg(6,9,2)?
		80	52	48	8^{35}	-4^{90}	
!	126	50	13	24	2^{105}	-13^{20}	§10.42; Goethals; unique by [222]; ↑
		75	48	39	12^{20}	-3^{105}	↑
+	126	60	33	24	12^{21}	-3^{104}	↑
		65	28	39	2^{104}	-13^{21}	pg(6,13,3)?; Taylor ↑
−	129	64	31	32	5.18^{64}	-6.18^{64}	† $v \neq a^2 + b^2$
+	130	48	20	16	8^{39}	-4^{90}	S(2,4,40); lines in PG(3, 3); $O_6^+(3)$
		81	48	54	3^{90}	-9^{39}	pg(10,9,6)?
?	133	24	5	4	5^{56}	-4^{76}	GQ(6, 3) does not exist (Dixmier & Zara [294])
		108	87	90	3^{76}	-6^{56}	
?	133	32	6	8	4^{76}	-6^{56}	
		100	75	75	5^{56}	-5^{76}	
?	133	44	15	14	6^{56}	-5^{76}	
		88	57	60	4^{76}	-7^{56}	
−	133	66	32	33	5.27^{66}	-6.27^{66}	† $v \neq a^2 + b^2$
+	135	64	28	32	4^{84}	-8^{50}	Cohen pg(9,8,4); see also [266]; ↓
		70	37	35	7^{50}	-5^{84}	§10.43; $O_8^+(2)$; from ETF (Fickus et al. [325]); ↓
?	136	30	8	6	6^{51}	-4^{84}	
		105	80	84	3^{84}	-7^{51}	
!	136	30	15	4	13^{16}	-2^{119}	$T(17)$
		105	78	91	1^{119}	-14^{16}	
+	136	60	24	28	4^{85}	-8^{50}	↑
		75	42	40	7^{50}	-5^{85}	$NO_5^+(4)$; from ETF (Fickus et al. [325]); ↑
+	136	63	30	28	7^{51}	-5^{84}	§10.44; $NO_8^-(2)$; ↑
		72	36	40	4^{84}	-8^{51}	↑
+	137	68	33	34	5.35^{68}	-6.35^{68}	Paley(137); ↓
−	141	70	34	35	5.44^{70}	-6.44^{70}	† $v \neq a^2 + b^2$
+	143	70	33	35	5^{77}	-7^{65}	qs 2-(78,36,30); pg(11,7,5)?; ↓
		72	36	36	6^{65}	-6^{77}	S(2,6,66); qs 2-(66,30,29); ↓
!	144	22	10	2	10^{22}	-2^{121}	12 × 12
		121	100	110	1^{121}	-11^{22}	OA(12, 11)?
+	144	33	12	6	9^{33}	-3^{110}	OA(12, 3)
		110	82	90	2^{110}	-10^{33}	OA(12, 10)?
+	144	39	6	12	3^{104}	-9^{39}	§10.45; $L_3(3)$ (rk 8)
		104	76	72	8^{39}	-4^{104}	
+	144	44	16	12	8^{44}	-4^{99}	OA(12, 4)
		99	66	72	3^{99}	-9^{44}	OA(12, 9)?
?	144	52	16	20	4^{91}	-8^{52}	
		91	58	56	7^{52}	-5^{91}	
+	144	55	22	20	7^{55}	-5^{88}	§10.46; OA(12, 5)
		88	52	56	4^{88}	-8^{55}	OA(12, 8)?
−	144	65	16	40	1^{135}	-25^{8}	† $q_{22}^2 < 0$; † Absolute bound
		78	52	30	24^{8}	-2^{135}	† $q_{11}^1 < 0$; † Absolute bound
+	144	65	28	30	5^{78}	-7^{65}	RSHCD$^-$; ↑
		78	42	42	6^{65}	-6^{78}	Fickus et al. [324]; ↑
+	144	66	30	30	6^{66}	-6^{77}	OA(12, 6); Wallis [718]; RSHCD$^+$; ↑
		77	40	42	5^{77}	-7^{66}	OA(12, 7); Wallis [718]; Goethals-Seidel [355]; ↑

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ex	v	k	λ	μ	r^f	s^g	comment
?	145	72	35	36	5.52^{72}	-6.52^{72}	↓?
?	147	66	25	33	3^{110}	-11^{36}	pg(7,11,3)?; ↓?
		80	46	40	10^{36}	-4^{110}	↓?
?	148	63	22	30	3^{111}	-11^{36}	↑?
		84	50	44	10^{36}	-4^{111}	↑?
?	148	70	36	30	10^{37}	-4^{110}	↑?
		77	36	44	3^{110}	-11^{37}	↑?
+	149	74	36	37	5.60^{74}	-6.60^{74}	Paley(149); ↓
!	153	32	16	4	14^{17}	-2^{135}	$T(18)$
		120	91	105	1^{135}	-15^{17}	pg(9,15,7)
?	153	56	19	21	5^{84}	-7^{68}	pg(9,7,3)?
		96	60	60	6^{68}	-6^{84}	
?	153	76	37	38	5.68^{76}	-6.68^{76}	↓?
?	154	48	12	16	4^{98}	-8^{55}	pg(7,8,2)?
		105	72	70	7^{55}	-5^{98}	
−	154	51	8	21	2^{132}	-15^{21}	† $q_{22}^2 < 0$
		102	71	60	14^{21}	-3^{132}	† $q_{11}^1 < 0$
?	154	72	26	40	2^{132}	-16^{21}	
		81	48	36	15^{21}	-3^{132}	
+	155	42	17	9	11^{30}	-3^{124}	S(2,3,31); lines in $\mathsf{PG}(4,2)$
		112	78	88	2^{124}	-12^{30}	
+	156	30	4	6	4^{90}	-6^{65}	§10.47; $\mathsf{O}_5(5)$; $\mathsf{Sp}_4(5)$; $\mathsf{GQ}(5,5)$
		125	100	100	5^{65}	-5^{90}	
+	157	78	38	39	5.76^{78}	-6.76^{78}	Paley(157); ↓
?	160	54	18	18	6^{75}	-6^{84}	pg(10,6,3) does not exist (no dual)
		105	68	70	5^{84}	-7^{75}	
−	161	80	39	40	5.84^{80}	-6.84^{80}	† $v \neq a^2 + b^2$
?	162	21	0	3	3^{105}	-6^{56}	
		140	121	120	5^{56}	-4^{105}	
?	162	23	4	3	5^{69}	-4^{92}	
		138	117	120	3^{92}	-6^{69}	
?	162	49	16	14	7^{63}	-5^{98}	
		112	76	80	4^{98}	-8^{63}	
!	162	56	10	24	2^{140}	-16^{21}	§10.48; $U_4(3)$; $q_{22}^2 = 0$
		105	72	60	15^{21}	-3^{140}	unique by Cameron, Goethals & Seidel [178]; subconstituent of McLaughlin graph; $q_{11}^1 = 0$
?	162	69	36	24	15^{23}	-3^{138}	
		92	46	60	2^{138}	-16^{23}	
+	165	36	3	9	3^{120}	-9^{44}	$\mathsf{U}_5(2)$; $\mathsf{GQ}(4,8)$
		128	100	96	8^{44}	-4^{120}	
−	165	82	40	41	5.92^{82}	-6.92^{82}	† $v \neq a^2 + b^2$
!	169	24	11	2	11^{24}	-2^{144}	13×13
		144	121	132	1^{144}	-12^{24}	$\mathrm{OA}(13,12)$
+	169	36	13	6	10^{36}	-3^{132}	$\mathrm{OA}(13,3)$
		132	101	110	2^{132}	-11^{36}	$\mathrm{OA}(13,11)$
?	169	42	5	12	3^{126}	-10^{42}	
		126	95	90	9^{42}	-4^{126}	
+	169	48	17	12	9^{48}	-4^{120}	$\mathrm{OA}(13,4)$
		120	83	90	3^{120}	-10^{48}	$\mathrm{OA}(13,10)$
?	169	56	15	20	4^{112}	-9^{56}	
		112	75	72	8^{56}	-5^{112}	
+	169	60	23	20	8^{60}	-5^{108}	$\mathrm{OA}(13,5)$
		108	67	72	4^{108}	-9^{60}	$\mathrm{OA}(13,9)$
?	169	70	27	30	5^{98}	-8^{70}	
		98	57	56	7^{70}	-6^{98}	
+	169	72	31	30	7^{72}	-6^{96}	$\mathrm{OA}(13,6)$
		96	53	56	5^{96}	-8^{72}	$\mathrm{OA}(13,8)$
+	169	84	41	42	6^{84}	-7^{84}	Paley(169); $\mathrm{OA}(13,7)$; ↓
+	170	78	35	36	6^{85}	-7^{84}	$\mathrm{OA}(13,7)^*$; ↑
		91	48	49	6^{84}	-7^{85}	S(2,7,85)?; ↑
!	171	34	17	4	15^{18}	-2^{152}	$T(19)$
		136	105	120	1^{152}	-16^{18}	
?	171	50	13	15	5^{95}	-7^{75}	

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ex	v	k	λ	μ	r^f	s^g	comment
		120	84	84	6^{75}	-6^{95}	
?	171	60	15	24	3^{132}	-12^{38}	pg(6,12,2)?
		110	73	66	11^{38}	-4^{132}	
+	173	86	42	43	6.08^{86}	-7.08^{86}	Paley(173); ↓
+	175	30	5	5	5^{84}	-5^{90}	Wallis [718]; $\mathsf{GQ}(6,4)$
		144	118	120	4^{90}	-6^{84}	
?	175	66	29	22	11^{42}	-4^{132}	
		108	63	72	3^{132}	-12^{42}	pg(10,12,6)?
+	175	72	20	36	2^{153}	-18^{21}	p. 269; edges of Hoffman-Singleton; Haemers pg(5,18,2); ↓
		102	65	51	17^{21}	-3^{153}	↓
?	176	25	0	4	3^{120}	-7^{55}	
		150	128	126	6^{55}	-4^{120}	
+	176	40	12	8	8^{55}	-4^{120}	§10.49; pg(11,4,2) does not exist (no dual)
		135	102	108	3^{120}	-9^{55}	$NU_5(2)$
+	176	45	18	9	12^{32}	-3^{143}	S(2,3,33)
		130	93	104	2^{143}	-13^{32}	pg(11,13,8)?
+	176	49	12	14	5^{98}	-7^{77}	§10.50; Higman symmetric 2-design; pg(8,7,2)?
		126	90	90	6^{77}	-6^{98}	
!	176	70	18	34	2^{154}	-18^{21}	§10.51; $S(4,7,23) \setminus S(3,6,22)$; $\mathsf{M}_{22}/\mathsf{A}_7$; ↑
		105	68	54	17^{21}	-3^{154}	Witt: qs 2-(22,7,16); ↑
?	176	70	24	30	4^{120}	-10^{55}	pg(8,10,3)?
		105	64	60	9^{55}	-5^{120}	
−	176	70	42	18	26^{10}	-2^{165}	† Absolute bound
		105	52	78	1^{165}	-27^{10}	† Absolute bound
+	176	85	48	34	17^{22}	-3^{153}	p. 218; Haemers; ↑
		90	38	54	2^{153}	-18^{22}	pg(6,18,3)?; ↑
−	177	88	43	44	6.15^{88}	-7.15^{88}	† $v \neq a^2 + b^2$
+	181	90	44	45	6.23^{90}	-7.23^{90}	Paley(181); ↓
?	183	52	11	16	4^{122}	-9^{60}	
		130	93	90	8^{60}	-5^{122}	
+	183	70	29	25	9^{60}	-5^{122}	S(2,5,61)
		112	66	72	4^{122}	-10^{60}	
−	184	48	2	16	2^{160}	-16^{23}	† $q_{22}^2 < 0$
		135	102	90	15^{23}	-3^{160}	† $q_{11}^1 < 0$
?	185	92	45	46	6.30^{92}	-7.30^{92}	↓?
?	189	48	12	12	6^{90}	-6^{98}	pg(9,6,2)?
		140	103	105	5^{98}	-7^{90}	
?	189	60	27	15	15^{28}	-3^{160}	
		128	82	96	2^{160}	-16^{28}	pg(9,16,6)?
?	189	88	37	44	4^{132}	-11^{56}	pg(9,11,4)?; ↓?
		100	55	50	10^{56}	-5^{132}	↓?
−	189	94	46	47	6.37^{94}	-7.37^{94}	† $v \neq a^2 + b^2$
!	190	36	18	4	16^{19}	-2^{170}	$T(20)$
		153	120	136	1^{170}	-17^{19}	pg(10,17,8)?
?	190	45	12	10	7^{75}	-5^{114}	pg(10,5,2) does not exist (no dual)
		144	108	112	4^{114}	-8^{75}	
?	190	84	33	40	4^{133}	-11^{56}	↑?
		105	60	55	10^{56}	-5^{133}	↑?
+	190	84	38	36	8^{75}	-6^{114}	S(2,6,76)
		105	56	60	5^{114}	-9^{75}	
?	190	90	45	40	10^{57}	-5^{132}	↑?
		99	48	55	4^{132}	-11^{57}	pg(10,11,5)?; ↑?
+	193	96	47	48	6.45^{96}	-7.45^{96}	Paley(193); ↓
+	195	96	46	48	6^{104}	-8^{90}	pg(13,8,6)?; ↓
		98	49	49	7^{90}	-7^{104}	S(2,7,91); ↓
!	196	26	12	2	12^{26}	-2^{169}	14×14
		169	144	156	1^{169}	-13^{26}	OA(14, 13)?
?	196	39	2	9	3^{147}	-10^{48}	
		156	125	120	9^{48}	-4^{147}	
+	196	39	14	6	11^{39}	-3^{156}	OA(14, 3)
		156	122	132	2^{156}	-12^{39}	OA(14, 12)?
?	196	45	4	12	3^{150}	-11^{45}	

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ex	v	k	λ	μ	r^f	s^g	comment
		150	116	110	10^{45}	-4^{150}	
+	196	52	18	12	10^{52}	-4^{143}	OA(14, 4)
		143	102	110	3^{143}	-11^{52}	OA(14, 11)?
+	196	60	14	20	4^{135}	-10^{60}	Huang-Huang-Lin, §8.4.3; pg(7,10,2)?
		135	94	90	9^{60}	-5^{135}	
+	196	60	23	16	11^{48}	-4^{147}	S(2,4,49); Huffman-Tonchev [445]: qs 2-(49,9,6)
		135	90	99	3^{147}	-12^{48}	
+	196	65	24	20	9^{65}	-5^{130}	OA(14, 5)
		130	84	90	4^{130}	-10^{65}	OA(14, 10)?
?	196	75	26	30	5^{120}	-9^{75}	
		120	74	72	8^{75}	-6^{120}	
+	196	78	32	30	8^{78}	-6^{117}	OA(14, 6)
		117	68	72	5^{117}	-9^{78}	OA(14, 9)?
?	196	81	42	27	18^{24}	-3^{171}	
		114	59	76	2^{171}	-19^{24}	pg(7,19,4)?
−	196	85	18	51	1^{187}	-34^{8}	† $q_{22}^2 < 0$; † Absolute bound
		110	75	44	33^{8}	-2^{187}	† $q_{11}^1 < 0$; † Absolute bound
?	196	90	40	42	6^{105}	-8^{90}	RSHCD$^-$?; ↑?
		105	56	56	7^{90}	-7^{105}	↑?
+	196	91	42	42	7^{91}	-7^{104}	OA(14, 7)?; RSHCD$^+$; ↑
		104	54	56	6^{104}	-8^{91}	OA(14, 8)?; ↑
+	197	98	48	49	6.52^{98}	-7.52^{98}	Paley(197); ↓
−	201	100	49	50	6.59^{100}	-7.59^{100}	† $v \neq a^2 + b^2$
?	204	28	2	4	4^{119}	-6^{84}	
		175	150	150	5^{84}	-5^{119}	
?	204	63	22	18	9^{68}	-5^{135}	
		140	94	100	4^{135}	-10^{68}	S(2,10,136)?
?	205	68	15	26	3^{164}	-14^{40}	
		136	93	84	13^{40}	-4^{164}	
?	205	96	50	40	14^{40}	-4^{164}	
		108	51	63	3^{164}	-15^{40}	
?	205	102	50	51	6.66^{102}	-7.66^{102}	↓?
?	208	45	8	10	5^{117}	-7^{90}	
		162	126	126	6^{90}	-6^{117}	
+	208	75	30	25	10^{64}	-5^{143}	§10.52; S(2,5,65); $NU_3(4)$
		132	81	88	4^{143}	-11^{64}	pg(13,11,8)?
?	208	81	24	36	3^{168}	-15^{39}	
		126	80	70	14^{39}	-4^{168}	
−	209	16	3	1	5^{76}	-3^{132}	† $\mu = 1$
		192	176	180	2^{132}	-6^{76}	
?	209	52	15	12	8^{76}	-5^{132}	
		156	115	120	4^{132}	-9^{76}	
+	209	100	45	50	5^{132}	-10^{76}	pg(11,10,5)?; ↓
		108	57	54	9^{76}	-6^{132}	↓
−	209	104	51	52	6.73^{104}	-7.73^{104}	† $v \neq a^2 + b^2$
?	210	33	0	6	3^{154}	-9^{55}	
		176	148	144	8^{55}	-4^{154}	
!	210	38	19	4	17^{20}	-2^{189}	$T(21)$
		171	136	153	1^{189}	-18^{20}	
?	210	76	26	28	6^{114}	-8^{95}	
		133	84	84	7^{95}	-7^{114}	
?	210	77	28	28	7^{99}	-7^{110}	
		132	82	84	6^{110}	-8^{99}	
?	210	95	40	45	5^{133}	-10^{76}	↑?
		114	63	60	9^{76}	-6^{133}	↑?
+	210	99	48	45	9^{77}	-6^{132}	§10.53; Klin et al. [494], S_7; ↑
		110	55	60	5^{132}	-10^{77}	pg(12,10,6)?; ↑
−	213	106	52	53	6.80^{106}	-7.80^{106}	† $v \neq a^2 + b^2$
+	216	40	4	8	4^{140}	-8^{75}	p. 266; Crnković et al. [243], $\mathsf{O}_6^-(2)$
		175	142	140	7^{75}	-5^{140}	
?	216	43	10	8	7^{86}	-5^{129}	
		172	136	140	4^{129}	-8^{86}	

continued...

ex	v	k	λ	μ	r^f	s^g	comment
−	216	70	40	14	28^{12}	-2^{203}	† Absolute bound
		145	88	116	1^{203}	-29^{12}	† Absolute bound
?	216	75	18	30	3^{175}	-15^{40}	pg(6,15,2)?
		140	94	84	14^{40}	-4^{175}	
?	216	86	40	30	14^{43}	-4^{172}	
		129	72	84	3^{172}	-15^{43}	
?	216	90	39	36	9^{80}	-6^{135}	S(2,6,81)?
		125	70	75	5^{135}	-10^{80}	
?	217	66	15	22	4^{154}	-11^{62}	pg(7,11,2)?
		150	105	100	10^{62}	-5^{154}	
?	217	88	39	33	11^{62}	-5^{154}	
		128	72	80	4^{154}	-12^{62}	
−	217	108	53	54	6.87^{108}	-7.87^{108}	† $v \neq a^2 + b^2$
?	220	72	22	24	6^{120}	-8^{99}	pg(10,8,3)?
		147	98	98	7^{99}	-7^{120}	
+	220	84	38	28	14^{44}	-4^{175}	Tonchev [703]: qs 2-(45,9,8)
		135	78	90	3^{175}	-15^{44}	pg(10,15,6)?
+	221	64	24	16	12^{51}	-4^{169}	S(2,4,52)
		156	107	117	3^{169}	-13^{51}	pg(13,13,9)
?	221	110	54	55	6.93^{110}	-7.93^{110}	↓?
+	222	51	20	9	14^{36}	-3^{185}	S(2,3,37)
		170	127	140	2^{185}	-15^{36}	
!	225	28	13	2	13^{28}	-2^{196}	15×15
		196	169	182	1^{196}	-14^{28}	OA(15, 14)?
+	225	42	15	6	12^{42}	-3^{182}	OA(15, 3)
		182	145	156	2^{182}	-13^{42}	OA(15, 13)?
?	225	48	3	12	3^{176}	-12^{48}	
		176	139	132	11^{48}	-4^{176}	
−	225	56	1	18	2^{200}	-19^{24}	† $q_{22}^2 < 0$
		168	129	114	18^{24}	-3^{200}	† $q_{11}^1 < 0$
+	225	56	19	12	11^{56}	-4^{168}	OA(15, 4)
		168	123	132	3^{168}	-12^{56}	OA(15, 12)?
?	225	64	13	20	4^{160}	-11^{64}	
		160	115	110	10^{64}	-5^{160}	
+	225	70	25	20	10^{70}	-5^{154}	OA(15, 5)
		154	103	110	4^{154}	-11^{70}	OA(15, 11)?
?	225	80	25	30	5^{144}	-10^{80}	pg(9,10,3)?
		144	93	90	9^{80}	-6^{144}	
+	225	84	33	30	9^{84}	-6^{140}	OA(15, 6)
		140	85	90	5^{140}	-10^{84}	OA(15, 10)?
−	225	96	19	57	1^{216}	-39^{8}	† $q_{22}^2 < 0$; † Absolute bound
		128	88	52	38^{8}	-2^{216}	† $q_{11}^1 < 0$; † Absolute bound
?	225	96	39	42	6^{128}	-9^{96}	
		128	73	72	8^{96}	-7^{128}	
?	225	96	51	33	21^{24}	-3^{200}	
		128	64	84	2^{200}	-22^{24}	
+	225	98	43	42	8^{98}	-7^{126}	OA(15, 7)?; Pasechnik (§8.12)
		126	69	72	6^{126}	-9^{98}	OA(15, 9)?
+	225	112	55	56	7^{112}	-8^{112}	ConfMat$(16)^2$; OA(15, 8)?; ↓
+	226	105	48	49	7^{113}	-8^{112}	ConfMat$(16)^{2*}$; ↑
		120	63	64	7^{112}	-8^{113}	S(2,8,113)?; ↑
+	229	114	56	57	7.07^{114}	-8.07^{114}	Paley(229); ↓
+	231	30	9	3	9^{55}	-3^{175}	§10.54; Cameron graph
		200	172	180	2^{175}	-10^{55}	
!	231	40	20	4	18^{21}	-2^{209}	$T(22)$
		190	153	171	1^{209}	-19^{21}	pg(11,19,9)?
?	231	70	21	21	7^{110}	-7^{120}	pg(11,7,3)?
		160	110	112	6^{120}	-8^{110}	
?	231	90	33	36	6^{132}	-9^{98}	pg(11,9,4)?
		140	85	84	8^{98}	-7^{132}	
?	232	33	2	5	4^{144}	-7^{87}	
		198	169	168	6^{87}	-5^{144}	
?	232	63	14	18	5^{144}	-9^{87}	pg(8,9,2)?

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ex	v	k	λ	μ	r^f	s^g	comment
		168	122	120	8^{87}	-6^{144}	
?	232	77	36	20	19^{28}	-3^{203}	
		154	96	114	2^{203}	-20^{28}	
?	232	81	30	27	9^{87}	-6^{144}	
		150	95	100	5^{144}	-10^{87}	S(2,10,145)?
+	233	116	57	58	7.13^{116}	-8.13^{116}	Paley(233); ↓
?	235	42	9	7	7^{94}	-5^{140}	
		192	156	160	4^{140}	-8^{94}	
?	235	52	9	12	5^{140}	-8^{94}	
		182	141	140	7^{94}	-6^{140}	
?	236	55	18	11	11^{59}	-4^{176}	
		180	135	144	3^{176}	-12^{59}	S(2,12,177)?
−	237	118	58	59	7.20^{118}	-8.20^{118}	† $v \neq a^2 + b^2$
?	238	75	20	25	5^{153}	-10^{84}	
		162	111	108	9^{84}	-6^{153}	
+	241	120	59	60	7.26^{120}	-8.26^{120}	Paley(241); ↓
+	243	22	1	2	4^{132}	-5^{110}	§10.55; Berlekamp-vanLint-Seidel; $[11,5]_3$ (wts $6,9$)
		220	199	200	4^{110}	-5^{132}	
?	243	66	9	21	3^{198}	-15^{44}	
		176	130	120	14^{44}	-4^{198}	
−	243	88	52	20	34^{11}	-2^{231}	† Absolute bound
		154	85	119	1^{231}	-35^{11}	† Absolute bound
+	243	110	37	60	2^{220}	-25^{22}	p. 311; Delsarte; $[55,5]_3$ (wts $36,45$)
		132	81	60	24^{22}	-3^{220}	
?	243	112	46	56	4^{182}	-14^{60}	pg(9,14,4)?; ↓?
		130	73	65	13^{60}	-5^{182}	↓?
?	244	108	42	52	4^{183}	-14^{60}	↑?
		135	78	70	13^{60}	-5^{183}	↑?
?	244	117	60	52	13^{61}	-5^{182}	↑?
		126	60	70	4^{182}	-14^{61}	↑?
?	245	52	3	13	3^{195}	-13^{49}	
		192	152	144	12^{49}	-4^{195}	
?	245	64	18	16	8^{100}	-6^{144}	
		180	131	135	5^{144}	-9^{100}	
?	245	108	39	54	3^{204}	-18^{40}	pg(7,18,3)?; ↓?
		136	81	68	17^{40}	-4^{204}	↓?
?	245	122	60	61	7.33^{122}	-8.33^{122}	↓?
?	246	85	20	34	3^{204}	-17^{41}	pg(6,17,2)?
		160	108	96	16^{41}	-4^{204}	
?	246	105	36	51	3^{205}	-18^{40}	↑?
		140	85	72	17^{40}	-4^{205}	↑?
?	246	119	64	51	17^{41}	-4^{204}	↑?
		126	57	72	3^{204}	-18^{41}	↑?
+	247	54	21	9	15^{38}	-3^{208}	S(2,3,39)
		192	146	160	2^{208}	-16^{38}	pg(13,16,10)?
?	249	88	27	33	5^{165}	-11^{83}	
		160	104	100	10^{83}	-6^{165}	
−	249	124	61	62	7.39^{124}	-8.39^{124}	† $v \neq a^2 + b^2$
?	250	81	24	27	6^{144}	-9^{105}	pg(10,9,3)?
		168	113	112	8^{105}	-7^{144}	
?	250	96	44	32	16^{45}	-4^{204}	
		153	88	102	3^{204}	-17^{45}	pg(10,17,6)?
!	253	42	21	4	19^{22}	-2^{230}	$T(23)$
		210	171	190	1^{230}	-20^{22}	
−	253	90	17	40	2^{230}	-25^{22}	† $q_{22}^2 < 0$
		162	111	90	24^{22}	-3^{230}	† $q_{11}^1 < 0$
+	253	112	36	60	2^{230}	-26^{22}	§10.56; $S(4,7,23)$; M_{23}
		140	87	65	25^{22}	-3^{230}	Witt: qs 2-(23,7,21)
−	253	126	62	63	7.45^{126}	-8.45^{126}	† $v \neq a^2 + b^2$
+	255	126	61	63	7^{135}	-9^{119}	$O_9(2)$; $Sp_8(2)$; pg(15,9,7); ↓
		128	64	64	8^{119}	-8^{135}	S(2,8,120); ↓
!	256	30	14	2	14^{30}	-2^{225}	16×16; $[10,4]_4$ (wts $4,8$); $[30,8]_2$ (wts $8,16$)

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ex	v	k	λ	μ	r^f	s^g	comment
		225	196	210	1^{225}	-15^{30}	OA(16, 15)
+	256	45	16	6	13^{45}	-3^{210}	§10.57; OA(16, 3); $H_2(2,4)$; Brouwer [112]; $[15,4]_4$ (wts 8, 12); $[45,8]_2$ (wts 16, 24)
		210	170	182	2^{210}	-14^{45}	OA(16, 14)
+	256	51	2	12	3^{204}	-13^{51}	vanLint-Schrijver, §7.3.1; $VO_4^-(4)$; $[17,4]_4$ (wts 12, 16)
		204	164	156	12^{51}	-4^{204}	vanLint-Schrijver, §7.3.1
+	256	60	20	12	12^{60}	-4^{195}	Jenrich (rk 4); OA(16, 4); Wallis [718]; $[20,4]_4$ (wts 12, 16); Brouwer [112]; $[60,8]_2$ (wts 24, 32)
		195	146	156	3^{195}	-13^{60}	OA(16, 13)
−	256	66	2	22	2^{231}	-22^{24}	† $q_{22}^2 < 0$
		189	144	126	21^{24}	-3^{231}	† $q_{11}^1 < 0$
+	256	68	12	20	4^{187}	-12^{68}	Brouwer [112]; $[68,8]_2$ (wts 32, 40)
		187	138	132	11^{68}	-5^{187}	
+	256	75	26	20	11^{75}	-5^{180}	OA(16, 5); $H_4(2,2)$; Wallis [718]; $VO_4^+(4)$; $[25,4]_4$ (wts 16, 20); $[75,8]_2$ (wts 32, 40)
		180	124	132	4^{180}	-12^{75}	OA(16, 12)
+	256	85	24	30	5^{170}	-11^{85}	vanLint-Schrijver, §7.3.1; $[85,8]_2$ (wts 40, 48)
		170	114	110	10^{85}	-6^{170}	vanLint-Schrijver, §7.3.1
+	256	90	34	30	10^{90}	-6^{165}	OA(16, 6); $[30,4]_4$ (wts 20, 24); $[90,8]_2$ (wts 40, 48)
		165	104	110	5^{165}	-11^{90}	OA(16, 11)
+	256	102	38	42	6^{153}	-10^{102}	§10.58; Liebeck $2^8.\mathsf{L}_2(17)$ (rk 3); vanLint-Schrijver, §7.3.1; $[34,4]_4$ (wts 24, 28)
		153	92	90	9^{102}	-7^{153}	vanLint-Schrijver, §7.3.1
+	256	105	44	42	9^{105}	-7^{150}	OA(16, 7); $[35,4]_4$ (wts 24, 28); Brouwer [112]; $[105,8]_2$ (wts 48, 56)
		150	86	90	6^{150}	-10^{105}	OA(16, 10)
+	256	119	54	56	7^{136}	-9^{119}	§10.59; $VO_8^-(2)$; $[119,8]_2$ (wts 56, 64); RSHCD$^-$; ↑
		136	72	72	8^{119}	-8^{136}	Fickus et al. [324]; ↑
+	256	120	56	56	8^{120}	-8^{135}	§10.60; OA(16, 8); Wallis [718]; $[40,4]_4$ (wts 28, 32); $[120,8]_2$ (wts 56, 64); RSHCD$^+$; ↑
		135	70	72	7^{135}	-9^{120}	OA(16, 9); Wallis [718]; Goethals-Seidel [355]; $VO_8^+(2)$; ↑
+	257	128	63	64	7.52^{128}	-8.52^{128}	Paley(257); ↓
?	259	42	5	7	5^{147}	-7^{111}	
		216	180	180	6^{111}	-6^{147}	
?	260	70	15	20	5^{168}	-10^{91}	pg(8,10,2)?
		189	138	135	9^{91}	-6^{168}	
?	261	52	11	10	7^{116}	-6^{144}	
		208	165	168	5^{144}	-8^{116}	
?	261	64	14	16	6^{144}	-8^{116}	pg(9,8,2)?
		196	147	147	7^{116}	-7^{144}	
?	261	80	25	24	8^{116}	-7^{144}	
		180	123	126	6^{144}	-9^{116}	
?	261	84	39	21	21^{29}	-3^{231}	
		176	112	132	2^{231}	-22^{29}	pg(9,22,6)?
?	261	130	64	65	7.58^{130}	-8.58^{130}	↓?
?	265	96	32	36	6^{159}	-10^{105}	
		168	107	105	9^{105}	-7^{159}	
?	265	132	65	66	7.64^{132}	-8.64^{132}	↓?
?	266	45	0	9	3^{209}	-12^{56}	
		220	183	176	11^{56}	-4^{209}	
+	269	134	66	67	7.70^{134}	-8.70^{134}	Paley(269); ↓
?	273	72	21	18	9^{104}	-6^{168}	pg(13,6,3)?
		200	145	150	5^{168}	-10^{104}	
?	273	80	19	25	5^{182}	-11^{90}	
		192	136	132	10^{90}	-6^{182}	
+	273	102	41	36	11^{90}	-6^{182}	S(2,6,91)
		170	103	110	5^{182}	-12^{90}	
?	273	136	65	70	6^{168}	-11^{104}	

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ex	v	k	λ	μ	r^f	s^g	comment
		136	69	66	10^{104}	-7^{168}	
$-$	273	136	67	68	7.76^{136}	-8.76^{136}	† $v \neq a^2+b^2$
!	275	112	30	56	2^{252}	-28^{22}	§10.61; $q_{22}^2 = 0$; no pg(5,28,2) (Östergård-Soicher [598]); ↓
		162	105	81	27^{22}	-3^{252}	McLaughlin graph, §10.61; $q_{11}^1 = 0$; ↓
!	276	44	22	4	20^{23}	-2^{252}	$T(24)$
		231	190	210	1^{252}	-21^{23}	pg(12,21,10)?
?	276	75	10	24	3^{230}	-17^{45}	
		200	148	136	16^{45}	-4^{230}	
?	276	75	18	21	6^{160}	-9^{115}	
		200	145	144	8^{115}	-7^{160}	
$-$	276	110	28	54	2^{253}	-28^{22}	† $q_{22}^2 < 0$; † Absolute bound
		165	108	84	27^{22}	-3^{253}	† $q_{11}^1 < 0$; † Absolute bound
?	276	110	52	38	18^{45}	-4^{230}	
		165	92	108	3^{230}	-19^{45}	
+	276	135	78	54	27^{23}	-3^{252}	p. 317; Conway / Goethals&Seidel; ↑
		140	58	84	2^{252}	-28^{23}	pg(6,28,3)?; ↑
+	277	138	68	69	7.82^{138}	-8.82^{138}	Paley(277); ↓
+	279	128	52	64	4^{216}	-16^{62}	pg(9,16,4)?; ↓
		150	85	75	15^{62}	-5^{216}	↓
+	280	36	8	4	8^{90}	-4^{189}	p. 287; $\mathsf{HJ.2} / 3.\mathsf{A}_6.2^2$ (rk 4); $\mathsf{U}_4(3)$; $\mathsf{GQ}(9,3)$
		243	210	216	3^{189}	-9^{90}	
?	280	62	12	14	6^{155}	-8^{124}	
		217	168	168	7^{124}	-7^{155}	
?	280	63	14	14	7^{135}	-7^{144}	pg(10,7,2)?
		216	166	168	6^{144}	-8^{135}	
+	280	117	44	52	5^{195}	-13^{84}	§10.62; pg(10,13,4)?
		162	96	90	12^{84}	-6^{195}	Mathon-Rosa S_9 (rk 5)
?	280	124	48	60	4^{217}	-16^{62}	↑?
		155	90	80	15^{62}	-5^{217}	↑?
+	280	135	70	60	15^{63}	-5^{216}	p. 287; $\mathsf{HJ.2} / 3.\mathsf{A}_6.2^2$ (rk 4); ↑
		144	68	80	4^{216}	-16^{63}	pg(10,16,5)?; ↑
+	281	140	69	70	7.88^{140}	-8.88^{140}	Paley(281); ↓
?	285	64	8	16	4^{209}	-12^{75}	
		220	171	165	11^{75}	-5^{209}	
$-$	285	142	70	71	7.94^{142}	-8.94^{142}	† $v \neq a^2+b^2$
?	286	95	24	35	4^{220}	-15^{65}	
		190	129	120	14^{65}	-5^{220}	
?	286	125	60	50	15^{65}	-5^{220}	
		160	84	96	4^{220}	-16^{65}	pg(11,16,6)?
?	287	126	45	63	3^{245}	-21^{41}	pg(7,21,3)?; ↓?
		160	96	80	20^{41}	-4^{245}	↓?
?	288	41	4	6	5^{164}	-7^{123}	
		246	210	210	6^{123}	-6^{164}	
?	288	42	6	6	6^{140}	-6^{147}	
		245	208	210	5^{147}	-7^{140}	
?	288	105	52	30	25^{27}	-3^{260}	
		182	106	130	2^{260}	-26^{27}	pg(8,26,5)?
?	288	112	36	48	4^{224}	-16^{63}	pg(8,16,3)?
		175	110	100	15^{63}	-5^{224}	
?	288	123	42	60	3^{246}	-21^{41}	↑?
		164	100	84	20^{41}	-4^{246}	↑?
?	288	140	76	60	20^{42}	-4^{245}	↑?
		147	66	84	3^{245}	-21^{42}	pg(8,21,4)?; ↑?
!	289	32	15	2	15^{32}	-2^{256}	17×17
		256	225	240	1^{256}	-16^{32}	$\mathrm{OA}(17,16)$
+	289	48	17	6	14^{48}	-3^{240}	$\mathrm{OA}(17,3)$
		240	197	210	2^{240}	-15^{48}	$\mathrm{OA}(17,15)$
$-$	289	54	1	12	3^{234}	-14^{54}	† Bondarenko-Radchenko [90]
		234	191	182	13^{54}	-4^{234}	
+	289	64	21	12	13^{64}	-4^{224}	$\mathrm{OA}(17,4)$
		224	171	182	3^{224}	-14^{64}	$\mathrm{OA}(17,14)$
?	289	72	11	20	4^{216}	-13^{72}	

continued...

ex	v	k	λ	μ	r^f	s^g	comment
		216	163	156	12^{72}	-5^{216}	
+	289	80	27	20	12^{80}	-5^{208}	OA(17, 5)
		208	147	156	4^{208}	-13^{80}	OA(17, 13)
?	289	90	23	30	5^{198}	-12^{90}	
		198	137	132	11^{90}	-6^{198}	
+	289	96	35	30	11^{96}	-6^{192}	OA(17, 6); vanLint-Schrijver, §7.3.1
		192	125	132	5^{192}	-12^{96}	OA(17, 12); vanLint-Schrijver, §7.3.1
?	289	108	37	42	6^{180}	-11^{108}	
		180	113	110	10^{108}	-7^{180}	
+	289	112	45	42	10^{112}	-7^{176}	OA(17, 7)
		176	105	110	6^{176}	-11^{112}	OA(17, 11)
−	289	120	21	70	1^{280}	-50^{8}	† $q_{22}^2 < 0$; † Absolute bound
		168	117	70	49^{8}	-2^{280}	† $q_{11}^1 < 0$; † Absolute bound
?	289	126	53	56	7^{162}	-10^{126}	
		162	91	90	9^{126}	-8^{162}	
+	289	128	57	56	9^{128}	-8^{160}	OA(17, 8)
		160	87	90	7^{160}	-10^{128}	OA(17, 10)
+	289	144	71	72	8^{144}	-9^{144}	Paley(289); OA(17, 9); ↓
+	290	136	63	64	8^{145}	-9^{144}	OA(17, 9)*; ↑
		153	80	81	8^{144}	-9^{145}	S(2,9,145)?; ↑
+	293	146	72	73	8.06^{146}	-9.06^{146}	Paley(293); ↓
+	297	40	7	5	7^{120}	-5^{176}	§10.63; lines in $\mathsf{U}_5(2)$; $\mathsf{GQ}(8, 4)$
		256	220	224	4^{176}	-8^{120}	
?	297	104	31	39	5^{208}	-13^{88}	pg(9,13,3)?
		192	126	120	12^{88}	-6^{208}	
?	297	128	64	48	20^{44}	-4^{252}	
		168	87	105	3^{252}	-21^{44}	pg(9,21,5)?
−	297	148	73	74	8.12^{148}	-9.12^{148}	† $v \neq a^2 + b^2$
?	300	26	4	2	6^{117}	-4^{182}	
		273	248	252	3^{182}	-7^{117}	
!	300	46	23	4	21^{24}	-2^{275}	$T(25)$
		253	210	231	1^{275}	-22^{24}	
+	300	65	10	15	5^{195}	-10^{104}	§10.64; $NO_5^{-\perp}(5)$
		234	183	180	9^{104}	-6^{195}	
?	300	69	18	15	9^{115}	-6^{184}	
		230	175	180	5^{184}	-10^{115}	
−	300	92	10	36	2^{276}	-28^{23}	† $q_{22}^2 < 0$; † Absolute bound
		207	150	126	27^{23}	-3^{276}	† $q_{11}^1 < 0$; † Absolute bound
+	300	104	28	40	4^{234}	-16^{65}	§10.64; $NO_5^{-}(5)$
		195	130	120	15^{65}	-5^{234}	
?	300	115	50	40	15^{69}	-5^{230}	
		184	108	120	4^{230}	-16^{69}	
?	300	117	60	36	27^{26}	-3^{273}	
		182	100	126	2^{273}	-28^{26}	
+	301	60	23	9	17^{42}	-3^{258}	S(2,3,43)
		240	188	204	2^{258}	-18^{42}	
?	301	108	27	45	3^{258}	-21^{42}	
		192	128	112	20^{42}	-4^{258}	
?	301	150	65	84	3^{258}	-22^{42}	
		150	83	66	21^{42}	-4^{258}	
−	301	150	74	75	8.17^{150}	-9.17^{150}	† $v \neq a^2 + b^2$
+	304	108	42	36	12^{95}	-6^{208}	S(2,6,96)
		195	122	130	5^{208}	-13^{95}	pg(16,13,10)?
+	305	76	27	16	15^{60}	-4^{244}	S(2,4,61)
		228	167	180	3^{244}	-16^{60}	
?	305	152	75	76	8.23^{152}	-9.23^{152}	↓?
?	306	55	4	11	4^{220}	-11^{85}	
		250	205	200	10^{85}	-5^{220}	
?	306	60	10	12	6^{170}	-8^{135}	
		245	196	196	7^{135}	-7^{170}	
−	309	154	76	77	8.29^{154}	-9.29^{154}	† $v \neq a^2 + b^2$
+	313	156	77	78	8.35^{156}	-9.35^{156}	Paley(313); ↓
+	317	158	78	79	8.40^{158}	-9.40^{158}	Paley(317); ↓

continued...

ex	v	k	λ	μ	r^f	s^g	comment
?	319	150	65	75	5^{231}	-15^{87}	pg(11,15,5)?; ↓?
		168	92	84	14^{87}	-6^{231}	↓?
?	320	87	22	24	7^{174}	-9^{145}	
		232	168	168	8^{145}	-8^{174}	
?	320	88	24	24	8^{154}	-8^{165}	
		231	166	168	7^{165}	-9^{154}	
?	320	99	18	36	3^{275}	-21^{44}	
		220	156	140	20^{44}	-4^{275}	
?	320	132	46	60	4^{255}	-18^{64}	
		187	114	102	17^{64}	-5^{255}	
?	320	145	60	70	5^{232}	-15^{87}	↑?
		174	98	90	14^{87}	-6^{232}	↑?
?	320	154	78	70	14^{88}	-6^{231}	↑?
		165	80	90	5^{231}	-15^{88}	pg(12,15,6)?; ↑?
−	321	160	79	80	8.46^{160}	-9.46^{160}	† $v \neq a^2 + b^2$
?	322	96	20	32	4^{252}	-16^{69}	pg(7,16,2)?
		225	160	150	15^{69}	-5^{252}	
+	323	160	78	80	8^{170}	-10^{152}	pg(17,10,8)?; ↓
		162	81	81	9^{152}	-9^{170}	S(2,9,153)?; ↓
!	324	34	16	2	16^{34}	-2^{289}	18×18
		289	256	272	1^{289}	-17^{34}	OA(18, 17)?
+	324	51	18	6	15^{51}	-3^{272}	OA(18, 3)
		272	226	240	2^{272}	-16^{51}	OA(18, 16)?
−	324	57	0	12	3^{266}	-15^{57}	† Gavrilyuk & Makhnev [336]; † Kaski & Östergård [483]
		266	220	210	14^{57}	-4^{266}	
?	324	68	7	16	4^{243}	-13^{80}	
		255	202	195	12^{80}	-5^{243}	
+	324	68	22	12	14^{68}	-4^{255}	OA(18, 4)
		255	198	210	3^{255}	-15^{68}	OA(18, 15)?
?	324	76	10	20	4^{247}	-14^{76}	
		247	190	182	13^{76}	-5^{247}	
+	324	85	28	20	13^{85}	-5^{238}	OA(18, 5)
		238	172	182	4^{238}	-14^{85}	OA(18, 14)?
?	324	95	22	30	5^{228}	-13^{95}	
		228	162	156	12^{95}	-6^{228}	
+	324	95	34	25	14^{80}	-5^{243}	S(2,5,81)
		228	157	168	4^{243}	-15^{80}	
+	324	102	36	30	12^{102}	-6^{221}	OA(18, 6)
		221	148	156	5^{221}	-13^{102}	OA(18, 13)?
?	324	114	36	42	6^{209}	-12^{114}	
		209	136	132	11^{114}	-7^{209}	
+	324	119	46	42	11^{119}	-7^{204}	OA(18, 7)
		204	126	132	6^{204}	-12^{119}	OA(18, 12)?
−	324	133	22	77	1^{315}	-56^{8}	† $q^2_{22} < 0$; † Absolute bound
		190	133	80	55^{8}	-2^{315}	† $q^1_{11} < 0$; † Absolute bound
?	324	133	52	56	7^{190}	-11^{133}	
		190	112	110	10^{133}	-8^{190}	
?	324	136	58	56	10^{136}	-8^{187}	OA(18, 8)?
		187	106	110	7^{187}	-11^{136}	OA(18, 11)?
+	324	152	70	72	8^{171}	-10^{152}	RSHCD$^-$; ↑
		171	90	90	9^{152}	-9^{171}	↑
+	324	153	72	72	9^{153}	-9^{170}	OA(18, 9)?; RSHCD$^+$; ↑
		170	88	90	8^{170}	-10^{153}	OA(18, 10)?; ↑
!	325	48	24	4	22^{25}	-2^{299}	$T(26)$
		276	231	253	1^{299}	-23^{25}	pg(13,23,11)?
?	325	54	3	10	4^{234}	-11^{90}	
		270	225	220	10^{90}	-5^{234}	
+	325	60	15	10	10^{104}	-5^{220}	§10.65; $NO_5^{+\perp}(5)$; Wallis [718]; pg(13,5,2)?
		264	213	220	4^{220}	-11^{104}	
+	325	68	3	17	3^{272}	-17^{52}	$q^2_{22} = 0$; $\mathsf{O}_6^-(4)$; $\mathsf{GQ}(4,16)$
		256	204	192	16^{52}	-4^{272}	$q^1_{11} = 0$
?	325	72	15	16	7^{168}	-8^{156}	pg(10,8,2)?

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ex	v	k	λ	μ	r^f	s^g	comment
		252	195	196	7^{156}	-8^{168}	
−	325	108	63	22	43^{12}	-2^{312}	† Absolute bound
		216	129	172	1^{312}	-44^{12}	† Absolute bound
+	325	144	68	60	14^{90}	-6^{234}	§10.65; $NO_5^+(5)$
		180	95	105	5^{234}	-15^{90}	pg(13,15,7)?
?	325	162	80	81	8.51^{162}	-9.51^{162}	↓?
?	329	40	3	5	5^{188}	-7^{140}	
		288	252	252	6^{140}	-6^{188}	
−	329	164	81	82	8.57^{164}	-9.57^{164}	† $v \neq a^2 + b^2$
+	330	63	24	9	18^{44}	-3^{285}	dist. 1 or 4 in J(11,4) - Mathon; S(2,3,45)
		266	211	228	2^{285}	-19^{44}	pg(15,19,12)?
?	330	105	40	30	15^{77}	-5^{252}	
		224	148	160	4^{252}	-16^{77}	pg(15,16,10)?
?	330	140	58	60	8^{175}	-10^{154}	pg(15,10,6)?
		189	108	108	9^{154}	-9^{175}	
?	333	166	82	83	8.62^{166}	-9.62^{166}	↓?
+	336	80	28	16	16^{63}	-4^{272}	Jenrich, p. 323; S(2,4,64); qs 2-(64,24,46); lines in $AG(3,4)$ (rk 4); Wallis [718]
		255	190	204	3^{272}	-17^{63}	pg(16,17,12)?
?	336	125	40	50	5^{245}	-15^{90}	
		210	134	126	14^{90}	-6^{245}	
?	336	135	54	54	9^{160}	-9^{175}	pg(16,9,6)?
		200	118	120	8^{175}	-10^{160}	
+	337	168	83	84	8.68^{168}	-9.68^{168}	Paley(337); ↓
?	340	108	30	36	6^{220}	-12^{119}	pg(10,12,3)?
		231	158	154	11^{119}	-7^{220}	
?	341	70	15	14	8^{154}	-7^{186}	pg(11,7,2)?
		270	213	216	6^{186}	-9^{154}	
?	341	84	19	21	7^{186}	-9^{154}	
		256	192	192	8^{154}	-8^{186}	
?	341	102	31	30	9^{154}	-8^{186}	
		238	165	168	7^{186}	-10^{154}	
−	341	170	84	85	8.73^{170}	-9.73^{170}	† $v \neq a^2 + b^2$
?	342	33	4	3	6^{152}	-5^{189}	
		308	277	280	4^{189}	-7^{152}	
?	342	66	15	12	9^{132}	-6^{209}	pg(12,6,2)?
		275	220	225	5^{209}	-10^{132}	
+	343	54	5	9	5^{216}	-9^{126}	Godsil [345]; $\mathsf{GQ}(6,8)$
		288	242	240	8^{126}	-6^{216}	
−	343	96	54	16	40^{14}	-2^{328}	† Absolute bound
		246	165	205	1^{328}	-41^{14}	† Absolute bound
?	343	102	21	34	4^{272}	-17^{70}	pg(7,17,2)?
		240	171	160	16^{70}	-5^{272}	
?	343	114	45	34	16^{76}	-5^{266}	
		228	147	160	4^{266}	-17^{76}	
+	343	150	53	75	3^{300}	-25^{42}	Godsil [345]; pg(7,25,3)?; ↓
		192	116	96	24^{42}	-4^{300}	↓
?	343	162	81	72	15^{90}	-6^{252}	
		180	89	100	5^{252}	-16^{90}	
?	344	147	50	72	3^{301}	-25^{42}	↑?
		196	120	100	24^{42}	-4^{301}	↑?
+	344	168	92	72	24^{43}	-4^{300}	↑
		175	78	100	3^{300}	-25^{43}	pg(8,25,4)?; Taylor ↑
?	345	120	35	45	5^{252}	-15^{92}	pg(9,15,3)?
		224	148	140	14^{92}	-6^{252}	
?	345	128	46	48	8^{184}	-10^{160}	
		216	135	135	9^{160}	-9^{184}	
−	345	172	85	86	8.79^{172}	-9.79^{172}	† $v \neq a^2 + b^2$
+	349	174	86	87	8.84^{174}	-9.84^{174}	Paley(349); ↓
?	351	50	13	6	11^{90}	-4^{260}	
		300	255	264	3^{260}	-12^{90}	
!	351	50	25	4	23^{26}	-2^{324}	$T(27)$
		300	253	276	1^{324}	-24^{26}	

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ex	v	k	λ	μ	r^f	s^g	comment
?	351	70	13	14	7^{182}	-8^{168}	
		280	223	224	7^{168}	-8^{182}	
?	351	110	37	33	11^{130}	-7^{220}	
		240	162	168	6^{220}	-12^{130}	
?	351	112	43	32	16^{78}	-5^{272}	
		238	157	170	4^{272}	-17^{78}	
+	351	126	45	45	9^{168}	-9^{182}	§10.66
		224	142	144	8^{182}	-10^{168}	$NO_7^-(3)$
?	351	140	49	60	5^{260}	-16^{90}	
		210	129	120	15^{90}	-6^{260}	
?	351	140	73	44	32^{26}	-3^{324}	
		210	113	144	2^{324}	-33^{26}	
–	351	150	81	51	33^{25}	-3^{325}	† Absolute bound
		200	100	132	2^{325}	-34^{25}	† Absolute bound
?	351	160	64	80	4^{285}	-20^{65}	pg(9,20,4)?; ↓?
		190	109	95	19^{65}	-5^{285}	↓?
?	352	26	0	2	4^{208}	-6^{143}	
		325	300	300	5^{143}	-5^{208}	
?	352	36	0	4	4^{231}	-8^{120}	
		315	282	280	7^{120}	-5^{231}	
?	352	39	6	4	7^{143}	-5^{208}	
		312	276	280	4^{208}	-8^{143}	
?	352	108	44	28	20^{54}	-4^{297}	
		243	162	180	3^{297}	-21^{54}	
?	352	117	36	40	7^{208}	-11^{143}	
		234	156	154	10^{143}	-8^{208}	
?	352	126	50	42	14^{99}	-6^{252}	
		225	140	150	5^{252}	-15^{99}	pg(16,15,10)?
–	352	130	78	30	50^{11}	-2^{340}	† Absolute bound
		221	120	170	1^{340}	-51^{11}	† Absolute bound
?	352	156	60	76	4^{286}	-20^{65}	↑?
		195	114	100	19^{65}	-5^{286}	↑?
?	352	171	90	76	19^{66}	-5^{285}	↑?
		180	84	100	4^{285}	-20^{66}	pg(10,20,5)?; ↑?
+	353	176	87	88	8.89^{176}	-9.89^{176}	Paley(353); ↓
+	357	100	35	25	15^{84}	-5^{272}	S(2,5,85); lines in $\mathsf{PG}(3,4)$; $\mathsf{O}_6^+(4)$
		256	180	192	4^{272}	-16^{84}	pg(17,16,12)?
–	357	178	88	89	8.95^{178}	-9.95^{178}	† $v \neq a^2 + b^2$
!	361	36	17	2	17^{36}	-2^{324}	19×19
		324	289	306	1^{324}	-18^{36}	OA(19, 18)
+	361	54	19	6	16^{54}	-3^{306}	OA(19, 3)
		306	257	272	2^{306}	-17^{54}	OA(19, 17)
+	361	72	23	12	15^{72}	-4^{288}	OA(19, 4)
		288	227	240	3^{288}	-16^{72}	OA(19, 16)
?	361	80	9	20	4^{280}	-15^{80}	
		280	219	210	14^{80}	-5^{280}	
+	361	90	29	20	14^{90}	-5^{270}	OA(19, 5)
		270	199	210	4^{270}	-15^{90}	OA(19, 15)
?	361	100	21	30	5^{260}	-14^{100}	
		260	189	182	13^{100}	-6^{260}	
+	361	108	37	30	13^{108}	-6^{252}	OA(19, 6)
		252	173	182	5^{252}	-14^{108}	OA(19, 14)
?	361	120	35	42	6^{240}	-13^{120}	
		240	161	156	12^{120}	-7^{240}	
+	361	126	47	42	12^{126}	-7^{234}	OA(19, 7)
		234	149	156	6^{234}	-13^{126}	OA(19, 13)
?	361	140	51	56	7^{220}	-12^{140}	
		220	135	132	11^{140}	-8^{220}	
+	361	144	59	56	11^{144}	-8^{216}	OA(19, 8)
		216	127	132	7^{216}	-12^{144}	OA(19, 12)
–	361	150	93	40	55^{10}	-2^{350}	† Absolute bound
		210	99	154	1^{350}	-56^{10}	† Absolute bound
?	361	160	69	72	8^{200}	-11^{160}	

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ex	v	k	λ	μ	r^f	s^g	comment
		200	111	110	10^{160}	-9^{200}	
+	361	162	73	72	10^{162}	-9^{198}	OA(19, 9); Pasechnik (§8.12)
		198	107	110	8^{198}	-11^{162}	OA(19, 11)
−	361	168	95	63	35^{24}	-3^{336}	† Absolute bound
		192	86	120	2^{336}	-36^{24}	† Absolute bound
+	361	180	89	90	9^{180}	-10^{180}	Paley(361); OA(19, 10); ↓
+	362	171	80	81	9^{181}	-10^{180}	OA(19, 10)*; ConfMat(20)^{2*}; ↑
		190	99	100	9^{180}	-10^{181}	S(2,10,181)?; ↑
?	363	170	73	85	5^{272}	-17^{90}	pg(11,17,5)?; ↓?
		192	106	96	16^{90}	-6^{272}	↓?
?	364	33	2	3	5^{195}	-6^{168}	
		330	299	300	5^{168}	-6^{195}	
?	364	66	20	10	14^{77}	-4^{286}	
		297	240	252	3^{286}	-15^{77}	
?	364	88	12	24	4^{286}	-16^{77}	
		275	210	200	15^{77}	-5^{286}	
+	364	120	38	40	8^{195}	-10^{168}	$\mathsf{O}_7(3)$; $\mathsf{Sp}_6(3)$; pg(13,10,4)?
		243	162	162	9^{168}	-9^{195}	
?	364	121	48	36	17^{77}	-5^{286}	
		242	156	170	4^{286}	-18^{77}	
?	364	165	68	80	5^{273}	-17^{90}	↑?
		198	112	102	16^{90}	-6^{273}	↑?
?	364	176	90	80	16^{91}	-6^{272}	↑?
		187	90	102	5^{272}	-17^{91}	↑?
?	365	182	90	91	9.05^{182}	-10.05^{182}	↓?
?	369	184	91	92	9.10^{184}	-10.10^{184}	↓?
+	371	120	44	36	14^{105}	-6^{265}	S(2,6,106)
		250	165	175	5^{265}	-15^{105}	
?	372	56	10	8	8^{155}	-6^{216}	
		315	266	270	5^{216}	-9^{155}	
+	373	186	92	93	9.16^{186}	-10.16^{186}	Paley(373); ↓
−	375	22	5	1	7^{110}	-3^{264}	† $\mu = 1$
		352	330	336	2^{264}	-8^{110}	
?	375	66	9	12	6^{220}	-9^{154}	
		308	253	252	8^{154}	-7^{220}	
?	375	68	13	12	8^{170}	-7^{204}	
		306	249	252	6^{204}	-9^{170}	
?	375	102	45	21	27^{34}	-3^{340}	
		272	190	216	2^{340}	-28^{34}	
?	375	110	25	35	5^{275}	-15^{99}	
		264	188	180	14^{99}	-6^{275}	
?	375	136	44	52	6^{255}	-14^{119}	
		238	153	147	13^{119}	-7^{255}	
?	375	154	53	70	4^{308}	-21^{66}	
		220	135	120	20^{66}	-5^{308}	
?	375	170	85	70	20^{68}	-5^{306}	
		204	103	120	4^{306}	-21^{68}	
?	375	176	94	72	26^{44}	-4^{330}	
		198	93	117	3^{330}	-27^{44}	
?	375	182	85	91	7^{234}	-13^{140}	pg(15,13,7)?; ↓?
		192	100	96	12^{140}	-8^{234}	↓?
?	376	105	32	28	11^{140}	-7^{235}	
		270	192	198	6^{235}	-12^{140}	
?	376	175	78	84	7^{235}	-13^{140}	↑?
		200	108	104	12^{140}	-8^{235}	↑?
?	376	180	88	84	12^{141}	-8^{234}	↑?
		195	98	104	7^{234}	-13^{141}	↑?
?	377	180	81	90	6^{260}	-15^{116}	pg(13,15,6)?; ↓?
		196	105	98	14^{116}	-7^{260}	↓?
?	377	188	93	94	9.21^{188}	-10.21^{188}	↓?
+	378	52	1	8	4^{273}	-11^{104}	Cossidente-Penttila [233]
		325	280	275	10^{104}	-5^{273}	
!	378	52	26	4	24^{27}	-2^{350}	$T(28)$

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ex	v	k	λ	μ	r^f	s^g	comment
		325	276	300	1^{350}	-25^{27}	pg(14,25,12)?
+	378	116	34	36	8^{203}	-10^{174}	Muzychuk [580]
		261	180	180	9^{174}	-9^{203}	
+	378	117	36	36	9^{182}	-9^{195}	§10.67; Wallis [718]; pg(14,9,4)?
		260	178	180	8^{195}	-10^{182}	$NO_7^+(3)$
?	378	174	75	84	6^{261}	-15^{116}	↑?
		203	112	105	14^{116}	-7^{261}	↑?
?	378	182	91	84	14^{117}	-7^{260}	↑?
		195	96	105	6^{260}	-15^{117}	pg(14,15,7)?; ↑?
?	381	114	29	36	6^{254}	-13^{126}	
		266	187	182	12^{126}	-7^{254}	
?	381	140	55	49	13^{126}	-7^{254}	S(2,7,127)?
		240	148	156	6^{254}	-14^{126}	
−	381	190	94	95	9.26^{190}	-10.26^{190}	† $v \neq a^2+b^2$
?	385	60	5	10	5^{252}	-10^{132}	
		324	273	270	9^{132}	-6^{252}	
?	385	168	77	70	14^{120}	-7^{264}	
		216	117	126	6^{264}	-15^{120}	
−	385	192	95	96	9.31^{192}	-10.31^{192}	† $v \neq a^2+b^2$
+	389	194	96	97	9.36^{194}	-10.36^{194}	Paley(389); ↓
?	391	140	39	56	4^{322}	-21^{68}	
		250	165	150	20^{68}	-5^{322}	
?	391	182	93	77	21^{68}	-5^{322}	
		208	102	120	4^{322}	-22^{68}	
?	392	46	0	6	4^{276}	-10^{115}	
		345	304	300	9^{115}	-5^{276}	
?	392	51	10	6	9^{136}	-5^{255}	
		340	294	300	4^{255}	-10^{136}	
+	392	69	26	9	20^{48}	-3^{343}	S(2,3,49)
		322	261	280	2^{343}	-21^{48}	
?	392	115	18	40	3^{345}	-25^{46}	$q_{22}^2 = 0$
		276	200	180	24^{46}	-4^{345}	$q_{11}^1 = 0$
?	392	136	60	40	24^{51}	-4^{340}	
		255	158	180	3^{340}	-25^{51}	
?	392	153	54	63	6^{272}	-15^{119}	
		238	147	140	14^{119}	-7^{272}	
−	392	184	66	104	2^{368}	-40^{23}	† Absolute bound
		207	126	90	39^{23}	-3^{368}	† Absolute bound
−	393	196	97	98	9.41^{196}	-10.41^{196}	† $v \neq a^2+b^2$
?	396	135	30	54	3^{351}	-27^{44}	pg(6,27,2)?
		260	178	156	26^{44}	-4^{351}	
?	396	150	51	60	6^{275}	-15^{120}	pg(11,15,4)?
		245	154	147	14^{120}	-7^{275}	
+	397	198	98	99	9.46^{198}	-10.46^{198}	Paley(397); ↓
+	399	198	97	99	9^{209}	-11^{189}	pg(19,11,9)?; ↓
		200	100	100	10^{189}	-10^{209}	S(2,10,190)?; ↓
?	400	21	2	1	5^{175}	-4^{224}	
		378	357	360	3^{224}	-6^{175}	
!	400	38	18	2	18^{38}	-2^{361}	20×20
		361	324	342	1^{361}	-19^{38}	$OA(20,19)$?
+	400	56	6	8	6^{224}	-8^{175}	$\mathsf{O}_5(7)$; $\mathsf{Sp}_4(7)$; $\mathsf{GQ}(7,7)$
		343	294	294	7^{175}	-7^{224}	
+	400	57	20	6	17^{57}	-3^{342}	$OA(20,3)$
		342	290	306	2^{342}	-18^{57}	$OA(20,18)$?
+	400	76	24	12	16^{76}	-4^{323}	$OA(20,4)$
		323	258	272	3^{323}	-17^{76}	$OA(20,17)$?
?	400	84	8	20	4^{315}	-16^{84}	
		315	250	240	15^{84}	-5^{315}	
+	400	95	30	20	15^{95}	-5^{304}	$OA(20,5)$
		304	228	240	4^{304}	-16^{95}	$OA(20,16)$?
−	400	102	2	34	2^{374}	-34^{25}	† $q_{22}^2 < 0$; † Absolute bound
		297	228	198	33^{25}	-3^{374}	† $q_{11}^1 < 0$; † Absolute bound
?	400	105	20	30	5^{294}	-15^{105}	pg(8,15,2)?

continued…

ex	v	k	λ	μ	r^f	s^g	comment
		294	218	210	14^{105}	-6^{294}	
−	400	114	8	42	2^{375}	-36^{24}	† $q_{22}^2 < 0$; † Absolute bound
		285	212	180	35^{24}	-3^{375}	† $q_{11}^1 < 0$; † Absolute bound
+	400	114	38	30	14^{114}	-6^{285}	OA(20, 6)
		285	200	210	5^{285}	-15^{114}	OA(20, 15)?
?	400	126	34	42	6^{273}	-14^{126}	pg(10,14,3)?
		273	188	182	13^{126}	-7^{273}	
?	400	133	42	45	8^{224}	-11^{175}	
		266	177	176	10^{175}	-9^{224}	
?	400	133	48	42	13^{133}	-7^{266}	OA(20, 7)?
		266	174	182	6^{266}	-14^{133}	OA(20, 14)?
?	400	147	50	56	7^{252}	-13^{147}	
		252	160	156	12^{147}	-8^{252}	
?	400	152	60	56	12^{152}	-8^{247}	OA(20, 8)?
		247	150	156	7^{247}	-13^{152}	OA(20, 13)?
?	400	156	74	52	26^{48}	-4^{351}	
		243	138	162	3^{351}	-27^{48}	pg(10,27,6)?
−	400	161	24	92	1^{391}	-69^{8}	† $q_{22}^2 < 0$; † Absolute bound
		238	168	102	68^{8}	-2^{391}	† $q_{11}^1 < 0$; † Absolute bound
?	400	168	68	72	8^{231}	-12^{168}	pg(15,12,6)?
		231	134	132	11^{168}	-9^{231}	
?	400	171	74	72	11^{171}	-9^{228}	OA(20, 9)?
		228	128	132	8^{228}	-12^{171}	OA(20, 12)?
+	400	189	88	90	9^{210}	-11^{189}	RSHCD$^-$; ↑
		210	110	110	10^{189}	-10^{210}	Fickus et al. [324]; ↑
−	400	189	108	72	39^{24}	-3^{375}	† Absolute bound
		210	92	130	2^{375}	-40^{24}	† Absolute bound
+	400	190	90	90	10^{190}	-10^{209}	OA(20, 10)?; Wallis [718]; RSHCD$^+$; ↑
		209	108	110	9^{209}	-11^{190}	OA(20, 11)?; Wallis [718]; Goethals-Seidel [355]; ↑
+	401	200	99	100	9.51^{200}	-10.51^{200}	Paley(401); ↓
−	405	84	3	21	3^{350}	-21^{54}	† $q_{22}^2 < 0$
		320	256	240	20^{54}	-4^{350}	† $q_{11}^1 < 0$
?	405	96	18	24	6^{264}	-12^{140}	pg(9,12,2)?
		308	235	231	11^{140}	-7^{264}	
?	405	132	63	33	33^{30}	-3^{374}	
		272	172	204	2^{374}	-34^{30}	pg(9,34,6)?
?	405	196	91	98	7^{260}	-14^{144}	pg(15,14,7)?; ↓?
		208	109	104	13^{144}	-8^{260}	↓?
?	405	202	100	101	9.56^{202}	-10.56^{202}	↓?
!	406	54	27	4	25^{28}	-2^{377}	$T(29)$
		351	300	325	1^{377}	-26^{28}	
?	406	108	30	28	10^{174}	-8^{231}	
		297	216	220	7^{231}	-11^{174}	
?	406	165	68	66	11^{174}	-9^{231}	
		240	140	144	8^{231}	-12^{174}	S(2,12,232)?
?	406	189	84	91	7^{261}	-14^{144}	↑?
		216	117	112	13^{144}	-8^{261}	↑?
?	406	195	96	91	13^{145}	-8^{260}	↑?
		210	105	112	7^{260}	-14^{145}	↑?
+	407	126	45	36	15^{110}	-6^{296}	S(2,6,111)
		280	189	200	5^{296}	-16^{110}	
?	408	110	28	30	8^{220}	-10^{187}	pg(12,10,3)?
		297	216	216	9^{187}	-9^{220}	
?	408	176	70	80	6^{288}	-16^{119}	pg(12,16,5)?
		231	134	126	15^{119}	-7^{288}	
+	409	204	101	102	9.61^{204}	-10.61^{204}	Paley(409); ↓
?	411	130	45	39	13^{137}	-7^{273}	
		280	188	196	6^{273}	-14^{137}	S(2,14,274)?
?	413	112	36	28	14^{118}	-6^{294}	
		300	215	225	5^{294}	-15^{118}	S(2,15,295)?
−	413	206	102	103	9.66^{206}	-10.66^{206}	† $v \neq a^2 + b^2$
?	414	63	12	9	9^{161}	-6^{252}	

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ex	v	k	λ	μ	r^f	s^g	comment
		350	295	300	5^{252}	-10^{161}	
–	414	140	22	60	2^{390}	-40^{23}	† $q_{22}^2 < 0$; † Absolute bound
		273	192	156	39^{23}	-3^{390}	† $q_{11}^1 < 0$; † Absolute bound
+	416	100	36	20	20^{65}	-4^{350}	§10.68; $G_2(4)$ in Suz tower
		315	234	252	3^{350}	-21^{65}	pg(16,21,12)?
?	416	165	64	66	9^{220}	-11^{195}	pg(16,11,6)?
		250	150	150	10^{195}	-10^{220}	
–	417	208	103	104	9.71^{208}	-10.71^{208}	† $v \neq a^2 + b^2$
?	418	147	56	49	14^{132}	-7^{285}	S(2,7,133)?
		270	171	180	6^{285}	-15^{132}	pg(19,15,12)?
+	421	210	104	105	9.76^{210}	-10.76^{210}	Paley(421); ↓
?	424	99	26	22	11^{159}	-7^{264}	
		324	246	252	6^{264}	-12^{159}	
+	425	72	27	9	21^{50}	-3^{374}	S(2,3,51)
		352	288	308	2^{374}	-22^{50}	pg(17,22,14)?
?	425	160	60	60	10^{204}	-10^{220}	pg(17,10,6)?
		264	163	165	9^{220}	-11^{204}	
?	425	212	105	106	9.81^{212}	-10.81^{212}	↓?
?	428	112	21	32	5^{320}	-16^{107}	pg(8,16,2)?
		315	234	225	15^{107}	-6^{320}	
?	429	108	27	27	9^{208}	-9^{220}	pg(13,9,3)?
		320	238	240	8^{220}	-10^{208}	
–	429	214	106	107	9.86^{214}	-10.86^{214}	† $v \neq a^2 + b^2$
?	430	39	8	3	9^{129}	-4^{300}	
		390	353	360	3^{300}	-10^{129}	
?	430	135	36	45	6^{300}	-15^{129}	pg(10,15,3)?
		294	203	196	14^{129}	-7^{300}	
?	430	165	68	60	15^{129}	-7^{300}	
		264	158	168	6^{300}	-16^{129}	
+	433	216	107	108	9.90^{216}	-10.90^{216}	Paley(433); ↓
!	435	56	28	4	26^{29}	-2^{405}	$T(30)$
		378	325	351	1^{405}	-27^{29}	pg(15,27,13)?
?	435	154	53	55	9^{231}	-11^{203}	pg(15,11,5)?
		280	180	180	10^{203}	-10^{231}	
?	435	182	73	78	8^{260}	-13^{174}	pg(15,13,6)?
		252	147	144	12^{174}	-9^{260}	
?	437	100	15	25	5^{322}	-15^{114}	
		336	260	252	14^{114}	-6^{322}	
–	437	218	108	109	9.95^{218}	-10.95^{218}	† $v \neq a^2 + b^2$
+	438	92	31	16	19^{72}	-4^{365}	S(2,4,73)
		345	268	285	3^{365}	-20^{72}	
!	441	40	19	2	19^{40}	-2^{400}	21×21
		400	361	380	1^{400}	-20^{40}	$\mathrm{OA}(21, 20)$?
+	441	56	7	7	7^{216}	-7^{224}	Wallis [718]; $\mathsf{GQ}(8, 6)$
		384	334	336	6^{224}	-8^{216}	
+	441	60	21	6	18^{60}	-3^{380}	$\mathrm{OA}(21, 3)$
		380	325	342	2^{380}	-19^{60}	$\mathrm{OA}(21, 19)$?
+	441	80	25	12	17^{80}	-4^{360}	$\mathrm{OA}(21, 4)$
		360	291	306	3^{360}	-18^{80}	$\mathrm{OA}(21, 18)$?
?	441	88	7	20	4^{352}	-17^{88}	
		352	283	272	16^{88}	-5^{352}	
?	441	88	35	13	25^{44}	-3^{396}	
		352	276	300	2^{396}	-26^{44}	
+	441	100	31	20	16^{100}	-5^{340}	$\mathrm{OA}(21, 5)$
		340	259	272	4^{340}	-17^{100}	$\mathrm{OA}(21, 17)$?
?	441	110	19	30	5^{330}	-16^{110}	
		330	249	240	15^{110}	-6^{330}	
–	441	120	15	39	3^{392}	-27^{48}	† $q_{22}^2 < 0$
		320	238	216	26^{48}	-4^{392}	† $q_{11}^1 < 0$
+	441	120	39	30	15^{120}	-6^{320}	$\mathrm{OA}(21, 6)$
		320	229	240	5^{320}	-16^{120}	$\mathrm{OA}(21, 16)$?
–	441	128	10	48	2^{416}	-40^{24}	† $q_{22}^2 < 0$; † Absolute bound
		312	231	195	39^{24}	-3^{416}	† $q_{11}^1 < 0$; † Absolute bound

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ex	v	k	λ	μ	r^f	s^g	comment
?	441	132	33	42	6^{308}	-15^{132}	
		308	217	210	14^{132}	-7^{308}	
+	441	140	49	42	14^{140}	-7^{300}	OA(21, 7)
		300	201	210	6^{300}	-15^{140}	OA(21, 15)?
?	441	152	43	57	5^{342}	-19^{98}	pg(9,19,3)?
		288	192	180	18^{98}	-6^{342}	
?	441	154	49	56	7^{286}	-14^{154}	
		286	187	182	13^{154}	-8^{286}	
?	441	160	61	56	13^{160}	-8^{280}	OA(21, 8)?
		280	175	182	7^{280}	-14^{160}	OA(21, 14)?
−	441	176	25	100	1^{432}	-76^{8}	† $q_{22}^2 < 0$; † Absolute bound
		264	187	114	75^{8}	-2^{432}	† $q_{11}^1 < 0$; † Absolute bound
?	441	176	67	72	8^{264}	-13^{176}	
		264	159	156	12^{176}	-9^{264}	
?	441	176	85	60	29^{48}	-4^{392}	
		264	147	174	3^{392}	-30^{48}	
?	441	180	75	72	12^{180}	-9^{260}	OA(21, 9)?
		260	151	156	8^{260}	-13^{180}	OA(21, 13)?
?	441	184	87	69	23^{72}	-5^{368}	
		256	140	160	4^{368}	-24^{72}	
?	441	190	89	76	19^{98}	-6^{342}	
		250	135	150	5^{342}	-20^{98}	
?	441	198	87	90	9^{242}	-12^{198}	
		242	133	132	11^{198}	-10^{242}	
?	441	200	91	90	11^{200}	-10^{240}	OA(21, 10)?
		240	129	132	9^{240}	-12^{200}	OA(21, 12)?
?	441	220	95	124	3^{396}	-32^{44}	
		220	123	96	31^{44}	-4^{396}	
+	441	220	109	110	10^{220}	-11^{220}	Mathon [544]; OA(21, 11)?; ↓
−	442	105	8	30	3^{390}	-25^{51}	† $q_{22}^2 < 0$
		336	260	240	24^{51}	-4^{390}	† $q_{11}^1 < 0$
?	442	210	99	100	10^{221}	-11^{220}	↑?
		231	120	121	10^{220}	-11^{221}	S(2,11,221)?; ↑?
?	445	222	110	111	10.05^{222}	-11.05^{222}	↓?
?	448	150	50	50	10^{216}	-10^{231}	pg(16,10,5)?
		297	196	198	9^{231}	-11^{216}	
?	448	162	66	54	18^{105}	-6^{342}	
		285	176	190	5^{342}	-19^{105}	pg(16,19,10)?
+	449	224	111	112	10.09^{224}	-11.09^{224}	Paley(449); ↓
?	451	130	33	39	7^{286}	-13^{164}	pg(11,13,3)?
		320	228	224	12^{164}	-8^{286}	
?	451	156	57	52	13^{164}	-8^{286}	
		294	189	196	7^{286}	-14^{164}	S(2,14,287)?
−	453	226	112	113	10.14^{226}	-11.14^{226}	† $v \neq a^2 + b^2$
−	456	35	10	2	11^{95}	-3^{360}	† $\mu = 2$
		420	386	396	2^{360}	-12^{95}	
?	456	65	10	9	8^{208}	-7^{247}	
		390	333	336	6^{247}	-9^{208}	
?	456	80	4	16	4^{360}	-16^{95}	
		375	310	300	15^{95}	-5^{360}	
−	456	91	2	22	3^{399}	-23^{56}	† $q_{22}^2 < 0$
		364	294	276	22^{56}	-4^{399}	† $q_{11}^1 < 0$
?	456	104	22	24	8^{247}	-10^{208}	
		351	270	270	9^{208}	-9^{247}	
?	456	130	24	42	4^{380}	-22^{75}	
		325	236	220	21^{75}	-5^{380}	
?	456	140	40	44	8^{266}	-12^{189}	
		315	218	216	11^{189}	-9^{266}	
?	456	140	58	36	26^{56}	-4^{399}	
		315	210	234	3^{399}	-27^{56}	
?	456	175	78	60	23^{75}	-5^{380}	
		280	164	184	4^{380}	-24^{75}	
?	456	182	73	72	11^{208}	-10^{247}	

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ex	v	k	λ	μ	r^f	s^g	comment
		273	162	165	9^{247}	-12^{208}	
?	456	195	74	90	5^{360}	-21^{95}	
		260	154	140	20^{95}	-6^{360}	
+	457	228	113	114	10.19^{228}	-11.19^{228}	Paley(457); ↓
?	459	208	82	104	4^{390}	-26^{68}	pg(9,26,4)?; ↓?
		250	145	125	25^{68}	-5^{390}	↓?
?	460	85	18	15	10^{184}	-7^{275}	
		374	303	308	6^{275}	-11^{184}	
?	460	99	18	22	7^{275}	-11^{184}	pg(10,11,2)?
		360	282	280	10^{184}	-8^{275}	
?	460	147	42	49	7^{299}	-14^{160}	
		312	213	208	13^{160}	-8^{299}	
−	460	153	32	60	3^{414}	-31^{45}	† Bondarenko et al. [88]
		306	212	186	30^{45}	-4^{414}	
?	460	204	78	100	4^{391}	-26^{68}	↑?
		255	150	130	25^{68}	-5^{391}	↑?
?	460	216	116	88	32^{45}	-4^{414}	
		243	114	144	3^{414}	-33^{45}	
?	460	225	120	100	25^{69}	-5^{390}	↑?
		234	108	130	4^{390}	-26^{69}	pg(10,26,5)?; ↑?
+	461	230	114	115	10.24^{230}	-11.24^{230}	Paley(461); ↓
!	465	58	29	4	27^{30}	-2^{434}	$T(31)$
		406	351	378	1^{434}	-28^{30}	
?	465	144	43	45	9^{248}	-11^{216}	
		320	220	220	10^{216}	-10^{248}	
?	465	192	72	84	6^{340}	-18^{124}	
		272	163	153	17^{124}	-7^{340}	
−	465	232	115	116	10.28^{232}	-11.28^{232}	† $v \neq a^2 + b^2$
−	469	234	116	117	10.33^{234}	-11.33^{234}	† $v \neq a^2 + b^2$
?	470	126	27	36	6^{329}	-15^{140}	
		343	252	245	14^{140}	-7^{329}	
−	473	236	117	118	10.37^{236}	-11.37^{236}	† $v \neq a^2 + b^2$
?	474	165	52	60	7^{315}	-15^{158}	
		308	202	196	14^{158}	-8^{315}	
?	475	90	25	15	15^{114}	-5^{360}	pg(19,5,3) does not exist (no dual)
		384	308	320	4^{360}	-16^{114}	
+	475	96	32	16	20^{75}	-4^{399}	S(2,4,76)
		378	297	315	3^{399}	-21^{75}	pg(19,21,15)?
?	476	133	42	35	14^{152}	-7^{323}	
		342	243	252	6^{323}	-15^{152}	
?	476	133	60	28	35^{34}	-3^{441}	
		342	236	270	2^{441}	-36^{34}	
?	477	140	31	45	5^{371}	-19^{105}	
		336	240	228	18^{105}	-6^{371}	
?	477	168	57	60	9^{264}	-12^{212}	
		308	199	198	11^{212}	-10^{264}	
?	477	238	118	119	10.42^{238}	-11.42^{238}	↓?
?	481	240	119	120	10.47^{240}	-11.47^{240}	↓?
?	483	240	118	120	10^{252}	-12^{230}	pg(21,12,10)?; ↓?
		242	121	121	11^{230}	-11^{252}	S(2,11,231)?; ↓?
!	484	42	20	2	20^{42}	-2^{441}	22×22
		441	400	420	1^{441}	-21^{42}	OA(22, 21)?
+	484	63	22	6	19^{63}	-3^{420}	OA(22, 3)
		420	362	380	2^{420}	-20^{63}	OA(22, 20)?
+	484	84	26	12	18^{84}	-4^{399}	OA(22, 4)
		399	326	342	3^{399}	-19^{84}	OA(22, 19)?
?	484	92	6	20	4^{391}	-18^{92}	
		391	318	306	17^{92}	-5^{391}	
?	484	105	14	25	5^{363}	-16^{120}	
		378	297	288	15^{120}	-6^{363}	
+	484	105	32	20	17^{105}	-5^{378}	OA(22, 5)
		378	292	306	4^{378}	-18^{105}	OA(22, 18)?
?	484	115	18	30	5^{368}	-17^{115}	

continued...

ex	v	k	λ	μ	r^f	s^g	comment
		368	282	272	16^{115}	-6^{368}	
?	484	126	40	30	16^{126}	-6^{357}	OA(22, 6)?
		357	260	272	5^{357}	-17^{126}	OA(22, 17)?
−	484	135	18	45	3^{435}	-30^{48}	† $q_{22}^2 < 0$
		348	257	232	29^{48}	-4^{435}	† $q_{11}^1 < 0$
?	484	138	32	42	6^{345}	-16^{138}	
		345	248	240	15^{138}	-7^{345}	
+	484	138	47	36	17^{120}	-6^{363}	S(2,6,121)
		345	242	255	5^{363}	-18^{120}	
?	484	147	50	42	15^{147}	-7^{336}	OA(22, 7)?
		336	230	240	6^{336}	-16^{147}	OA(22, 16)?
?	484	161	48	56	7^{322}	-15^{161}	
		322	216	210	14^{161}	-8^{322}	
?	484	168	62	56	14^{168}	-8^{315}	OA(22, 8)?
		315	202	210	7^{315}	-15^{168}	OA(22, 15)?
?	484	184	66	72	8^{299}	-14^{184}	
		299	186	182	13^{184}	-9^{299}	
?	484	189	76	72	13^{189}	-9^{294}	OA(22, 9)?
		294	176	182	8^{294}	-14^{189}	OA(22, 14)?
?	484	207	86	90	9^{276}	-13^{207}	
		276	158	156	12^{207}	-10^{276}	
?	484	210	92	90	12^{210}	-10^{273}	OA(22, 10)?
		273	152	156	9^{273}	-13^{210}	OA(22, 13)?
?	484	230	108	110	10^{253}	-12^{230}	RSHCD$^-$?; ↑?
		253	132	132	11^{230}	-11^{253}	↑?
?	484	231	110	110	11^{231}	-11^{252}	OA(22, 11)?; RSHCD$^+$?; ↑?
		252	130	132	10^{252}	-12^{231}	OA(22, 12)?; ↑?
?	485	242	120	121	10.51^{242}	-11.51^{242}	↓?
?	486	97	16	20	7^{291}	-11^{194}	
		388	310	308	10^{194}	-8^{291}	
?	486	100	22	20	10^{210}	-8^{275}	
		385	304	308	7^{275}	-11^{210}	
−	486	165	36	66	3^{440}	-33^{45}	† Makhnev [534]
		320	220	192	32^{45}	-4^{440}	
?	486	194	67	84	5^{388}	-22^{97}	
		291	180	165	21^{97}	-6^{388}	
?	486	210	99	84	21^{100}	-6^{385}	
		275	148	165	5^{385}	-22^{100}	
−	489	244	121	122	10.56^{244}	-11.56^{244}	† $v \neq a^2 + b^2$
?	490	144	28	48	4^{414}	-24^{75}	pg(7,24,2)?
		345	248	230	23^{75}	-5^{414}	
?	490	165	56	55	11^{225}	-10^{264}	
		324	213	216	9^{264}	-12^{225}	
?	490	192	92	64	32^{49}	-4^{440}	
		297	168	198	3^{440}	-33^{49}	pg(10,33,6)?
?	493	246	122	123	10.60^{246}	-11.60^{246}	↓?
?	494	85	12	15	7^{285}	-10^{208}	
		408	337	336	9^{208}	-8^{285}	
?	495	38	1	3	5^{285}	-7^{209}	
		456	420	420	6^{209}	-6^{285}	
+	495	78	29	9	23^{54}	-3^{440}	S(2,3,55)
		416	346	368	2^{440}	-24^{54}	
?	495	104	28	20	14^{143}	-6^{351}	
		390	305	315	5^{351}	-15^{143}	
?	495	190	53	85	3^{450}	-35^{44}	
		304	198	168	34^{44}	-4^{450}	
?	495	190	85	65	25^{76}	-5^{418}	
		304	178	200	4^{418}	-26^{76}	
?	495	208	86	88	10^{260}	-12^{234}	
		286	165	165	11^{234}	-11^{260}	
−	495	208	130	56	76^{10}	-2^{484}	† Absolute bound
		286	133	209	1^{484}	-77^{10}	† Absolute bound
?	495	234	93	126	3^{450}	-36^{44}	

continued...

ex	v	k	λ	μ	r^f	s^g	comment
		260	151	120	35^{44}	-4^{450}	
+	495	238	109	119	7^{340}	-17^{154}	§10.69; $\mathsf{O}^-_{10}(2)$; pg(15,17,7)?; ↓
		256	136	128	16^{154}	-8^{340}	↓
?	496	54	4	6	6^{279}	-8^{216}	
		441	392	392	7^{216}	-7^{279}	
!	496	60	30	4	28^{31}	-2^{464}	$T(32)$
		435	378	406	1^{464}	-29^{31}	pg(16,29,14)?
?	496	110	18	26	6^{341}	-14^{154}	
		385	300	294	13^{154}	-7^{341}	
?	496	135	38	36	11^{216}	-9^{279}	pg(16,9,4)?
		360	260	264	8^{279}	-12^{216}	
–	496	165	80	42	41^{30}	-3^{465}	† Absolute bound
		330	206	246	2^{465}	-42^{30}	† Absolute bound
?	496	198	80	78	12^{216}	-10^{279}	
		297	176	180	9^{279}	-13^{216}	
?	496	231	102	112	7^{341}	-17^{154}	↑?
		264	144	136	16^{154}	-8^{341}	↑?
+	496	240	120	112	16^{155}	-8^{340}	Wallis [718]; ↑
		255	126	136	7^{340}	-17^{155}	$NO^+_{10}(2)$; Goethals-Seidel [355]; pg(16,17,8)?; ↑
?	497	186	55	78	4^{426}	-27^{70}	
		310	201	180	26^{70}	-5^{426}	
?	497	240	127	105	27^{70}	-5^{426}	
		256	120	144	4^{426}	-28^{70}	
–	497	248	123	124	10.65^{248}	-11.65^{248}	† $v \neq a^2+b^2$
?	498	161	64	46	23^{83}	-5^{414}	
		336	220	240	4^{414}	-24^{83}	
–	501	250	124	125	10.69^{250}	-11.69^{250}	† $v \neq a^2+b^2$
?	505	84	3	16	4^{404}	-17^{100}	
		420	351	340	16^{100}	-5^{404}	
+	505	120	39	25	19^{100}	-5^{404}	S(2,5,101)
		384	288	304	4^{404}	-20^{100}	
?	505	180	53	70	5^{404}	-22^{100}	
		324	213	198	21^{100}	-6^{404}	
?	505	224	108	92	22^{100}	-6^{404}	
		280	147	165	5^{404}	-23^{100}	
?	505	252	125	126	10.74^{252}	-11.74^{252}	↓?
?	506	100	18	20	8^{275}	-10^{230}	pg(11,10,2)?
		405	324	324	9^{230}	-9^{275}	
?	507	44	1	4	5^{308}	-8^{198}	
		462	421	420	7^{198}	-6^{308}	
?	507	46	5	4	7^{230}	-6^{276}	
		460	417	420	5^{276}	-8^{230}	
?	507	138	49	33	21^{92}	-5^{414}	
		368	262	280	4^{414}	-22^{92}	
?	507	154	41	49	7^{338}	-15^{168}	
		352	246	240	14^{168}	-8^{338}	
?	507	176	70	56	20^{110}	-6^{396}	
		330	209	225	5^{396}	-21^{110}	
–	507	184	36	84	2^{483}	-50^{23}	† $q^2_{22} < 0$; † Absolute bound
		322	221	175	49^{23}	-3^{483}	† $q^1_{11} < 0$; † Absolute bound
?	507	184	71	64	15^{168}	-8^{338}	S(2,8,169)?
		322	201	210	7^{338}	-16^{168}	
?	507	198	57	90	3^{462}	-36^{44}	
		308	199	168	35^{44}	-4^{462}	
?	507	230	121	90	35^{46}	-4^{460}	
		276	135	168	3^{460}	-36^{46}	
?	507	240	106	120	6^{380}	-20^{126}	pg(13,20,6)?; ↓?
		266	145	133	19^{126}	-7^{380}	↓?
?	508	234	100	114	6^{381}	-20^{126}	↑?
		273	152	140	19^{126}	-7^{381}	↑?
?	508	247	126	114	19^{127}	-7^{380}	↑?
		260	126	140	6^{380}	-20^{127}	↑?
+	509	254	126	127	10.78^{254}	-11.78^{254}	Paley(509); ↓

continued...

ex	v	k	λ	μ	r^f	s^g	comment
?	511	68	15	8	12^{146}	-5^{364}	
		442	381	390	4^{364}	-13^{146}	
?	511	78	5	13	5^{364}	-13^{146}	
		432	366	360	12^{146}	-6^{364}	
+	512	70	6	10	6^{315}	-10^{196}	$\mathsf{GQ}(7,9)$; $[10,3]_8$ (wts $8, 10$)
		441	380	378	9^{196}	-7^{315}	
+	512	73	12	10	9^{219}	-7^{292}	Fiedler-Klin [326]; $[73,9]_2$ (wts $32, 40$)
		438	374	378	6^{292}	-10^{219}	
−	512	126	70	18	54^{16}	-2^{495}	† Absolute bound
		385	276	330	1^{495}	-55^{16}	† Absolute bound
+	512	133	24	38	5^{399}	-19^{112}	Godsil [345]; pg(8,19,2)?
		378	282	270	18^{112}	-6^{399}	
−	512	189	96	54	45^{28}	-3^{483}	† Absolute bound
		322	186	230	2^{483}	-46^{28}	† Absolute bound
+	512	196	60	84	4^{441}	-28^{70}	pg(8,28,3); $[28,3]_8$ (wts $24, 28$)
		315	202	180	27^{70}	-5^{441}	
+	512	219	106	84	27^{73}	-5^{438}	Fiedler-Klin [326]; $[219,9]_2$ (wts $96, 112$)
		292	156	180	4^{438}	-28^{73}	

Table 12.1: Parameters of strongly regular graphs

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Parameter Index

Index of the numerical parameter sets (v, k, λ, μ) for strongly regular graphs mentioned in the text. A dagger (†) denotes that no such graph exists, a question mark (?) that none is known. See also

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