Cooperative games

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Chapter 1

Ten Examples of TU-games

Let $N = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$, be a fixed set of ***players*** called the ***grand coalition***. We call (N, v) a ***cooperative TU-game*** (alternatively, a ***coalitional TU-game***) if v is a function from the powerset $\mathcal{P}(N) := \{A \mid A \subseteq N\}$ of N to $\mathbb{R}$ such that $v(\emptyset) = 0$. A non-empty subset of N is called a ***coalition***.

Intuitively, $v(S)$ is the worth the coalition S can achieve through cooperation. TU stands for ***transferable utility***, which indicates that the worth is expressed in terms of a utility that can be arbitrarily transferred between the members of a coalition. An example of such an transferable utility is arbitrarily divisible money. A non-example are units of some non-divisible good, for instance compact discs.

To clarify the intuition behind such a simple definition we now consider a number of examples. In each example we assume that $v(\emptyset) = 0$ and explain the worth $v(S)$ for each coalition. For each $i \in N$ we abbreviate $v(\{i\})$ to $v(i)$. Also, we write $v(12)$ instead of $v(\{1, 2\})$, etc. We begin with three simple ones.

Example 1 [Majority game]
Assume three players take a decision by a majority vote. We model this by assigning to a 'winning coalition' 1 and to the other ones 0:

$$v(S) := \begin{cases} 1 & \text{if } |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

□

Example 2 **[One seller, two buyers]**
Suppose $N = \{1, 2, 3\}$, where player 1 is a seller of an object, say a house, and players 2 and 3 are buyers. Players 1, 2 and 3 value the house at 1, 2 and 3 (say, hundred thousand of euros), respectively.

If player 1 sells the house to player 2 at a price of x, then $1 \leq x \leq 2$ (otherwise the transaction does not take place), player 1 profit is $x-1$, while player 2 profit is $2-x$. So their joint profit is $(x-1)+(2-x) = 1$. Similarly, in case players 1 and 3 trade, their joint profit is 2. Moreover, no profit can be achieved by a single player nor by players 2 and 3 alone, since both are buyers. Finally, the profit of players 1,2 and 3 together is 2 and is achieved through a trade between players 1 and 3.

This explains the following definition of v:

$$v(S) := \begin{cases} 1 & \text{if } S = \{1, 2\} \\ 2 & \text{if } S = \{1, 3\} \text{ or } S = \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

□

Example 3 **[Glove game]**
Suppose $N = L \dot{\cup} R$, where the members of L own a left glove and the member of R own a right glove. We want to assign to each coalition the number of pairs of (left-right) gloves. So we put

$$v(S) := min\{|L \cap S|, |R \cap S|\}.$$

□

Next, consider some more general examples.

Example 4 **[Weighted majority game]**
Assume that n players adopt a bill by voting and that player's i vote has ***weight*** w_i. Suppose further that the decision is adopted if the sum of the weights (weakly) exceeds the ***threshold*** q. Alternatively, we may think of a parliament of $\sum_{i=1}^{n} w_i$ members in which each party i is represented by w_i members, who always vote 'en bloc', and such that q votes are required to adopt a bill.

This is modelled by the following game:

$$v(S) := \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

that we abbreviate to

$$[q; w_1, \ldots, w_n].$$

This example is a generalization of Example 1. Indeed, the majority game there considered is $[2; 1, 1, 1]$. □

Example 5 [Production economy]
Suppose that player 1 is a capitalist who owns a factory and that players $2, \ldots, n$ are workers. If the capitalist employs k workers, together they can produce an output worth $f(k)$. Workers alone can produce nothing. This is modelled by the following game:

$$v(S) := \begin{cases} f(|S|-1) & \text{if } 1 \in S \\ 0 & \text{otherwise} \end{cases}$$

□

Example 6 [Market game]
Consider a market with k continuous goods (for example, deposits of some metal) such that each player (agent) has an initial ***endowment*** of these goods represented by a vector $\omega_i \in \mathbb{R}^k_+$. The valuation of agent i of each bundle of these goods is represented by a function $f_i : \mathbb{R}^k_+ \to \mathbb{R}$. Agents can increase their profit by trading. This is captured by the following game in which to each coalition a maximal aggregate profit that can be achieved through trading is assigned:

$$v(S) := max\{\sum_{i \in S} f_i(\mathbf{x}_i) \mid \mathbf{x}_i \in \mathbb{R}^k_+,\ \sum_{i \in S} \mathbf{x}_i = \sum_{i \in S} \omega_i\}.$$

Note that for a coalition S, $\sum_{i \in S} \omega_i$ denotes its aggregate endowment. The restriction $\mathbf{x}_i \in \mathbb{R}^k_+$ ensures that only 'non-negative' bundles of goods are admitted, while the equality $\sum_{i \in S} \mathbf{x}_i = \sum_{i \in S} \omega_i$ between the vector sums ensures that we indeed limit ourself to trading.

We assume that for each coalition S the maximum used in the definition of $v(S)$ exists. In Chapter 3 we shall be more specific about the conditions that ensure this property. □

Example 7 [Assignment game]
This example is a generalization of Example 2 and is concerned with an arbitrary ***two-sided market*** that consists of sellers and buyers. Denote the set of sellers by *Sell* and the set of buyers by *Buy* and let $N = Sell \,\dot{\cup}\, Buy$.

Each seller owns an object, for example, a house. Assume that each seller i values his house at $s_i \geq 0$ and that each buyer j values the house of seller i at $b_{ij} \geq 0$. If $b_{ij} \geq s_i$, the transaction between seller i and buyer j can take place. The resulting profit from this transaction is $b_{ij} - s_i$ (see Example 2). If $b_{ij} < s_i$, no transaction can take place and so the resulting profit is 0.

We represent this situation by an $k \times l$ matrix, where $k = |Sell|$ and $l = |Buy|$, with the entries $p_{ij} := max\{0, b_{ij} - s_i\}$. Each assignment of houses from sellers to buyers corresponds to a matching between *Sell* and *Buy* (that is, a 1-1 correspondence between some members of *Sell* and some members of *Buy*). Given a set X of buyers and a set Y of sellers we denote the set of all matchings between X and Y by $M(X, Y)$.

The corresponding game is defined by:

$$v(S) := max\{\textstyle\sum_{(i,j)\in\mu} p_{ij} \mid \mu \in M(S \cap Sell, S \cap Buy)\}.$$

So to each coalition we assign the maximal aggregate profit that can be achieved by means of house sales within this coalition. This definition implies that coalitions that consists solely of sellers or solely of buyers have worth 0.

To be more specific, consider the case of three sellers and three buyers and the following 3×3 matrix of profits:

$$\begin{pmatrix} 5 & \mathbf{8} & 2 \\ 7 & 9 & \mathbf{6} \\ \mathbf{2} & 3 & 0 \end{pmatrix}$$

There are six maximal matchings between sellers and buyers and the the matching for which the resulting profit is maximal is the one with the entries in bold. So in this case worth of the grand coalition is $2 + 8 + 6$, i.e., $v(N) = 16$. □

Example 8 [Böhm-Bawerk horse market]
This example is a special case of the assignment game in which there is no product differentation. So the objects for sale are identical and for historical reasons one talks then about a horse market. So in this case each buyer j has just one valuation, b_j, for each object and, as before, each seller i has the reservation price s_i for the object he owns.

If $b_j \geq s_i$, seller i and buyer j can trade and the resulting profit is $b_j - s_i$. Otherwise no transaction between those two players can take place and the profit is 0. So in this case we model the situation by an $k \times l$ matrix, where

$k = |Sell|$ and $l = |Buy|$, with the entries $p_{ij} := max\{0, b_j - s_i\}$ and the resulting game is defined as above.

To be more specific, consider the case of three sellers, with the respective reservation prices 3, 5, 7 and four buyers with the valuations 8, 6, 4, 2. We get then the following 3×4 matrix of profits:

$$\begin{pmatrix} \mathbf{5} & 3 & 1 & 0 \\ 3 & \mathbf{1} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For example, the upper left corner represents the entry $p_{1,1} = max\{0, 8 - 5\}$ for the first seller (with price 3) and the first buyer (with valuation 8). Ignoring the matchings that yield a zero entry in the matrix there are two matchings with the maximal profit. One of them is indicated by the two entries in bold. So in this case $v(N) = 6$. □

We conclude with two examples that are concerned with cost sharing. Mathematically, there is no difference as the resulting function, now c, just as v, also assigns a real value to each coalition. However, $c(S)$ stands for the *cost* incurred by the coalition S in sharing a joint facility and not the worth S can achieve, so the underlying intuition is different. The resulting game, (N, c), is called a ***cost-sharing game***.

Example 9 [Sharing a water supply system] Suppose that a company considers building a a water supply system that is to be shared between three villages. The costs of the construction depend on for whom the system is to be built and are as follows:

$$\begin{aligned} c(1) &:= 120, \\ c(2) &:= 140, \\ c(3) &:= 120, \\ c(12) &:= 170, \\ c(13) &:= 160, \\ c(23) &:= 190, \\ c(123) &:= 255. \end{aligned}$$

That is, if the water supply system is to be provided only for villages 1 and 2, then the total cost will be 170, etc. □

Example 10 [**Airport game**] Suppose n airlines share a runway. To serve the planes of company i the runway must be of a length resulting in the construction cost c_i. This yields the following cost-sharing game:

$$c(S) := max\{c_i \mid i \in S\}.$$

So $c(S)$ is the cost of building a runway that can serve the planes of the companies that are members of S. □

It is clear that cost-sharing games, even though they take a different perspective on what a coalition can achieve, are closely related to TU-games. In fact, given a cost-sharing game (N, c) we can associate with it in a canonic way a TU-game (N, v) by putting

$$v(S) := \sum_{i \in S} c(i) - c(S).$$

(We abbreviate here $c(\{i\})$ to $c(i)$.) Then $v(S)$ is simply the cost saving for coalition S. Note that by definition for all i we have then $v(i) = 0$. This coincides with the intuition that cost saving for each player acting alone is 0; savings can arise only by forming multi-player coalitions.