Principles of Constraint Programming

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Chapter 6
Some Incomplete Constraint Solvers

Objectives

Introduce incomplete constraint solvers, for

- equality and disequality constraints,
- Boolean constraints,
- linear constraints over integer intervals and over finite integer domains, and
- the arithmetic constraints over integer intervals,
- the arithmetic constraints over reals.

Equality Rules

Consider equality and disequality constraints over arbitrary domains.

EQUALITY 1

$$\frac{\langle x = x \;;\; x \in D\rangle}{\langle \;;\; x \in D\rangle}$$

EQUALITY 2

$$\frac{\langle x = y \;;\; x \in D_x, y \in D_y\rangle}{\langle x = y \;;\; x \in D_x \cap D_y, y \in D_x \cap D_y\rangle}$$

Disequality Rules

DISEQUALITY 1

$$\frac{\langle x \neq x \ ; \ x \in D \rangle}{\langle \ ; \ x \in \emptyset \rangle}$$

DISEQUALITY 2

$$\frac{\langle x \neq y \ ; \ x \in D_x, y \in D_y \rangle}{\langle \ ; \ x \in D_x, y \in D_y \rangle}$$

where $D_x \cap D_y = \emptyset$,

DISEQUALITY 3

$$\frac{\langle x \neq y \ ; \ x \in D, y = a \rangle}{\langle \ ; \ x \in D - \{a\}, y = a \rangle}$$

where $a \in D$, and similarly with $x \neq y$ replaced by $y \neq x$.

Characterization Result

Theorem A CSP with only equality and disequality constraints is hyper-arc consistent iff it is closed under the applications of the *EQUALITY 1-2* and *DISEQUALITY 1-3* rules.

Boolean Constraints

Boolean variables: range over $\{0, 1\}$.

Boolean domain expression: $x \in D$ with $D \subseteq \{0, 1\}$.

Boolean expression: built out of Boolean variables using $\neg$ (*negation*), $\wedge$ (*conjunction*) and $\vee$ (*disjunction*).

Boolean constraint:

$$s = t$$

where s, t are Boolean expressions.

Simple Boolean constraints

- $x = y$,
- $\neg x = y$,
- $x \wedge y = z$,
- $x \vee y = z$.

Rules

Transformation Rules

Reduce Boolean constraints to simple constraints

Example

$$\frac{x \wedge s = z}{x \wedge y = z, s = y}$$

where s is not a variable or is $\equiv$ to x or z.

Rules for Simple Constraints

Example

$$\frac{\langle x \wedge y = z \ ; \ x \in D_x, y \in D_y, z \in \{1\}\rangle}{\langle \ ; \ x \in D_x \cap \{1\}, y \in D_y \cap \{1\}, z \in \{1\}\rangle}$$

Write as

$$x \wedge y = z, z = 1 \rightarrow x = 1, y = 1.$$

Domain Reduction Rules: *BOOL*

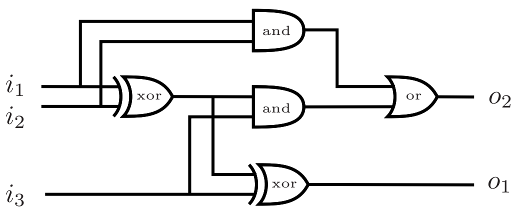
EQU	*1*	$x = y, x = 1 \longrightarrow y = 1$
EQU	*2*	$x = y, y = 1 \longrightarrow x = 1$
EQU	*3*	$x = y, x = 0 \longrightarrow y = 0$
EQU	*4*	$x = y, y = 0 \longrightarrow x = 0$
NOT	*1*	$\neg x = y, x = 1 \longrightarrow y = 0$
NOT	*2*	$\neg x = y, x = 0 \longrightarrow y = 1$
NOT	*3*	$\neg x = y, y = 1 \longrightarrow x = 0$
NOT	*4*	$\neg x = y, y = 0 \longrightarrow x = 1$
AND	*1*	$x \wedge y = z, x = 1, y = 1 \longrightarrow z = 1$
AND	*2*	$x \wedge y = z, x = 1, z = 0 \longrightarrow y = 0$
AND	*3*	$x \wedge y = z, y = 1, z = 0 \longrightarrow x = 0$
AND	*4*	$x \wedge y = z, x = 0 \longrightarrow z = 0$
AND	*5*	$x \wedge y = z, y = 0 \longrightarrow z = 0$
AND	*6*	$x \wedge y = z, z = 1 \longrightarrow x = 1, y = 1$
OR	*1*	$x \vee y = z, x = 1 \longrightarrow z = 1$
OR	*2*	$x \vee y = z, x = 0, y = 0 \longrightarrow z = 0$
OR	*3*	$x \vee y = z, x = 0, z = 1 \longrightarrow y = 1$
OR	*4*	$x \vee y = z, y = 0, z = 1 \longrightarrow x = 1$
OR	*5*	$x \vee y = z, y = 1 \longrightarrow z = 1$
OR	*6*	$x \vee y = z, z = 0 \longrightarrow x = 0, y = 0$

Characterization Result

Theorem A non-failed Boolean CSP is hyper-arc consistent iff it is closed under the applications of the rules of $BOOL$.

Constraint Propagation using *BOOL*: Example

- **Full Adder Circuit**



It computes the binary sum $i_1 + i_2 + i_3$ in the binary word $o_2 o_1$.

- **Example**: $1 + 1 + 0$ yields 10.
- **Problem**: Deduce that

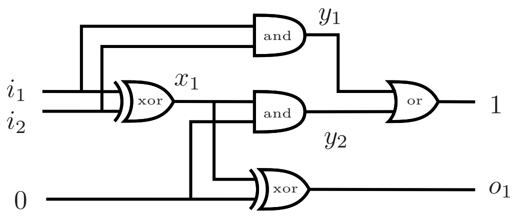
  $i_1 = 1, i_2 = 1$ and $o_1 = 0$

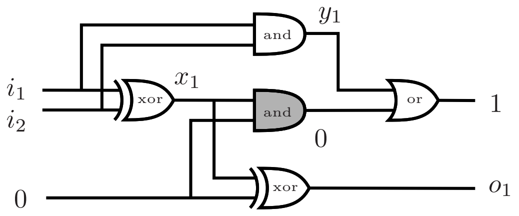
  follows from

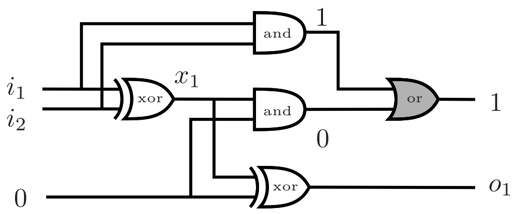
  $i_3 = 0$ and $o_2 = 1$.
- **Two proof rules for** *XOR*

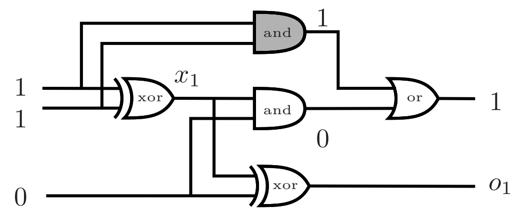
XOR 1	$x \oplus y = z, x = 1, y = 1 \rightarrow z = 0$
XOR 2	$x \oplus y = z, x = 0, y = 0 \rightarrow z = 0$

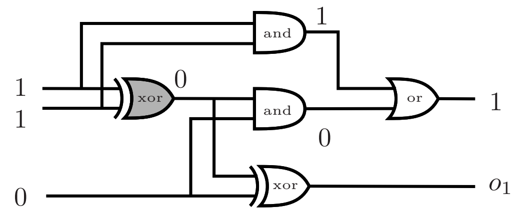
Constraint Propagation in Full Adder Circuit

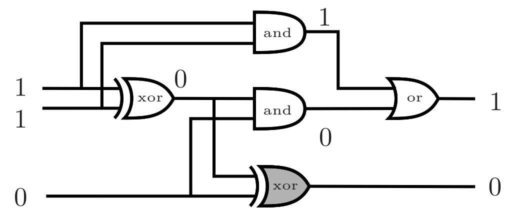












Linear Constraints on Integer Intervals

Consider the language with

- two constants 0 and 1,
- the unary minus function "$-$",
- two binary functions "$+$" and "$-$".

Linear expression:a term in this language.
Linear constraint: a formula

$$s \ op \ t,$$

where s and t are linear expressions and $op \in \{<, \leq, =, \neq, \geq, >\}$.

Abbreviations:

Terms of the form

$$\underbrace{1 + \ldots + 1}_{n \text{ times}}$$

to n, terms of the form

$$\underbrace{x + \ldots + x}_{n \text{ times}}$$

to nx and analogously for -1 and $-x$.

Simple Disequality Rules

SIMPLE DISEQUALITY 1

$$\frac{\langle x \neq y \; ; \; x \in [a..b], y \in [c..d] \rangle}{\langle \; ; \; x \in [a..b], y \in [c..d] \rangle}$$

where $b < c$ or $d < a$

SIMPLE DISEQUALITY 2

$$\frac{\langle x \neq y \; ; \; x \in [a..b], y = a \rangle}{\langle \; ; \; x \in [a+1..b], y = a \rangle}$$

SIMPLE DISEQUALITY 3

$$\frac{\langle x \neq y \; ; \; x \in [a..b], y = b \rangle}{\langle \; ; \; x \in [a..b-1], y = b \rangle}$$

Domain Reduction for Inequality Constraints

Intuitive example

Consider

$$3x + 4y - 5z \leq 7$$

with $x \in [l_x..h_x], y \in [l_y..h_y], z \in [l_z..h_z]$.

Rewrite as

$$x \leq \frac{7 - 4y + 5z}{3}$$

Any value of x that satisfies it also satisfies

$$x \leq \frac{7 - 4l_y + 5h_z}{3}$$

We seek integer solutions, so

$$x \leq \lfloor \frac{7 - 4l_y + 5h_z}{3} \rfloor$$

So we can reduce $[l_x..h_x]$ to

$$[l_x..min(\lfloor \frac{7-4l_y+5h_z}{3} \rfloor, h_x)].$$

Domain Reduction for *LINEAR EQUALITY* rule

$$\frac{\langle \Sigma_{i \in POS}\, a_i x_i - \Sigma_{i \in NEG}\, a_i x_i = b \ ; \ x_1 \in [l_1..h_1], \ldots, x_n \in [l_n..h_n] \rangle}{\langle \Sigma_{i \in POS}\, a_i x_i - \Sigma_{i \in NEG}\, a_i x_i = b \ ; \ x_1 \in [l'_1..h'_1], \ldots, x_n \in [l'_n..h'_n] \rangle}$$

where for $j \in POS$

$$l'_j := max(l_j, \lceil \gamma_j \rceil), \ h'_j := min(h_j, \lfloor \alpha_j \rfloor),$$

for $j \in NEG$

$$l'_j := max(l_j, \lceil \beta_j \rceil), \ h'_j := min(h_j, \lfloor \delta_j \rfloor),$$

and

$$\alpha_j := \frac{b - \Sigma_{i \in POS - \{j\}}\, a_i l_i + \Sigma_{i \in NEG}\, a_i h_i}{a_j}$$

$$\beta_j := \frac{-b + \Sigma_{i \in POS}\, a_i l_i - \Sigma_{i \in NEG - \{j\}}\, a_i h_i}{a_j}$$

$$\gamma_j := \frac{b - \Sigma_{i \in POS - \{j\}}\, a_i h_i + \Sigma_{i \in NEG}\, a_i l_i}{a_j}$$

$$\delta_j := \frac{-b + \Sigma_{i \in POS}\, a_i h_i - \Sigma_{i \in NEG - \{j\}}\, a_i l_i}{a_j}$$

Example: SEND + MORE = MONEY

```
   SEND
 + MORE
 ------
  MONEY
```

Originally:
$[S,E,N,D,M,O,R,Y] \in [0..9]$

After constraint propagation:
$S = 9$, $E \in [4..7]$, $N \in [5..8]$, $D \in [2..8]$,
$M = 1$, $O = 0$, $R \in [2..8]$, $Y \in [2..8]$.

1. Use the transformation rules to transform "SEND + MORE = MONEY" constraint to

$9000 \cdot M + 900 \cdot O + 90 \cdot N + Y$ -
$(91 \cdot E + D + 1000 \cdot S + 10 \cdot R) = 0.$

2. Apply *LINEAR EQUALITY* reduction rule.
$S = 9$, $E \in [0..9]$, $N \in [0..9]$, $D \in [0..9]$,
$M = 1$, $O \in [0..1]$, $R \in [0..9]$, $Y \in [0..9]$.
Repeated use yields no new outcome.

3. Apply *SIMPLE DISEQUALITY* rule to $M \neq O$ to conclude $O = 0$.

4. Repeatedly use $M = 1, O = 0, S = 9$ and *SIMPLE DISEQUALITY* rules. This eventually yields
S = 9, E $\in$ [2..8], N $\in$ [2..8], D $\in$ [2..8],
M = 1, O = 0, R $\in$ [2..8], Y $\in$ [2..8].

5. 5 iterations of *LINEAR EQUALITY* rule yield
E $\in$ [2..7], N $\in$ [3..8],
E $\in$ [3..7], N $\in$ [3..8],
E $\in$ [3..7], N $\in$ [4..8],
E $\in$ [4..7], N $\in$ [4..8],
E $\in$ [4..7], N $\in$ [5..8].
The other ranges remain unchanged.

The stabilising derivation after step 1. consists of 24 steps.

Arithmetic Constraints on Integer Intervals

Consider the language with

- two constants 0 and 1,
- the unary minus function "$-$",
- three binary functions "$+$", "$-$", and "$\cdot$" (**new**).

Arithmetic constraint: a formula

$$s \; op \; t,$$

where s and t are terms and $op \in \{<, \leq, =, \neq, \geq, >\}$.

Example

$$x^5 \cdot y^2 \cdot z^4 + 3x \cdot y^3 \cdot z^5 \leq$$
$$10 + 4x^4 \cdot y^6 \cdot z^2 - y^2 \cdot x^5 \cdot z^4$$

is an arithmetic constraint.

Approach Based on Atomic Arithmetic Constraints

Atomic Arithmetic Constraint:

- a linear constraint,
- $x \cdot y = z$.

Note Every arithmetic constraint can be reduced to a sequence of atomic constraints.

Example transformation rule

$$\frac{\Sigma_{i=1}^{n} m_i \; op \; b}{\Sigma_{i=1}^{n} v_i \; op \; b, \; m_1 = v_1, \ldots, \; m_n = v_n}$$

where $v_1, \ldots, v_n$ are auxiliary variables.

Interval Multiplication

X, Y sets of integers.

- multiplication:

$$X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\},$$

Note X, Y integer intervals.
$X \cdot Y$ does not have to be an interval.

Example: $[0..2] \cdot [1..2] = \{0, 1, 2, 4\}$.

A: a set of integers

$$int(A) := \begin{cases} \text{smallest int. interval} \supseteq A & \text{if it exists} \\ \mathcal{Z} & \text{otherwise.} \end{cases}$$

Multiplication Rule 1

MULTIPLICATION 1

$$\frac{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y, z \in D_z \cap int(D_x \cdot D_y) \rangle}$$

Example Consider

$$\langle x \cdot y = z \; ; \; x \in [0..2], y \in [1..2], z \in [4..6] \rangle.$$

$int([0..2] \cdot [1..2]) = [0..4]$
and $[4..6] \cap [0..4] = [4..4]$, so we get

$$\langle x \cdot y = z \; ; \; x \in [0..2], y \in [1..2], z \in [4..4] \rangle.$$

Multiplication Rules 2,3

- Interval division:

$$Z/Y = \{x \in \mathcal{Z} \mid \exists y \in Y \exists z \in Z \; x \cdot y = z\}.$$

Note Z, Y integer intervals.
Z/Y does not have to be an interval.

Example:
$[3..5]/[-1..2] = \{-5, -4, -3, 2, 3, 4, 5\}.$

MULTIPLICATION 2

$$\frac{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y, z \in D_z\rangle}{\langle x \cdot y = z \; ; \; x \in D_x \cap int(D_z/D_y), y \in D_y, z \in D_z\rangle}$$

MULTIPLICATION 3

$$\frac{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y, z \in D_z\rangle}{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y \cap int(D_z/D_x), z \in D_z\rangle}$$

Example

Consider

$$\langle x \cdot y = z \; ; \; x \in [1..20], y \in [9..11], z \in [155..161] \rangle.$$

Applying ***MULTIPLICATION 2*** rule yields

$$\langle x \cdot y = z \; ; \; x \in [16..16], y \in [9..11], z \in [155..161] \rangle$$

since $[155..161]/[9..11]) = [16..16]$
and $[1..20] \cap int([16..16]) = [16..16]$.

Applying ***MULTIPLICATION 3*** rule yields

$$\langle x \cdot y = z \; ; \; x \in [16..16], y \in [10..10], z \in [155..161] \rangle$$

since $[155..161]/[16..16] = [10..10]$
and $[9..11] \cap int([10..10]) = [10..10]$.

Applying ***MULTIPLICATION 1*** rule yields

$$\langle x \cdot y = z \; ; \; x \in [16..16], y \in [10..10], z \in [160..160] \rangle$$

since $[16..16] \cdot [10..10] = [160..160]$
and $[155..161] \cap int([160..160]) = [160..160]$.

Arithmetic Constraints on Reals

Consider the language with

- each real number as a constant, (**new**)
- the unary minus function "$-$",
- three binary functions "$+$", "$-$", and "$\cdot$".

Arithmetic constraint: a formula

$$s \; op \; t,$$

where s and t are terms and $op \in \{<, \leq, =, \neq, \geq, >\}$.

Example

$$2.4 \cdot x^5 \cdot y^2 \cdot z^4 + 3.6 \cdot x \cdot y^3 \cdot z^5$$
$$\leq 10.1 + 4.2 \cdot x^4 \cdot y^6 \cdot z^2$$

is an arithmetic constraint.

Domains: Extended Intervals

$$\mathcal{R}^+ := \mathcal{R} \cup \{-\infty, \infty\}.$$

Extend $<$ from $\mathcal{R}$ to $\mathcal{R}^+$ as expected.

Extended interval: expression

$$\langle a, b\rangle$$

where $a, b \in \mathcal{R}^+$.

Meaning:

$$\langle a, b\rangle = \{r \in \mathcal{R} \mid a \leq r \leq b\}.$$

Note For $a, b \in \mathcal{R}$

$$\langle a, a\rangle = \{a\},$$
$$\langle a, b\rangle = \{r \in \mathcal{R} \mid a \leq r \leq b\},$$
$$\langle -\infty, b\rangle = \{r \in \mathcal{R} \mid r \leq b\},$$
$$\langle a, \infty\rangle = \{r \in \mathcal{R} \mid a \leq r\},$$
$$\langle -\infty, \infty\rangle = \mathcal{R}.$$

Interval Arithmetic

X, Y sets of reals.

- addition:

$$X + Y := \{x + y \mid x \in X, y \in Y\},$$

- subtraction:

$$X - Y := \{x - y \mid x \in X, y \in Y\},$$

- multiplication:

$$X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\},$$

- division:

$$X/Y := \{u \in \mathcal{R} \mid \exists x \in X \exists y \in Y \ u \cdot y = x\}.$$

For real r and $op \in \{+, -, \cdot, /\}$

$$r \; op \; X := \{r\} \; op \; X$$

$$X \; op \; r := X \; op \; \{r\}.$$

Interval Arithmetics, ctd

Note X, Y extended intervals, r a real.

- $X \cap Y$, $X + Y$, $X - Y$ and $X \cdot Y$ are extended intervals.
- $X/\{r\}$ is an extended interval.
- X/Y does not have to be an extended interval.

Example:

$\langle 2, 16\rangle / \langle -\infty, -2\rangle = \{r \in \mathcal{R} \mid -8 \leq r < 0\}$.

A: a set of reals.

$int(A) :=$ smallest extended interval containing A.

Atomic Arithmetic Constraints

- $$\sum_{i=1}^{n} a_i x_i = b,$$
  - $n > 0$,
  - $a_1, \ldots, a_n$ non-zero reals,
  - $x_1, \ldots, x_n$ different variables,
  - b is a real,
- $x \neq y$,
- $x \cdot y = z$.

Note Every arithmetic constraint can be reduced to a sequence of atomic constraints.

Domain Reduction Rules

Intuition

$$\sum_{i=1}^{n} a_i x_i = b$$

implies that for $j \in [1..n]$

$$x_j = \frac{b - \Sigma_{i \in [1..n]-\{j\}} a_i x_i}{a_j}.$$

$$\mathcal{R}\text{-}\mathit{LINEAR\ EQUALITY}$$

$$\frac{\langle \Sigma_{i=1}^{n} a_i x_i = b \ ; \ x_1 \in D_1, \ldots, x_n \in D_n \rangle}{\langle \Sigma_{i=1}^{n} a_i x_i = b \ ; \ \ldots, x_j \in D'_j, \ldots \rangle}$$

where $j \in [1..n]$ and

$$D'_j := D_j \cap \frac{b - \Sigma_{i \in [1..n]-\{j\}} a_i \cdot D_i}{a_j}$$

$$\mathit{DISEQUALITY\ 2}$$

$$\frac{\langle x \neq y \ ; \ x \in D_x, y \in D_y \rangle}{\langle \ ; \ x \in D_x, y \in D_y \rangle}$$

where $D_x \cap D_y = \emptyset$.

Multiplication Rules

$\mathcal{R}$-*MULTIPLICATION 1*

$$\frac{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y, z \in D_z \cap D_x \cdot D_y \rangle}$$

$\mathcal{R}$-*MULTIPLICATION 2*

$$\frac{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \; ; \; x \in D_x \cap int(D_z/D_y), y \in D_y, z \in D_z \rangle}$$

$\mathcal{R}$-*MULTIPLICATION 3*

$$\frac{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \; ; \; x \in D_x, y \in D_y \cap int(D_z/D_x), z \in D_z \rangle}$$

Example

Consider

$\langle x \cdot y = z \; ; \; x \in \langle -\infty, -1\rangle, y \in \langle -\infty, -2\rangle, z \in \langle -\infty, 161\rangle\rangle.$

By ***$\mathcal{R}$-MULTIPLICATION 1*** rule

$\langle x \cdot y = z \; ; \; x \in \langle -\infty, -1\rangle, y \in \langle -\infty, -2\rangle, z \in \boldsymbol{\langle 2, 161\rangle}\rangle$

since

$$\langle -\infty, -1\rangle \cdot \langle -\infty, -2\rangle = \langle 2, \infty\rangle$$

and hence

$$\langle -\infty, 161\rangle \cap \langle 2, \infty\rangle = \langle 2, 161\rangle.$$

By ***$\mathcal{R}$-MULTIPLICATION 3*** rule

$\langle x \cdot y = z \; ; \; x \in \langle -\infty, -1\rangle, y \in \boldsymbol{\langle -161, -2\rangle}, z \in \langle 2, 161\rangle\rangle$

since

$$\langle 2, 161\rangle / \langle -\infty, -1\rangle = \{r \in \mathcal{R} \mid -161 \leq r < 0\}$$

and hence

$$int(\langle 2, 161\rangle / \langle -\infty, -1\rangle) = \langle -161, 0\rangle$$

and

$$\langle -\infty, -2\rangle \cap \langle -161, 0\rangle = \langle -161, -2\rangle.$$

Example, ctd

$\langle x \cdot y = z \; ; \; x \in \langle -\infty, -1\rangle, y \in \langle -161, -2\rangle, z \in \langle 2, 161\rangle\rangle.$

By ***$\mathcal{R}$-MULTIPLICATION 2*** rule

$\langle x \cdot y = z \; ; \; x \in \boldsymbol{\langle -80.5, -1\rangle}, y \in \langle -161, -2\rangle, z \in \langle 2, 161\rangle\rangle$

since

$$\langle 2, 161\rangle / \langle -161, -2\rangle = \langle -80.5, -2/161\rangle$$

and hence

$$\langle -\infty, -1\rangle \cap int(\langle -80.5, -2/161\rangle) = \langle -80.5, -1\rangle.$$

Last CSP is closed under the applications of the ***MULTIPLICATION*** rules.

Arithmetic Constraints on Reals: Implementation Issues

Step 1: Extend arithmetic operations from $\mathcal{R}$ to $\mathcal{R}^+$.

$\perp$: undefined operation,
PR: a positive real,
NR a negative real,
$\mathcal{R}$: outcome can be an arbitrary real.

		x				
	$x+y$	$-\infty$	NR	0	PR	∞
	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$\perp$
	NR		NR	NR	$\mathcal{R}$	∞
y	0			0	PR	∞
	PR				PR	∞
	∞					∞

		x				
	$x-y$	$-\infty$	NR	0	PR	∞
	$-\infty$	$\perp$	∞	∞	∞	∞
	NR	$-\infty$	$\mathcal{R}$	PR	PR	∞
y	0	$-\infty$	NR	0	PR	∞
	PR	$-\infty$	NR	NR	$\mathcal{R}$	∞
	∞	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$\perp$

		x				
	$x \cdot y$	$-\infty$	NR	0	PR	∞
	$-\infty$	∞	∞	$\perp$	$-\infty$	$-\infty$
	NR		PR	0	NR	$-\infty$
y	0			0	0	$\perp$
	PR				PR	∞
	∞					∞

		x				
	x/y	$-\infty$	NR	0	PR	∞
	$-\infty$	$\perp$	0	0	0	$\perp$
	NR	∞	PR	0	NR	$-\infty$
y	0	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
	PR	$-\infty$	NR	0	PR	∞
	∞	$\perp$	0	0	0	$\perp$

Examples

$\infty + (-\infty) = \perp$,
$\infty / PR = \infty$,
$PR - NR = \mathcal{R}$.

Implementation Issues, ctd

Step 2: Implement intersection, addition and subtraction of extended intervals.

Note $\langle a, b\rangle$ and $\langle c, d\rangle$: non-empty extended intervals. Then

- $\langle a, b\rangle \cap \langle c, d\rangle = \langle max(a, c), min(b, d)\rangle$.
- $\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$.
- $\langle a, b\rangle - \langle c, d\rangle = \langle a - d, b - c\rangle$.

Classification of Non-Empty Extended Intervals

Depends on the position of 0 w.r.t. such an interval.

class of $\langle a, b\rangle$	at least one negative	at least one positive	signs of endpoints
M	yes	yes	$a < 0 \wedge b > 0$
Z	no	no	$a = 0 \wedge b = 0$
P	no	yes	$a \geq 0 \wedge b > 0$
P_0	no	yes	$a = 0 \wedge b > 0$
P_1	no	yes	$a > 0 \wedge b > 0$
N	yes	no	$a < 0 \wedge b \leq 0$
N_0	yes	no	$a < 0 \wedge b = 0$
N_1	yes	no	$a < 0 \wedge b < 0$

Implementation of Multiplication

Step 3: Implement multiplication of extended intervals.

$\langle a, b\rangle$ and $\langle c, d\rangle$: non-empty extended intervals.

class of $\langle a, b\rangle$	class of $\langle c, d\rangle$	$\langle a, b\rangle \cdot \langle c, d\rangle$
P	P	$\langle a \cdot c, b \cdot d\rangle$
P	M	$\langle b \cdot c, b \cdot d\rangle$
P	N	$\langle b \cdot c, a \cdot d\rangle$
M	P	$\langle a \cdot d, b \cdot d\rangle$
M	M	$\langle min(a \cdot d, b \cdot c), max(a \cdot c, b \cdot d)\rangle$
M	N	$\langle b \cdot c, a \cdot c\rangle$
N	P	$\langle a \cdot d, b \cdot c\rangle$
N	M	$\langle a \cdot d, a \cdot c\rangle$
N	N	$\langle b \cdot d, a \cdot c\rangle$
Z	P, M, N, Z	$\langle 0, 0\rangle$
P, M, N	Z	$\langle 0, 0\rangle$

Example

Consider

$$\langle -3, 2\rangle \cdot \langle -4, 5\rangle.$$

Both intervals are of class M, so the entry

class of $\langle a, b\rangle$	class of $\langle c, d\rangle$	$\langle a, b\rangle \cdot \langle c, d\rangle$
M	M	$\langle min(a \cdot d, b \cdot c), max(a \cdot c, b \cdot d)\rangle$

applies. Thus

$$\begin{aligned}
&\langle -3, 2\rangle \cdot \langle -4, 5\rangle = \\
&\langle min((-3) \cdot 5, 2 \cdot (-4)), max((-3) \cdot (-4), 2 \cdot 5)\rangle = \\
&\langle min(-15, -8), max(12, 10)\rangle = \\
&\langle -15, 12\rangle.
\end{aligned}$$

Implementation of Division

Step 4: Implement division of extended intervals.

$\langle a, b \rangle$ and $\langle c, d \rangle$: non-empty extended intervals.

class of $\langle a,b\rangle$	class of $\langle c,d\rangle$	$\langle a,b\rangle/\langle c,d\rangle$
P_1	P_1	$\langle a/d, b/c\rangle \setminus \{0\}$
P_1	P_0	$\langle a/d, \infty\rangle \setminus \{0\}$
P_0	P_1	$\langle 0, b/c\rangle$
M	P_1	$\langle a/c, b/c\rangle$
N_0	P_1	$\langle a/c, 0\rangle$
N_1	P_1	$\langle a/c, b/d\rangle \setminus \{0\}$
N_1	P_0	$\langle -\infty, b/d\rangle \setminus \{0\}$
P_1	M	$(\langle -\infty, a/c\rangle \cup \langle a/d, \infty\rangle) \setminus \{0\}$
M, Z, P_0, N_0	M, Z, P_0, N_0	$\langle -\infty, +\infty\rangle$
N_1	M	$(\langle -\infty, b/d\rangle \cup \langle b/c, \infty\rangle) \setminus \{0\}$
P_1	N_1	$\langle b/d, a/c\rangle \setminus \{0\}$
P_1	N_0	$\langle -\infty, a/c\rangle \setminus \{0\}$
P_0	N_1	$\langle b/d, 0\rangle$
M	N_1	$\langle b/d, a/d\rangle$
N_0	N_1	$\langle 0, a/d\rangle$
N_1	N_1	$\langle b/c, a/d\rangle \setminus \{0\}$
N_1	N_0	$\langle b/c, \infty\rangle \setminus \{0\}$
Z	P_1, N_1	$\langle 0, 0\rangle$
P_1, N_1	Z	$\emptyset$

Example

Consider

$$\langle 2, 16\rangle / \langle -\infty, -2\rangle.$$

The intervals are of class P_1 and N_1, so the entry

class of $\langle a, b\rangle$	class of $\langle c, d\rangle$	$\langle a, b\rangle / \langle c, d\rangle$
P_1	N_1	$\langle b/d, a/c\rangle \setminus \{0\}$

applies. Thus

$\langle 2, 16\rangle / \langle -\infty, -2\rangle =$
$\langle 16/(-2), 2/(-\infty)\rangle \setminus \{0\} =$
$\{r \in \mathcal{R} \mid -8 \leq r < 0\}.$

Using Floating-point Numbers

Step 5: Introduce Floating-Point Numbers

Motivation We want to represent solutions to $9 \cdot x^2 = 1$ over $\langle -1, 1\rangle$ as

$x \in \langle -0.33334, -0.33333\rangle$

and

$x \in \langle 0.33333, 0.33334\rangle$.

Assume finite subset $\mathcal{F}$ of $\mathcal{R}^+$ containing $-\infty$ and ∞.

Elements of $\mathcal{F}$: **floating-point numbers**.

Floating-point interval:

$$\langle a, b\rangle,$$

a, b floating-point numbers.

$\Gamma(A)$:

the least floating-point interval containing A.

$\Gamma(A)$ always exists.

Amended Multiplication Rules

$\mathcal{F}$-*MULTIPLICATION 1*

$$\frac{\langle x \cdot y = z \ ; \ x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \ ; \ x \in D_x, y \in D_y, z \in D_z \cap \Gamma(D_x \cdot D_y) \rangle}$$

$\mathcal{F}$-*MULTIPLICATION 2*

$$\frac{\langle x \cdot y = z \ ; \ x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \ ; \ x \in D_x \cap \Gamma(D_z / D_y), y \in D_y, z \in D_z \rangle}$$

$\mathcal{F}$-*MULTIPLICATION 3*

$$\frac{\langle x \cdot y = z \ ; \ x \in D_x, y \in D_y, z \in D_z \rangle}{\langle x \cdot y = z \ ; \ x \in D_x, y \in D_y \cap \Gamma(D_z / D_x), z \in D_z \rangle}$$

- Combined with the implementation $\Gamma(X \cdot Y)$ and $\Gamma(X/Y)$ for floating-point intervals X, Y.
- Similar modification of other domain reduction rules.

Objectives

Introduce incomplete constraint solvers, for

- equality and disequality constraints,
- Boolean constraints,
- linear constraints over integer intervals and over finite integer domains, and
- the arithmetic constraints over integer intervals,
- the arithmetic constraints over reals.