Coordination Games on Graphs

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Based on joint work with Mona Rahn, Guido Schäfer and Sunil Simon
Coordination Games on Graphs: Definition

- Assume a finite graph.
- Each node has a set of colours available to it.
- Suppose that each node selects a colour from its set of colours.
- The payoff to a node is the number of neighbours who chose the same colour.
Example

A graph with a colour assignment.
Example, ctd

Consider the red joint strategy.

- The payoffs to the nodes on the square: 2, 1, 2, 1.
- The payoffs to each source node: 1.
Motivation

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- Coordination games on graphs are specific coordination games in the absence of common strategies.
- They also capture the idea of influence. Each node influences its neighbours to follow its choice.
- The purpose of cluster analysis is to partition in a meaningful way the nodes of a graph.
- Suppose the colours as the names of the clusters. Then a Nash equilibrium corresponds to a ‘satisfactory’ clustering.
Strategic Games: Definition

<table>
<thead>
<tr>
<th>Strategic game for ( n \geq 2 ) players</th>
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<tbody>
<tr>
<td>- a non-empty set ( S_i ) of strategies,</td>
</tr>
<tr>
<td>- payoff function ( p_i : S_1 \times \cdots \times S_n \to \mathbb{R} ),</td>
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<td>for each player ( i ).</td>
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- **Notation:** \((S_1, \ldots, S_n, p_1, \ldots, p_n)\).
- **Basic assumption:** the players choose their strategies simultaneously.
Related Classes of Games

- **Graphical Games** *(Kearns, Littman, Singh ’01)*
  - Given is a graph on the set of players.
  - Payoff for player $i$ is a function
    \[
    p_i : \times_{j \in \text{neigh}(i) \cup \{i\}} S_j \to \mathbb{R}.
    \]
  - **Intuition.**
    The payoff of each player depends only on his strategy and the strategies of its neighbours.

- **Polymatrix Games** *(Janovskaya ’68)*
  - $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ is called polymatrix if for all pairs of players $i$ and $j$ there exists a partial payoff function $p_{ij}$ such that
    \[
    p_i(s_i) := \sum_{j \neq i} p_{ij}(s_i, s_j).
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    Each pair of players plays a separate game. The payoffs in the main game aggregate the payoffs in these separate games.

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Some Properties of Games

Reminder

- $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.
- We sometimes write $(s_i, s_{-i})$ for $s$.

- **Positive Population Monotonicity (PPM)** (Konishi, Le Breton ’97)
  - $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ satisfies the positive population monotonicity (PPM) if for all $s$ and players $i, j$
    \[
    p_i(s) \leq p_i(s_i, s_{-j}).
    \]
  - Intuition.
    If more players (here player $j$) choose player’s $i$ strategy, then player’s $i$ payoff weakly increases.

- **Join the crowd property** (Simon, Apt ’13)
  - A game satisfies the join the crowd property if the payoff of each player weakly increases when more players choose his strategy.
  - Note.
    Every join the crowd game satisfies PPM.
Reminder: Nash Equilibrium

**Best response**

A strategy $s_i$ of player $i$ is a best response to a joint strategy $s_{−i}$ if for all $s'_i$, $p_i(s'_i, s_{−i}) \leq p_i(s_i, s_{−i})$.

**Nash equilibrium**

A joint strategy $s$ is a Nash equilibrium if for all players $i$, $s_i$ is the best response to $s_{−i}$.
Exact Potentials

- Assume $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.
- A **profitable deviation**: a pair $(s, s')$ of joint strategies such that $p_i(s') > p_i(s)$, where $s' = (s'_i, s_{-i})$.
- An **exact potential** for $G$: a function

\[ P : S_1 \times \cdots \times S_n \rightarrow \mathbb{R} \]

such that for every profitable deviation $(s, s')$, where $s' = (s'_i, s_{-i})$,

\[ P(s') - P(s) = p_i(s') - p_i(s). \]

**Note**

Every finite game with an exact potential has a Nash equilibrium.
Price of Anarchy and of Stability

- Social welfare: $SW(s) = \sum_{j=1}^{n} p_j(s)$.
- Price of anarchy
  \[
  \frac{\max_{s \in S} SW(s)}{\min_{s \in S, s \text{ is a NE}} SW(s)}
  \]
- Price of stability
  \[
  \frac{\max_{s \in S} SW(s)}{\max_{s \in S, s \text{ is a NE}} SW(s)}
  \]
Price of Anarchy and of Stability

**Theorem**

(i) Every coordination game on a graph has an exact potential.

(ii) The price of stability is 1.

(iii) For every graph there is a colour assignment such that the price of anarchy of the corresponding coordination game is $\infty$.

**Proof.**

(i) $F(s) := \frac{1}{2} SW(s)$ is an exact potential.

(ii) Assign to each node in a graph $(V, E)$ two colours: one private and one common.

The maximal social welfare is $2|E|$.

A bad Nash equilibrium: each node chooses a private node. The resulting social welfare is then 0.
Strong Equilibrium

- A **coalition**: a non-empty set of players.
- Given a joint strategy $s$ and $K = \{k_1, \ldots, k_m\} \subseteq \{1, \ldots, n\}$ we abbreviate $(s_{k_1}, \ldots, s_{k_m})$ to $s_K$.
- $p_K(s') > p_K(s)$: $p_i(s') > p_i(s)$ for all $i \in K$.
- Coalition $K$ can profitably deviate from $s$ if for some $s'$ such that
  - $s'_i \neq s_i$ for $i \in K$ and
  - $s'_i = s_i$ for $i \notin K$,
  $$p_K(s') > p_K(s).$$
- Notation: $s^K \rightarrow s'$.
- $s$ is a **strong equilibrium** if no coalition of players can profitably deviate from $s$.
- $G$ has the **c-FIP** if every sequence of profitable deviations by coalitions is finite.
Generalized Ordinal c-Potentials

- A generalized ordinal c-potential for $G$: a function

$$P : S_1 \times \cdots \times S_n \rightarrow A$$

such that for some strict partial ordering $(P(S_1 \times \cdots \times S_n), \succ)$

if $s^K \rightarrow s'$ for some $K$, then $P(s') \succ P(s)$.

Note

If a finite game has a generalized ordinal c-potential, then it has the c-FIP.
Crucial Lemma

Take a coordination game on \( G := (V, E) \) and a joint strategy \( s \).

- \( E_s^+ \) is the set of edges \( (i, j) \in E \) such that \( s_i = s_j \). These are the unicolour edges.
- An edge set \( F \subseteq E \) is a feedback edge set of \( G \) if \( G \setminus F \) is acyclic.
- For \( K \subseteq V \), \( G[K] \) is the subgraph of \( G \) induced by \( K \).

Lemma

Suppose \( s \xrightarrow{K} s' \) is a profitable deviation. Let \( F \) be a feedback edge set of \( G[K] \). Then

\[
SW(s') - SW(s) > 2|F \cap E_s^+| - 2|F \cap E_{s'}^+|.
\]
Consequences

Fix a graph $G := (V, E)$.

Corollary 1
Suppose $s^K \rightarrow s'$ is a profitable deviation such that $G[K]$ is a forest. Then $SW(s') > SW(s)$.

Corollary 2
Suppose $s^K \rightarrow s'$ is a profitable deviation such that $G[K]$ is a connected graph with exactly one cycle. Then $SW(s') \geq SW(s)$. 
The case of a ring

Example.

Social welfare: $6 \cdot 1 = 6$.

After the profitable deviation of the nodes on the triangle to $d$ the social welfare remains 6.
Can the social welfare decrease?

Example.

The payoffs to the nodes on the square: 2, 1, 2, 1.

Social welfare: $6 \cdot 1 + 2 + 1 + 2 + 1 = 12$. 
Example, ctd

From the previous joint strategy the nodes on the square can all profitably deviate to $e$:

- The payoffs to the nodes on the square: 3, 2, 3, 2.
- Social welfare is now $3 + 2 + 3 + 2 = 10$, so it decreased.
Strong Equilibria in Coordination Games

- A **pseudoforest**: a graph in which each connected component contains at most one cycle.

**Theorem**
Consider a coordination game on a graph that is a pseudoforest. Then the game has the c-FIP.

**Proof.**
- Consider $P(s) := (SW(s), \sum_{C \text{ is a cycle in } G} SW_C(s))$.
- $P$ is a generalized ordinal c-potential when we take the lexicographic ordering $>_\text{lex}$ on pairs of reals.
Other Positive Results

Theorem

Every coordination game in which only two colours are used has the c-FIP.

Proof.

SW is a generalized ordinal c-potential.

Theorem

Every coordination game whose underlying graph is complete has the c-FIP.

Proof.

- Given a sequence \( \theta \in \mathbb{R}^n \) let \( \theta^* \) be its reordering from the largest to the smallest element.
- Consider \( P(s) := (p_1(s), \ldots, p_n(s))^* \).
- \( P \) is a generalized ordinal c-potential when we take the lexicographic ordering \( >_{lex} \) on the sequences of reals.
General Case

Strong equilibria do not need to exist.

Example.
c-Weakly Acyclic Games

- A c-improvement path: a maximal sequence of profitable deviations of coalitions of players.
- A game is c-weakly acyclic if for every joint strategy there exists a finite c-improvement path that starts at it.

Note
There exist colouring games that do not have the c-FIP but are c-weakly acyclic.

Proof. In the last example add to each player a new colour $d$. 
Strong Price of Anarchy

**Theorem**

For all \( k > 1 \), the \( k \)-price of anarchy is between \( \frac{n-1}{k-1} \) and \( 2 \frac{n-1}{k-1} \).

The strong price of anarchy is 2.

**Proof idea.**

- An example that uses a complete graph shows that the \( k \)-price of anarchy is at least \( \frac{n-1}{k-1} \).
- Suppose that a game has a \( k \)-equilibrium \( s \). Let \( \sigma \) be a social optimum. Choose a coalition \( K \) of size \( k \).
- **Step 1.** Show that \( SW_K(\sigma) \leq 2SW_K(s) + |E_\sigma^+ \cap \delta(K)| \).
- **Step 2.** Summing over all \( K \) of size \( k \) one gets
  
  \[
  \binom{n-1}{k-1}SW(\sigma) \leq 2\binom{n-1}{k-1}SW(s) + \binom{n-2}{k-1}SW(\sigma).
  \]
- **Step 3.** This implies that the \( k \)-price of anarchy is at most
  
  \[
  \frac{2(n-1)}{(n-1) - \binom{n-2}{k-1}} = 2 \frac{n-1}{k-1}.
  \]
Final Comment
Thank you