Social Network Games with Obligatory Product Selection

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Joint work with Sunil Simon
Social networks

**Essential components of our model**

- Finite set of agents.
- Influence of “friends”.
- Finite product set for each agent.
- Resistance level in (threshold for) adopting a product.
Social networks

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![Diagram showing social network connections and resistance levels.](image-url)
The model

Social network [Apt, Markakis 2011]

- **Weighted directed graph:** \(G = (V, \rightarrow, w)\), where
  - \(V\): a finite set of agents,
  - \(w_{ij} \in (0, 1]\): weight of the edge \(i \rightarrow j\).
- **Products:** A finite set of products \(P\).
- **Product assignment:** \(P : V \rightarrow 2^P \setminus \{\emptyset\}\);
  - assigns to each agent a non-empty set of products.
- **Threshold function:** \(\theta(i, t) \in (0, 1]\), for each agent \(i\) and product \(t \in P(i)\).

- **Neighbours** of node \(i\): \(\{j \in V \mid j \rightarrow i\}\).
- **Source nodes:** Agents with no neighbours.
The associated strategic game

Interaction between agents: Each agent $i$ can adopt a product from the set $P(i)$.

Social network games

- **Players:** Agents in the network.
- **Strategies:** Set of strategies for player $i$ is $P(i)$.
- **Payoff:** Fix $c_0 > 0$.
  Given a joint strategy $s$ and an agent $i$, 

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Given a joint strategy $s$ and an agent $i$,

- if $i \in \text{source}(S)$,
  \[ p_i(s) = c_0 \]

- if $i \notin \text{source}(S)$,
  \[ p_i(s) = \sum_{j \in \mathcal{N}_i^t(s)} w_{ji} - \theta(i, t) \text{ if } s_i = t \text{ for some } t \in P(i) \]

\(\mathcal{N}_i^t(s)\): the set of neighbours of $i$ who adopted in $s$ the product $t$. 
Example

Threshold is 0.3 for all the players.

\[ P = \{ \bullet, \bullet, \bullet \} \]
Example

Threshold is 0.3 for all the players.

\[ \mathcal{P} = \{\bullet, \bullet, \bullet\} \]

Payoff:

\[ p_4(s) = p_5(s) = p_6(s) = c \]
Threshold is 0.3 for all the players.

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- \( p_4(s) = p_5(s) = p_6(s) = c \)
- \( p_1(s) = 0.4 - 0.3 = 0.1 \)
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Payoff:
- \[ p_4(s) = p_5(s) = p_6(s) = c \]
- \[ p_1(s) = 0.4 - 0.3 = 0.1 \]
- \[ p_2(s) = 0.5 - 0.3 = 0.2 \]
- \[ p_3(s) = 0.4 - 0.3 = 0.1 \]
Social network games

Properties

- **Graphical game:** The payoff for each player depends only on the choices made by his neighbours.

- **Join the crowd property:** The payoff of each player weakly increases if more players choose the same strategy.
Solution concept – Nash equilibrium

**Best response**

A strategy \( s_i \) of player \( i \) is a **best response** to a joint strategy \( s_{-i} \) if for all \( s_i' \),

\[
p_i(s_i', s_{-i}) \leq p_i(s_i, s_{-i}).
\]

**Nash equilibrium**

A strategy profile \( s \) is a Nash equilibrium if for all players \( i \), \( s_i \) is the best response to \( s_{-i} \).
Nash equilibrium: simple cycles

Does a Nash equilibrium always exist?
Nash equilibrium: simple cycles

Does a Nash equilibrium always exist?

No
Nash equilibrium: simple cycles

Does a Nash equilibrium always exist?

No

**Theorem** Consider a social network $S$ whose underlying graph is a simple cycle. It takes $O(n \cdot |P|^4)$ time to decide whether the game $G(S)$ has a Nash equilibrium.
Nash equilibrium: arbitrary case

**Theorem** Deciding whether for a social network $S$ the game $G(S)$ has a Nash equilibrium is NP-complete.

**Proof idea.**
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1. Use a specific social network game with no Nash equilibrium.
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The preceding example of a social network.
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The preceding example of a social network.
2. Use a specific NP-complete problem.
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**Theorem** Deciding whether for a social network $S$ the game $G(S)$ has a Nash equilibrium is NP-complete.

**Proof idea.**
1. Use a specific social network game with no Nash equilibrium.
2. Use a specific NP-complete problem.

The PARTITION problem

**Input:** $n$ positive rational numbers $(a_1, \ldots, a_n)$ such that $\sum_i a_i = 1$.

**Question:** Is there a set $S \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in S} a_i = \sum_{i \notin S} a_i = \frac{1}{2}.$$
Weakly acyclic games (1)

(Young '93, Milchtaich '96)

- $s'_i$ is a better response given $s$ if $p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})$.
- A profitable deviation: a pair $(s, s')$ of joint strategies such that $s' = (s'_i, s_{-i})$ for some $s'_i$ and $p_i(s') > p_i(s)$.
- An improvement path: a maximal sequence of profitable deviations.
- $G$ is weakly acyclic if for any joint strategy there exists a finite improvement path that starts at it.
Weakly acyclic games (2)

- For an arbitrary network $S$, deciding whether the game $\mathcal{G}(S)$ is weakly acyclic is co-NP hard.
- For a network $S$ whose underlying graph has no source nodes, deciding whether the game $\mathcal{G}(S)$ is weakly acyclic is also co-NP hard.
The more options one has, the more possibilities for experiencing conflict arise, and the more difficult it becomes to compare the options. There is a point where more options, products, and choices hurt both seller and consumer.
Paradox 1: vulnerable networks

Addition of a product to a social network can affect negatively everybody.

More specifically: a social network exists such that for some Nash equilibrium $s$ an addition of a product will trigger a sequence of changes that will always lead the agents from $s$ to a new Nash equilibrium that is worse for everybody.
Example

Nodes 1 and 2 prefer red over brown, and nodes 3 and 4 prefer green over blue.
Example

Nodes 1 and 2 prefer red over brown, and nodes 3 and 4 prefer green over blue.

The weights and thresholds are so chosen that this is a Nash equilibrium.
Nodes 1 and 2 prefer red over brown, and nodes 3 and 4 prefer green over blue.

This is not a Nash equilibrium.
Nodes 1 and 2 prefer red over brown, and nodes 3 and 4 prefer green over blue.

This is not a Nash equilibrium.
Example

Nodes 1 and 2 prefer **red** over **brown**, and nodes 3 and 4 prefer **green** over **blue**.

This is **not** a Nash equilibrium.
Example

Nodes 1 and 2 prefer red over brown, and nodes 3 and 4 prefer green over blue.

This is not a Nash equilibrium.
Nodes 1 and 2 prefer red over brown, and nodes 3 and 4 prefer green over blue. This is not a Nash equilibrium.
Example

Nodes 1 and 2 prefer red over brown, and nodes 3 and 4 prefer green over blue.

This is a Nash equilibrium. The payoff to each player is now strictly worse.
Paradox 2: inefficient networks

Removal of a product from a social network can affect positively everybody.

More specifically: a social network exists such that for some Nash equilibrium $s$ a removal of a product will trigger a sequence of changes that will always lead the agents from $s$ to a new Nash equilibrium that is better for everybody.
Cost $\theta$ is product independent.

The weight of each edge is $w$, where $w > \theta$.

Note Each node has two incoming edges.
Cost $\theta$ is product independent.
The weight of each edge is $w$, where $w > \theta$.
This is a Nash equilibrium. The payoff to each player is $w - \theta$. 
Cost $\theta$ is product independent.
The weight of each edge is $w$, where $w > \theta$.
This is not a legal joint strategy.
Cost $\theta$ is product independent.

The weight of each edge is $w$, where $w > \theta$.

This is not a Nash equilibrium.
Example

- Cost $\theta$ is product independent.
- The weight of each edge is $w$, where $w > \theta$.
- This is a Nash equilibrium. The payoff to each player is $2w - \theta$. 

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Other Paradoxes

- A social network $S$ is **fragile** if $G(S)$ has a Nash equilibrium while for some expansion $S'$ of $S$, $G(S')$ does not.
- A social network $S$ **unsafe** if $G(S)$ has a Nash equilibrium, while for some contraction $S'$ of $S$, $G(S')$ does not.
Thank you
Molte grazie
Dziękuję za uwagę