Strategic Games: Social Optima and Nash Equilibria

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Part I

- Strategic games.
- Nash equilibrium.
- Social optimum.
- Price of anarchy.
- Price of stability.

Strategic Games

Strategic game for $|N| \geq 2$ players:

$$G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N}).$$

For each player i

- (possibly infinite) set S_i of strategies,
- payoff function $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$.

Basic assumptions

- Players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others' rationality.

Three Examples (1)

The Battle of the Sexes

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

Matching Pennies

	H	T
H	$1,-1$	$-1,\ 1$
T	$-1,\ 1$	$1,-1$

Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Main Concepts

- Notation: $s_i, s'_i \in S_i$,
  $s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n$.
- s is a Nash equilibrium if

$$\forall i \in \{1, \ldots, n\}\ \forall s'_i \in S_i\ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- Social welfare of s:

$$SW(s) := \sum_{j=1}^{n} p_j(s).$$

- s is a social optimum if $SW(s)$ is maximal.

Intuitions

- Nash equilibrium:
  Every player is ‘happy’
  (played his best response).
- Social optimum:
  The desired state of affairs for the society.
- Main problem:
  Social optima may not be Nash equilibria.

Three Examples (2)

The Battle of the Sexes: Two Nash equilibria.

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

Matching Pennies: No Nash equilibrium.

	H	T
H	$1,-1$	$-1,\ 1$
T	$-1,\ 1$	$1,-1$

Prisoner's Dilemma: One Nash equilibrium.

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Price of Anarchy and of Stability

- Price of Anarchy (Koutsoupias, Papadimitriou, 1999):

$$\frac{\text{SW of social optimum}}{\text{SW of the worst Nash equilibrium}}$$

- Price of Stability (Schulz, Moses, 2003):

$$\frac{\text{SW of social optimum}}{\text{SW of the best Nash equilibrium}}$$

Examples

A 3×3 game

	L	M	R
T	$2, 2$	$4, 1$	$1, 0$
C	$1, 4$	$3, 3$	$1, 0$
B	$0, 1$	$0, 1$	$1, 1$

PoA = $\frac{6}{2} = 3$.
PoS = $\frac{6}{4} = 1.5$.

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

PoA = PoA = 2.

Cournot Competition (1838)

- One infinitely divisible product (oil),
- n companies decide simultaneously how much to produce,
- price is decreasing in total output.

Each $S_i = \mathbb{R}_+$,

$$p_i(s) := s_i\left(a - b\sum_{j=1}^{n} s_j\right) - cs_i$$

for some a, b, c, where $a > c$ and $b > 0$.

The price of the product: $a - b\sum_{j=1}^{n} s_j$.

The production cost: cs_i.

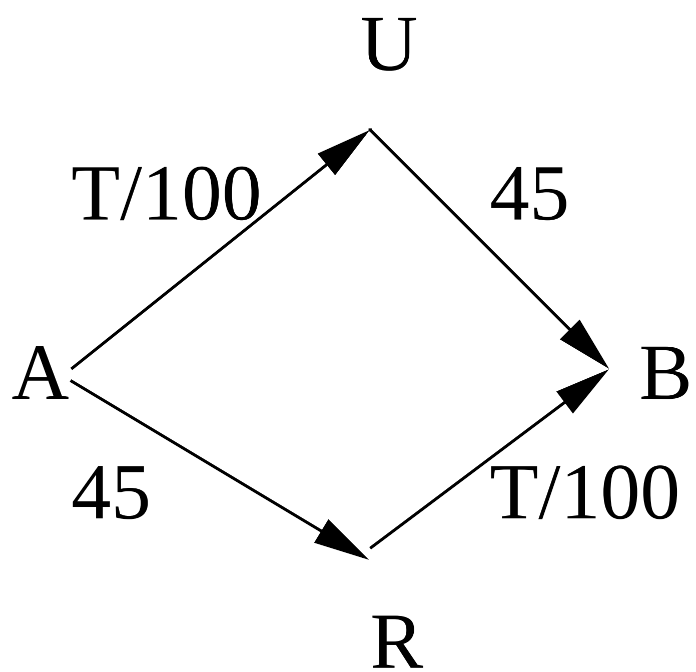
Cournot Competition (ctd)

- $p_i(s) := s_i\left(a - b\sum_{j=1}^{n} s_j\right) - cs_i$.
- Unique Nash equilibrium:
  s, with each $s_i = \frac{a-c}{b(n+1)}$.
  $SW(s) = \frac{(a-c)^2}{b} \cdot \frac{n}{(n+1)^2}$.
- Social optimum:
  when $\sum_{j=1}^{n} s_j = \frac{a-c}{2b}$.
  $SW(s) = \frac{(a-c)^2}{4b}$.
- Note
  PoA (= PoS) = $\frac{(n+1)^2}{4n}$.

Congestion Games: Example

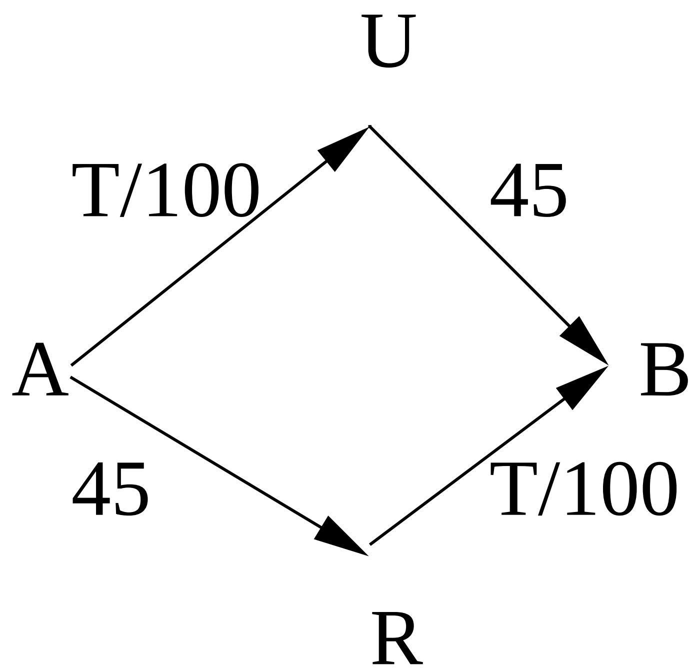
Assumptions:

- 4000 drivers drive from A to B.
- Each driver has 2 possibilities (strategies).



Problem: Find a Nash equilibrium (T = number of drivers).

Nash Equilibrium

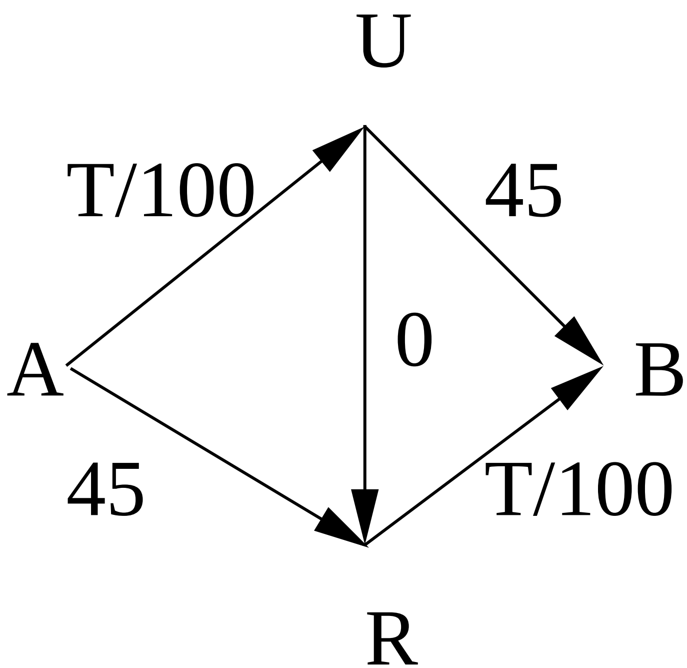


Answer: 2000/2000.

Travel time: 2000/100 + 45 = 45 + 2000/100 = 65.

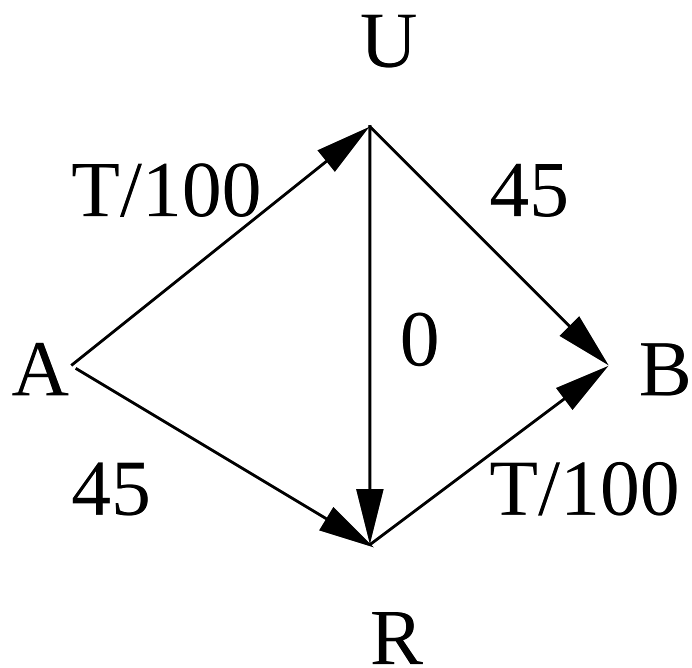
Braess Paradox

- Add a fast road from U to R.
- Each drives has now 3 possibilities (strategies):
  A - U - B,
  A - R - B,
  A - U - R - B.



Problem: Find a Nash equilibrium.

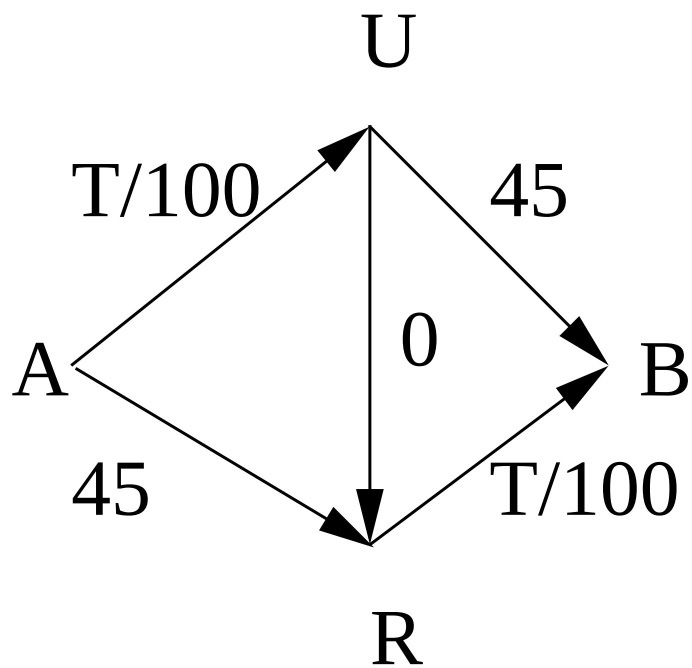
Nash Equilibrium



Answer: Each driver will choose the road A - U - R - B.

Why?: The road A - U - R - B is always a best response.

Bad News



- Travel time: 4000/100 + 4000/100 = 80!
- PoA (and PoS) went up from 1 to 80/65.

Does it Happen?

From Wikipedia ('Braess Paradox'):

- In Seoul, South Korea, a speeding-up in traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon restoration project.
- In Stuttgart, Germany after investments into the road network in 1969, the traffic situation did not improve until a section of newly-built road was closed for traffic again.
- In 1990 the closing of 42nd street in New York City reduced the amount of congestion in the area.
- In 2008 Youn, Gastner and Jeong demonstrated specific routes in Boston, New York City and London where this might actually occur and pointed out roads that could be closed to reduce predicted travel times.

Part II

- Altruistic games.
- Selfishness level.
  (Based on
  Selfishness level of strategic games,
  K.R. Apt and G. Schäfer)

Altruistic Games

- Given $G := (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$ and $\alpha \geq 0$.
- $G(\alpha) := (N, \{S_i\}_{i \in N}, \{r_i\}_{i \in N})$, where

$$r_i(s) := p_i(s) + \alpha SW(s).$$

- When $\alpha > 0$ the payoff of each player in $G(\alpha)$ depends on the social welfare of the players.
- $G(\alpha)$ is an altruistic version of G.

Selfishness Level (1)

- G is α-selfish if a Nash equilibrium of $G(\alpha)$ is a social optimum of $G(\alpha)$.
- If for no $\alpha \geq 0$, G is α-selfish, then its selfishness level is ∞.
- Suppose G is finite.
  If for some $\alpha \geq 0$, G is α-selfish, then

  $$\min_{\alpha \in \mathbb{R}_+} (G \text{ is } \alpha\text{-selfish})$$

  is the selfishness level of G.

Selfishness Level (2)

- Suppose G is infinite.
  - If for some $\alpha \geq 0$, G is α-selfish and

    $$\min_{\alpha \in \mathbb{R}_+} (G \text{ is } \alpha\text{-selfish})$$

    exists, then it is the selfishness level of G.
  - Otherwise the selfishness level of G is undefined.

Three Examples (1)

The Battle of the Sexes

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

Matching Pennies

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Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Three Examples (2)

The Battle of the Sexes: selfishness level is 0.

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

Matching Pennies: selfishness level is ∞.

	H	T
H	$1,-1$	$-1,\ \ 1$
T	$-1,\ \ 1$	$1,-1$

Prisoner's Dilemma: selfishness level is 1.

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

	C	D
C	$6,6$	$3,6$
D	$6,3$	$3,3$

Selfishness Level vs Price of Stability

- Note
  Selfishness level of a finite game is 0 iff price of stability is 1.
- Theorem
  For every finite $\alpha > 0$ and $\beta > 1$ there is a finite game with selfishness level α and price of stability β.

Example: Traveler's Dilemma

Two players, $S_i = \{2, \ldots, 100\}$,

$$p_i(s) := \begin{cases} s_i & \text{if } s_i = s_{-i} \\ s_i + 2 & \text{if } s_i < s_{-i} \\ s_{-i} - 2 & \text{otherwise.} \end{cases}$$

Problem: Find a Nash equilibrium.

Proposition Selfishness level is $\frac{1}{2}$.

Cournot Competition

- Note Price of anarchy (and of stability) converges with n to ∞.
- Proposition For each $n > 1$ the selfishness level is ∞.

Tragedy of the Commons

- Contiguous common resource (shared bandwidth),
- Each $S_i = [0,1]$,
- s_i: chosen fraction of the common resource
- payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Intuition: the payoff degrades when the resource is overused.
- Proposition For each $n > 1$ the selfishness level is ∞.

Congestion Games

Congestion game: $G = (N, E, \{S_i\}_{i \in N}, \{d_e\}_{e \in E})$, where

- E is a finite set of facilities,
- $S_i \subseteq 2^E$ is the set of facility subsets available to player i,
- $d_e \in \mathbb{N}$ is the delay function for facility $e \in E$.
- Let $x_e(s)$ be the number of players using facility e in s.
- The goal of a player is to minimize his individual cost $c_i(s) := \sum_{e \in s_i} d_e(x_e(s))$.
- Social cost: $SC(s) = \sum_{i=1}^{n} c_i(s)$.

Linear Congestion Games

- Linear congestion game: each delay function is of the form $d_e(x) = a_e x + b_e$, where $a_e, b_e \in \mathbb{N}$.
- Let L be the maximum number of facilities that any player can choose:
  $L := \max_{i \in N,\ s_i \in S_i} |s_i|$.
- $\Delta_{\max} := \max_{e \in E}(a_e + b_e)$,
  $\Delta_{\min} := \min_{e \in E}(a_e + b_e)$.

Proposition Selfishness level of a linear congestion game is $\leq \frac{1}{2}(L \cdot \Delta_{\max} - \Delta_{\min} - 1)$.

Note This bound does not depend on the number of players.

Take Home Message

- Price of anarchy and price of stability are descriptive concepts.
- Selfishness level is a normative concept.

Some Quotations

- Dalai Lama:

  `The intelligent way to be selfish is to work for the welfare of others.`

  Microeconomics: Behavior, Institutions, and Evolution, S. Bowles '04.

- `An excellent way to promote cooperation in a society is to teach people to care about the welfare of others.`

  The Evolution of Cooperation, R. Axelrod, '84.

THANK YOU

Dziękuję za uwagę