Strategic games

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Introduction

Mathematical game theory, as launched by Von Neumann and Morgenstern in their seminal book von Neumann and Morgenstern [1944], followed by Nash' contributions Nash [1950,1951], has become a standard tool in Economics for the study and description of various economic processes, including competition, cooperation, collusion, strategic behaviour and bargaining. Since then it has also been successfuly used in Biology, Political Sciences, Psychology and Sociology. With the advent of the Internet game theory became increasingly relevant in Computer Science.

One of the main areas in game theory are ***strategic games***, (sometimes also called ***non-cooperative games***), which form a simple model of interaction between profit maximizing players. In strategic games each player has a payoff function that he aims to maximize and the value of this function depends on the decisions taken *simultaneously* by all players. Such a simple description is still amenable to various interpretations, depending on the assumptions about the existence of *private information.* The purpose of these lecture notes is to provide a simple introduction to the most common concepts used in strategic games and most common types of such games.

Many books provide introductions to various areas of game theory, including strategic games. Most of them are written from the perspective of applications to Economics. In the nineties the leading textbooks were Myerson [1991], Binmore [1991], Fudenberg and Tirole [1991] and Osborne and Rubinstein [1994].

Moving to the next decade, Osborne [2005] is an excellent, broad in its scope, undergraduate level textbook, while Peters [2008] is probably the best book on the market for the graduate level. Undeservedly less known is the short and lucid Tijs [2003]. An elementary, short introduction, focusing on the concepts, is Shoham and Leyton-Brown [2008]. In turn, Ritzberger [2001] is a comprehensive book on strategic games that also extensively discusses ***extensive games***, i.e., games in which the players choose actions in turn. Finally, Binmore [2007] is thoroughly revised version of Binmore [1991].

Several textbooks on microeconomics include introductory chapters on game theory, including strategic games. Two good examples are Mas-Collel, Whinston and Green [1995] and Jehle and Reny [2000]. Finally, Nisan et al. [2007] is a recent collection of surveys and introductions to the computational aspects of game theory, with a number of articles concerned with strategic games and mechanism design.

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Chapter 1

Nash Equilibrium

Assume a set $\{1, \ldots, n\}$ of players, where $n > 1$. A ***strategic game*** (or ***non-cooperative game***) for n players, written as $(S_1, \ldots, S_n, p_1, \ldots, p_n)$, consists of

- a non-empty (possibly infinite) set S_i of ***strategies***,
- a ***payoff function*** $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$,

for each player i.

We study strategic games under the following basic assumptions:

- players choose their strategies *simultaneously*; subsequently each player receives a payoff from the resulting joint strategy,
- each player is ***rational***, which means that his objective is to maximize his payoff,
- players have ***common knowledge*** of the game and of each others' rationality.[1]

Here are three classic examples of strategic two-player games to which we shall return in a moment. We represent such games in the form of a bimatrix, the entries of which are the corresponding payoffs to the row and column players. So for instance in the Prisoner's Dilemma game, when the row player chooses C (cooperate) and the column player chooses D (defect),

[1]Intuitively, common knowledge of some fact means that everybody knows it, everybody knows that everybody knows it, etc. This notion can be formalized using epistemic logic.

then the payoff for the row player is 0 and the payoff for the column player is 3.

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

We introduce now some basic notions that will allow us to discuss and analyze strategic games in a meaningful way. Fix a strategic game

$$(S_1, \ldots, S_n, p_1, \ldots, p_n).$$

We denote $S_1 \times \ldots \times S_n$ by S, call each element $s \in S$ a ***joint strategy***, or a ***strategy profile***, denote the ith element of s by s_i, and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. Occasionally we write (s_i, s_{-i}) instead of s. Finally, we abbreviate $\times_{j \neq i} S_j$ to S_{-i} and use the '$_{-i}$' notation for other sequences and Cartesian products.

We call a strategy s_i of player i a ***best response*** to a joint strategy s_{-i} of his opponents if

$$\forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

Next, we call a joint strategy s a ***Nash equilibrium*** if each s_i is a best response to s_{-i}, that is, if

$$\forall i \in \{1, \ldots, n\} \; \forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

So a joint strategy is a Nash equilibrium if no player can achieve a higher payoff by *unilaterally* switching to another strategy. Intuitively, a Nash equilibrium is a situation in which each player is a posteriori satisfied with his choice.

Let us return now the three above introduced games.

Re: Prisoner's Dilemma

The Prisoner's Dilemma game has a unique Nash equilibrium, namely (D, D). One of the peculiarities of this game is that in its unique Nash equilibrium each player is worse off than in the outcome (C, C). We shall return to this game once we have more tools to study its characteristics.

To clarify the importance of this game we now provide a couple of simple interpretations of it. The first one, due to Aumann, is the following.

> Each player decides whether he will receive 1000 dollars or the other will receive 2000 dollars. The decisions are simultaneous and independent.

So the entries in the bimatrix of the Prisoner's Dilemma game refer to the thousands of dollars each player will receive. For example, if the row player asks to give 2000 dollars to the other player, and the column player asks for 1000 dollar for himself, the row player gets nothing while column player gets 3000 dollars. This contingency corresponds to the 0,3 entry in the bimatrix.

The original interpretation of this game that explains its name refers to the following story.

> Two suspects are taken into custody and separated. The district attorney is certain that they are guilty of a specific crime, but he does not have adequate evidence to convict them at a trial. He points out to each prisoner that each has two alternatives: to confess to the crime the police are sure they have done (C), or not to confess (N).
>
> If they both do not confess, then the district attorney states he will book them on some very minor trumped-up charge such as petty larceny or illegal possession of weapon, and they will both receive minor punishment; if they both confess they will be prosecuted, but he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor

will receive lenient treatment for turning state's evidence whereas the latter will get "the book" slapped at him.

This is represented by the following bimatrix, in which each negative entry, for example -1, corresponds to the 1 year prison sentence ('the lenient treatment' referred to above):

	C	N
C	$-5,-5$	$-1,-8$
N	$-8,-1$	$-2,-2$

The negative numbers are used here to be compatible with the idea that each player is interested in maximizing his payoff, so, in this case, of receiving a lighter sentence. So for example, if the row suspect decides to confess, while the column suspect decides not to confess, the row suspect will get 1 year prison sentence (the 'lenient treatment'), the other one will get 8 years of prison (' "the book" slapped at him').

Many other natural situations can be viewed as a Prisoner's Dilemma game. This allows us to explain the underlying, undesidered phenomena.

Consider for example the arms race. For each of two warring, equally strong countries, it is beneficial not to arm instead of to arm. Yet both countries end up arming themselves. As another example consider a couple seeking a divorce. Each partner can choose an inexpensive (bad) or an expensive (good) layer. In the end both partners end up choosing expensive lawyers. Next, suppose that two companies produce a similar product and may choose between low and high advertisement costs. Both end up heavily advertising.

Re: Matching Pennies game

Next, consider the Matching Pennies game. This game formalizes a game that used to be played by children. Each of two children has a coin and simultaneously shows heads (H) or tails (T). If the coins match then the first child wins, otherwise the second child wins. This game has no Nash equilibrium. This corresponds to the intuition that for no outcome both players are satisfied. Indeed, in each outcome the losing player regrets his choice. Moreover, the social welfare of each outcome is 0. Such games are called ***zero sum games*** and we shall return to them later. Also, we shall return to this game once we have introduced mixed strategies.

Re: Battle of the Sexes game

Finally, consider the Battle of the Sexes game. The interpretation of this game is as follows. A couple has to decide whether to go out for a football match (F) or a ballet (B). The man, the row player prefers a football match over the ballet, while the woman, the column player, the other way round. Moreover, each of them prefers to go out together than to end up going out separately. This game has two Nash equilibria, namely (F, F) and (B, B). Clearly, there is a problem how the couple should choose between these two satisfactory outcomes. Games of this type are called ***coordination games***.

Obviously, all three games are very simplistic. They deal with two players and each player has to his disposal just two strategies. In what follows we shall introduce many interesting examples of strategic games. Some of them will deal with many players and some games will have several, sometimes an infinite number of strategies.

To close this chapter we consider two examples of more interesting games, one for two players and another one for an arbitrary number of players.

Example 1 (Traveler's dilemma)

Suppose that two travellers have identical luggage, for which they both paid the same price. Their luggage is damaged (in an identical way) by an airline. The airline offers to recompense them for their luggage. They may ask for any dollar amount between \$2 and \$100. There is only one catch. If they ask for the same amount, then that is what they will both receive. However, if they ask for different amounts —say one asks for $\$m$ and the other for $\$m'$, with $m < m'$— then whoever asks for $\$m$ (the lower amount) will get $\$(m + 2)$, while the other traveller will get $\$(m - 2)$. The question is: what amount of money should each traveller ask for?

We can formalize this problem as a two-player strategic game, with the set $\{2, \ldots, 100\}$ of natural numbers as possible strategies. The following payoff function[2] formalizes the conditions of the problem:

$$p_i(s) := \begin{cases} s_i & \text{if } s_i = s_{-i} \\ s_i + 2 & \text{if } s_i < s_{-i} \\ s_{-i} - 2 & \text{otherwise} \end{cases}$$

It is easy to check that $(2, 2)$ is a Nash equilibrium. To check for other Nash equilibria consider any other combination of strategies (s_i, s_{-i}) and

[2]We denote in two-player games the opponent of player i by $-i$, instead of $3 - i$.

suppose that player i submitted a larger or equal amount, i.e., $s_i \geq s_{-i}$. Then player's i payoff is s_{-i} if $s_i = s_{-i}$ or $s_{-i} - 2$ if $s_i > s_{-i}$.

In the first case he will get a strictly higher payoff, namely $s_{-i} + 1$, if he submits instead the amount $s_{-i} - 1$. (Note that $s_i = s_{-i}$ and $(s_i, s_{-i}) \neq (2, 2)$ implies that $s_{-i} - 1 \in \{2, \ldots, 100\}$.) In turn, in the second case he will get a strictly higher payoff, namely s_{-i}, if he submits instead the amount s_{-i}.

So in each joint strategy $(s_i, s_{-i}) \neq (2, 2)$ at least one player has a strictly better alternative, i.e., his strategy is not a best response. This means that $(2, 2)$ is a unique Nash equilibrium. This is a paradoxical conclusion, if we recall that informally a Nash equilibrium is a state in which both players are satisfied with their choice. □

Example 2 Consider the following ***beauty contest game***. In this game there are $n > 2$ players, each with the set of strategies equal $\{1, \ldots, 100\}$, Each player submits a number and the payoff to each player is obtained by splitting 1 equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average. For example, if the submissions are $29, 32, 29$, then the payoffs are respectively $\frac{1}{2}, 0, \frac{1}{2}$.

Finding Nash equilibria of this game is not completely straightforward. At this stage we only observe that the joint strategy $(1, \ldots, 1)$ is clearly a Nash equilibrium. We shall answer the question of whether there are more Nash equilibria once we introduce some tools to analyze strategic games. □

Exercise 1 Find all Nash equilibria in the following games:

Stag hunt

	S	R
S	$2, 2$	$0, 1$
R	$1, 0$	$1, 1$

Coordination

	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$1, 1$

Pareto Coordination

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

Hawk-dove

	H	D
H	$0, 0$	$3, 1$
D	$1, 3$	$2, 2$

□

Exercise 2 Consider the following ***inspection game***.

There are two players: a worker and the boss. The worker can either Shirk or put an Effort, while the boss can either Inspect or Not. Finding a shirker has a benefit b while the inspection costs c, where $b > c > 0$. So if the boss carries out an inspection his benefit is $b - c > 0$ if the worker shirks and $-c < 0$ otherwise.

The worker receives 0 if he shirks and is inspected, and g if he shirks and is not found. Finally, the worker receives w, where $g > w > 0$ if he puts in the effort.

This leads to the following bimatrix:

	I	N
S	$0, b - c$	$g, 0$
E	$w, -c$	$w, 0$

Analyze the best responses in this game. What can we conclude from it about the Nash equilibria of this game?

□