Chapter 10

Elimination by Mixed Strategies

The notions of dominance apply in particular to mixed extensions of finite strategic games. But we can also consider dominance of a *pure* strategy by a *mixed* strategy. Given a finite strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$, we say that a (pure) strategy s_i of player i is ***strictly dominated by*** a mixed strategy m_i if

$$\forall s_{-i} \in S_{-i} \; p_i(m_i, s_{-i}) > p_i(s_i, s_{-i}),$$

and that s_i is ***weakly dominated by*** a mixed strategy m_i if

$$\forall s_{-i} \in S_{-i} \; p_i(m_i, s_{-i}) \geq p_i(s_i, s_{-i}) \text{ and } \exists s_{-i} \in S_{-i} \; p_i(m_i, s_{-i}) > p_i(s_i, s_{-i}).$$

In what follows we discuss for these two forms of dominance the counterparts of the results presented in Chapters 3 and 4.

10.1 Elimination of strictly dominated strategies

Strict dominance by a mixed strategy leads to a stronger form of strategy elimination. For example, in the game

	L	R
T	$2,1$	$0,1$
M	$0,1$	$2,1$
B	$0,1$	$0,1$

the strategy B is strictly dominated neither by T nor M but is strictly dominated by $\frac{1}{2} \cdot T + \frac{1}{2} \cdot M$.

We now focus on iterated elimination of pure strategies that are strictly dominated by a mixed strategy. As in Chapter 3 we would like to clarify whether it affects the Nash equilibria, in this case equilibria in mixed strategies. We denote the corresponding reduction relation between restrictions of a finite strategic game by $\rightarrow_{SM}$.

First, we introduce the following notation. Given two mixed strategies m_i, m'_i and a strategy s_i we denote by $m_i[s_i/m'_i]$ the mixed strategy obtained from m_i by substituting the strategy s_i by m'_i and by 'normalizing' the resulting sum. For example, given $m_i = \frac{1}{3}H + \frac{2}{3}T$ and $m'_i = \frac{1}{2}H + \frac{1}{2}T$ we have $m_i[H/m'_i] = \frac{1}{3}(\frac{1}{2}H + \frac{1}{2}T) + \frac{2}{3}T = \frac{1}{6}H + \frac{5}{6}T$.

We also use the following identification of mixed strategies over two sets of strategies S'_i and S_i such that $S'_i \subseteq S_i$. We view a mixed strategy $m_i \in \Delta S_i$ such that $support(m_i) \subseteq S'_i$ as a mixed strategy 'over' the set S'_i, i.e., as an element of $\Delta S'_i$, by limiting the domain of m_i to S'_i. Further, we view each mixed strategy $m_i \in \Delta S'_i$ as a mixed strategy 'over' the set S_i, i.e., as an element of ΔS_i, by assigning the probability 0 to the elements in $S_i \setminus S'_i$.

Next, we establish the following auxiliary lemma.

Lemma 37 (Persistence) *Given a finite strategic game G consider two restrictions R and R' of G such that $R \rightarrow_{SM} R'$.*

Suppose that a strategy $s_i \in R_i$ is strictly dominated in R by a mixed strategy from R. Then s_i is strictly dominated in R by a mixed strategy from R'.

Proof. We shall use the following, easy to establish, two properties of strict dominance by a mixed strategy in a given restriction:

(a) for all $\alpha \in (0, 1]$, if s_i is strictly dominated by $(1 - \alpha)s_i + \alpha\, m_i$, then s_i is strictly dominated by m_i,

(b) if s_i is strictly dominated by m_i and s'_i is strictly dominated by m'_i, then s_i is strictly dominated by $m_i[s'_i/m'_i]$.

Suppose that $R_i \setminus R'_i = \{t^1_i, \ldots, t^k_i\}$. By definition for all $j \in \{1, \ldots, k\}$ there exists in R a mixed strategy m^j_i such that t^j_i is strictly dominated in R

by m_i^j. We first prove by complete induction that for all $j \in \{1, \ldots, k\}$ there exists in R a mixed strategy n_i^j such that

$$t_i^j \text{ is strictly dominated in } R \text{ by } n_i^j \text{ and } \mathit{support}(n_i^j) \cap \{t_i^1, \ldots, t_i^j\} = \emptyset. \tag{10.1}$$

For some $\alpha \in (0, 1]$ and a mixed strategy n_i^1 with $t_i^1 \notin \mathit{support}(n_i^1)$ we have

$$m_i^1 = (1 - \alpha)t_i^1 + \alpha\, n_i^1.$$

By assumption t_i^1 is strictly dominated in R by m_i^1, so by (a) t_i^1 is strictly dominated in R by n_i^1, which proves (10.1) for $j = 1$.

Assume now that $\ell < k$ and that (10.1) holds for all $j \in \{1, \ldots, \ell\}$. By assumption $t_i^{\ell+1}$ is strictly dominated in R by $m_i^{\ell+1}$.

Let

$$m_i'' := m_i^{\ell+1}[t_i^1/n_i^1]\ldots[t_i^\ell/n_i^\ell].$$

By the induction hypothesis and (b) $t_i^{\ell+1}$ is strictly dominated in R by m_i'' and $\mathit{support}(m_i'') \cap \{t_i^1, \ldots, t_i^\ell\} = \emptyset$.

For some $\alpha \in (0, 1]$ and a mixed strategy $n_i^{\ell+1}$ with $t_i^{\ell+1} \notin \mathit{support}(n_i^{\ell+1})$ we have

$$m_i'' = (1 - \alpha)t_i^{\ell+1} + \alpha\, n_i^{\ell+1}.$$

By (a) $t_i^{\ell+1}$ is strictly dominated in R by $n_i^{\ell+1}$. Also $\mathit{support}(n_i^{\ell+1}) \cap \{t_i^1, \ldots, t_i^{\ell+1}\} = \emptyset$, which proves (10.1) for $j = \ell + 1$.

Suppose now that the strategy s_i is strictly dominated in R by a mixed strategy m_i from R. Define

$$m_i' := m_i[t_i^1/n_i^1]\ldots[t_i^k/n_i^k].$$

Then by (b) and (10.1) s_i is strictly dominated in R by m_i' and $\mathit{support}(m_i') \subseteq R_i'$, i.e., m_i' is a mixed strategy in R'. $\square$

The following is a counterpart of the Strict Elimination Lemma 1 and will be used in a moment.

Lemma 38 (Strict Mixed Elimination) *Given a finite strategic game G consider two restrictions R and R' of G such that $R \rightarrow_{SM} R'$.*

Then m is a Nash equilibrium of R iff it is a Nash equilibrium of R'.

Proof. Let

$$R := (R_1, \ldots, R_n, p_1, \ldots, p_n),$$

and

$$R' := (R'_1, \ldots, R'_n, p_1, \ldots, p_n).$$

($\Rightarrow$) It suffices to show that m is also a joint mixed strategy in R', i.e., that for all $i \in \{1, \ldots, n\}$ we have $support(m_i) \subseteq R'_i$.

Suppose otherwise. Then for some $i \in \{1, \ldots, n\}$ a strategy $s_i \in support(m_i)$ is strictly dominated by a mixed strategy $m'_i \in \Delta R_i$. So

$$p_i(m'_i, m''_{-i}) > p_i(s_i, m''_{-i}) \text{ for all } m''_{-i} \in \times_{j \neq i} \Delta R_j.$$

In particular

$$p_i(m'_i, m_{-i}) > p_i(s_i, m_{-i}).$$

But m is a Nash equilibrium of R and $s_i \in support(m_i)$ so by the Characterization Lemma 28

$$p_i(m) = p_i(s_i, m_{-i}).$$

Hence

$$p_i(m'_i, m_{-i}) > p_i(m),$$

which contradicts the choice of m.

($\Leftarrow$) Suppose m is not a Nash equilibrium of R. Then by the Characterization Lemma 28 for some $i \in \{1, \ldots, n\}$ and $s'_i \in R_i$

$$p_i(s'_i, m_{-i}) > p_i(m).$$

The strategy s'_i is eliminated since m is a Nash equilibrium of R'. So s'_i is strictly dominated in R by some mixed strategy in R. By the Persistence Lemma 37 s'_i is strictly dominated in R by some mixed strategy m'_i in R'. So

$$p_i(m'_i, m''_{-i}) \geq p_i(s'_i, m''_{-i}) \text{ for all } m''_{-i} \in \times_{j \neq i} \Delta R_j.$$

In particular

$$p_i(m'_i, m_{-i}) \geq p_i(s'_i, m_{-i})$$

and hence by the choice of s'_i

$$p_i(m'_i, m_{-i}) > p_i(m).$$

Since $m'_i \in \Delta R'_i$ this contradicts the assumption that m is a Nash equilibrium of R'. □

Instead of the lengthy wording 'the iterated elimination of strategies strictly dominated by a mixed strategy' we write ***IESDMS***. We have then the following counterpart of the IESDS Theorem 2, where we refer to Nash equilibria in mixed strategies. Given a restriction G' of G and a joint mixed strategy m of G, when we say that m is a Nash equilibrium of G' we implicitly stipulate that all supports of all m_is consist of strategies from G'.

We then have the following counterpart of the IESDS Theorem 2.

Theorem 39 (IESDMS) *Suppose that G is a finite strategic game.*

(i) If G' is an outcome of IESDMS from G, then m is a Nash equilibrium of G iff it is a Nash equilibrium of G'.

(ii) If G is solved by IESDMS, then the resulting joint strategy is a unique Nash equilibrium of G (in, possibly, mixed strategies).

Proof. By the Strict Mixed Elimination Lemma 38. □

To illustrate the use of this result let us return to the beauty contest game discussed in Examples 2 of Chapter 1 and 10 in Chapter 4. We explained there that $(1, \ldots, 1)$ is *a* Nash equilibrium. Now we can draw a stronger conclusion.

Example 19 One can show that the beauty contest game is solved by IESDMS in 99 rounds. In each round the highest strategy of each player is removed and eventually each player is left with the strategy 1. On the account of the above theorem we now conclude that $(1, \ldots, 1)$ is a *unique* Nash equilibrium. □

As in the case of strict dominance by a pure strategy we now address the question whether the outcome of IESDMS is unique. The answer is positive. To establish this result we proceed as before and establish the following lemma first. Recall that the notion of hereditarity was defined in the Appendix of Chapter 3.

Lemma 40 (Hereditarity III) *The relation of being strictly dominated by a mixed strategy is hereditary on the set of restrictions of a given finite game.*

Proof. This is an immediate consequence of the Persistence Lemma 37. Indeed, consider a finite strategic game G and two restrictions R and R' of G such that $R \rightarrow_{SM} R'$.

Suppose that a strategy $s_i \in R'_i$ is strictly dominated in R by a mixed strategy in R. By the Persistence Lemma 37 s_i is strictly dominated in R by a mixed strategy in R'. So s_i is also strictly dominated in R' by a mixed strategy in R'. $\Box$

This brings us to the following conclusion.

Theorem 41 (Order independence III) *All iterated eliminations of strategies strictly dominated by a mixed strategy yield the same outcome.*

Proof. By Theorem 5 and the Hereditarity III Lemma 40. $\Box$

10.2 Elimination of weakly dominated strategies

Next, we consider iterated elimination of pure strategies that are weakly dominated by a mixed strategy.

As already noticed in Chapter 4 an elimination by means of weakly dominated strategies can result in a loss of Nash equilibria. Clearly, the same observation applies here. On the other hand, as in the case of pure strategies, we can establish a partial result, where we refer to the reduction relation $\rightarrow_{WM}$ with the expected meaning.

Lemma 42 (Mixed Weak Elimination) *Given a finite strategic game G consider two restrictions R and R' of G such that $R \rightarrow_{WM} R'$.*

If m is a Nash equilibrium of R', then it is a Nash equilibrium of R.

Proof. It suffices to note that both the proofs of the Persistence Lemma 37 and of the ($\Leftarrow$) implication of the Strict Mixed Elimination Lemma 38 apply without any changes to weak dominance, as well. $\Box$

This brings us to the following counterpart of the IEWDS Theorem 9, where we refer to Nash equilibria in mixed strategies. Instead of 'the iterated elimination of strategies weakly dominated by a mixed strategy' we write ***IEWDMS***.

Theorem 43 (IEWDMS) *Suppose that G is a finite strategic game.*

(i) If G' is an outcome of IEWDMS from G and m is a Nash equilibrium of G', then m is a Nash equilibrium of G.

(ii) If G is solved by IEWDMS, then the resulting joint strategy is a Nash equilibrium of G.

Proof. By the Mixed Weak Elimination Lemma 42. □

Here is a simple application of this theorem.

Corollary 44 *Every mixed extension of a finite strategic game has a Nash equilibrium such that no strategy used in it is weakly dominated by a mixed strategy.*

Proof. It suffices to apply Nash Theorem 30 to an outcome of IEWDMS and use item (i) of the above theorem. □

Finally, observe that the outcome of IEWMDS does not need to be unique. In fact, Example 11 applies here, as well. It is instructive to note where the proof of the Order independence III Theorem 41 breaks down. It happens in the very last step of the proof of the Hereditarity III Lemma 40. Namely, if $R \rightarrow_{WM} R'$ and a strategy $s_i \in R'_i$ is weakly dominated in R by a mixed strategy in R', then we cannot conclude that s_i is weakly dominated in R' by a mixed strategy in R'.

10.3 Rationalizability

Finally, we consider iterated elimination of strategies that are never best responses to a joint mixed strategy of the opponents. Strategies that survive such an elimination process are called rationalizable strategies.

Formally, we define rationalizable strategies as follows. Consider a restriction R of a finite strategic game G. Let

$$\mathcal{RAT}(R) := (S'_1, \ldots, S'_n),$$

where for all $i \in \{1, \ldots, n\}$

$$S'_i := \{s_i \in R_i \mid \exists m_{-i} \in \times_{j \neq i} \Delta R_j \ s_i \text{ is a best response to } m_{-i} \text{ in } G\}.$$

Note the use of G instead of R in the definition of S'_i. We shall comment on it in below.

Consider now the outcome $G_{\mathcal{RAT}}$ of iterating $\mathcal{RAT}$ starting with G. We call then the strategies present in the restriction $G_{\mathcal{RAT}}$ ***rationalizable***.

We have the following counterpart of the IESDMS Theorem 39.

Theorem 45 *Assume a finite strategic game G.*

(i) Then m is a Nash equilibrium of G iff it is a Nash equilibrium of $G_{\mathcal{RAT}}$.

(ii) If each player has in $G_{\mathcal{RAT}}$ exactly one strategy, then the resulting joint strategy is a unique Nash equilibrium of G.

In the context of rationalizability a joint mixed strategy of the opponents is referred to as a ***belief***. The definition of rationalizability is generic in the class of beliefs w.r.t. which best responses are collected. For example, we could use here joint pure strategies of the opponents, or probability distributions over the Cartesian product of the opponents' strategy sets, so the elements of the set ΔS_{-i} (extending in an expected way the payoff functions). In the first case we talk about ***point beliefs*** and in the second case about ***correlated beliefs***.

In the case of point beliefs we can apply the elimination procedure entailed by $\mathcal{RAT}$ to arbitrary games. To avoid discussion of the outcomes reached in the case of infinite iterations we focus on a result for a limited case. We refer here to Nash equilibria in pure strategies.

Theorem 46 *Assume a strategic game G. Consider the definition of the $\mathcal{RAT}$ operator for the case of point beliefs and suppose that the outcome $G_{\mathcal{RAT}}$ is reached in finitely many steps.*

(i) Then s is a Nash equilibrium of G iff it is a Nash equilibrium of $G_{\mathcal{RAT}}$.

(ii) If each player is left in $G_{\mathcal{RAT}}$ with exactly one strategy, then the resulting joint strategy is a unique Nash equilibrium of G.

A subtle point is that when G is infinite, the restriction $G_{\mathcal{RAT}}$ may have empty strategy sets (and hence no joint strategy).

Example 20 ***Bertrand competition*** is a game concerned with a simultaneous selection of prices for the same product by two firms. The product is then sold by the firm that chose a lower price. In the case of a tie the product is sold by both firms and the profits are split.

Consider a version in which the range of possible prices is the left-open real interval $(0, 100]$ and the demand equals $100 - p$, where p is the lower price. So in this game G there are two players, each with the set $(0, 100]$ of strategies and the payoff functions are defined by:

$$p_1(s_1, s_2) := \begin{cases} s_1(100 - s_1) & \text{if } s_1 < s_2 \\ \dfrac{s_1(100 - s_1)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 > s_2 \end{cases}$$

$$p_2(s_1, s_2) := \begin{cases} s_2(100 - s_2) & \text{if } s_2 < s_1 \\ \dfrac{s_2(100 - s_2)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_2 > s_1 \end{cases}$$

Consider now each player's best responses to the strategies of the opponent. Since $s_1 = 50$ maximizes the value of $s_1(100 - s_1)$ in the interval $(0, 100]$, the strategy 50 is the unique best response of the first player to any strategy $s_2 > 50$ of the second player. Further, no strategy is a best response to a strategy $s_2 \leq 50$. By symmetry the same holds for the strategies of the second player.

So the elimination of never best responses leaves each player with a single strategy, 50. In the second round we need to consider the best responses to these two strategies in the *original* game G. In G the strategy $s_1 = 49$ is a better response to $s_2 = 50$ than $s_1 = 50$ and symmetrically for the second player. So in the second round of elimination both strategies 50 are eliminated and we reach the restriction with the empty strategy sets. By Theorem 46 we conclude that the original game G has no Nash equilibrium. □

Note that if we defined S'_i in the definition of the operator $\mathcal{RAT}$ using the restriction R instead of the original game G, the iteration would stop in the above example after the first round. Such a modified definition of the $\mathcal{RAT}$ operator is actually an instance of the IENBR (iterated elimination of never best responses) in which at each stage all never best responses are eliminated. So for the above game G we can then conclude by the IENBR

Theorem 11(i) that it has at most one equilibrium, namely $(50, 50)$, and then check separately that in fact it is not a Nash equilibrium.

Exercise 13 Show that the beauty contest game is indeed solved by IESDMS in 99 rounds. □

10.4 A comparison between the introduced notions

We introduced so far the notions of strict dominance, weak dominance, and a best response, and related them to the notion of a Nash equilibrium. To conclude this section we clarify the connections between the notions of dominance and of best response.

Clearly, if a strategy is strictly dominated, then it is a never best response. However, the converse fails. Further, there is no relation between the notions of weak dominance and never best response. Indeed, in the game considered in Section 4.2 strategy C is a never best response, yet it is neither strictly nor weakly dominated. Further, in the game given in Example 11 strategy M is weakly dominated and is also a best response to B.

The situation changes in the case of mixed extensions of two-player finite games. Below by a ***totally mixed strategy*** we mean a mixed strategy with full support, i.e., one in which each strategy is used with a strictly positive probability. We have the following results.

Theorem 47 *Consider a finite two-player strategic game.*

(i) A pure strategy is strictly dominated by a mixed strategy iff it is not a best response to a mixed strategy.

(ii) A pure strategy is weakly dominated by a mixed strategy iff it is not a best response to a totally mixed strategy.

We only prove here part (i).

We shall use the following result.

Theorem 48 (Separating Hyperplane) *Let A and B be disjoint convex subsets of $\mathbb{R}^k$. Then there exists a nonzero $c \in \mathbb{R}^k$ and $d \in \mathbb{R}$ such that*

$$c \cdot x \geq d \text{ for all } x \in A,$$

$$c \cdot y \leq d \text{ for all } y \in B.$$

Proof of Theorem 47(i).

Clearly, if a pure strategy is strictly dominated by a mixed strategy, then it is not a best response to a mixed strategy. To prove the converse fix a two-player strategic game (S_1, S_2, p_1, p_2). Also fix $i \in \{1, 2\}$.

Suppose that a strategy $s_i \in S_i$ is not strictly dominated by a mixed strategy. Let

$$A := \{x \in \mathbb{R}^{|S_{-i}|} \mid \forall s_{-i} \in S_{-i} \; x_{s_{-i}} > 0\}$$

and

$$B := \{(p_i(m_i, s_{-i}) - p_i(s_i, s_{-i}))_{s_{-i} \in S_{-i}} \mid m_i \in \Delta S_i\}.$$

By the choice of s_i the sets A and B are disjoint. Moreover, both sets are convex subsets of $\mathbb{R}^{|S_{-i}|}$.

By the Separating Hyperplane Theorem 48 for some nonzero $c \in \mathbb{R}^{|S_{-i}|}$ and $d \in \mathbb{R}$

$$c \cdot x \geq d \text{ for all } x \in A, \tag{10.2}$$

$$c \cdot y \leq d \text{ for all } y \in B. \tag{10.3}$$

But $\mathbf{0} \in B$, so by (10.3) $d \geq 0$. Hence by (10.2) and the definition of A for all $s_{-i} \in S_{-i}$ we have $c_{s_{-i}} \geq 0$. Again by (10.2) and the definition of A this excludes the contingency that $d > 0$, i.e., $d = 0$. Hence by (10.3)

$$\textstyle\sum_{s_{-i} \in S_{-i}} c_{s_{-i}} p_i(m_i, s_{-i}) \leq \sum_{s_{-i} \in S_{-i}} c_{s_{-i}} p_i(s_i, s_{-i}) \text{ for all } m_i \in \Delta S_i. \tag{10.4}$$

Let $\bar{c} := \sum_{s_{-i} \in S_{-i}} c_{s_{-i}}$. By the assumption $\bar{c} \neq 0$. Take

$$m_{-i} := \sum_{s_{-i} \in S_{-i}} \frac{c_{s_{-i}}}{\bar{c}} s_{-i}.$$

Then (10.4) can be rewritten as

$$p_i(m_i, m_{-i}) \leq p_i(s_i, m_{-i}) \text{ for all } m_i \in \Delta S_i,$$

i.e., s_i is a best response to m_{-i}. $\square$