Chapter 12

Pre-Bayesian Games

Mechanism design, as introduced in the previous chapter, can be explained in game-theoretic terms using pre-Bayesian games In strategic games, after each player selected his strategy, each player knows the payoff of *every other player*. This is not the case in pre-Bayesian games in which each player has a private type on which he can condition his strategy. This distinguishing feature of pre-Bayesian games explains why they form a class of ***games with incomplete information***. Formally, they are defined as follows.

Assume a set $\{1, \ldots, n\}$ of players, where $n > 1$. A ***pre-Bayesian game*** for n players consists of

- a non-empty set A_i of ***actions***,
- a non-empty set Θ_i of ***types***,
- a ***payoff function*** $p_i : A_1 \times \ldots \times A_n \times \Theta_i \rightarrow \mathbb{R}$,

for each player i.

Let $A := A_1 \times \ldots \times A_n$. In a pre-Bayesian game Nature (an external agent) moves first and provides each player i with a type $\theta_i \in \Theta_i$. Each player knows only his type. Subsequently the players simultaneously select their actions. The payoff function of each player now depends on his type, so after all players selected their actions, each player knows his payoff but does not know the payoffs of the other players. Note that given a pre-Bayesian game, every joint type $\theta \in \Theta$ uniquely determines a strategic game, to which we refer below as a θ-game.

A ***strategy*** for player i in a pre-Bayesian game is a function $s_i : \Theta_i \rightarrow A_i$. A strategy $s_i(\cdot)$ for player i is called

- ***best response*** to the joint strategy $s_{-i}(\cdot)$ of the opponents of i if for all $a_i \in A_i$ and $\theta \in \Theta$

$$p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq p_i(a_i, s_{-i}(\theta_{-i}), \theta_i),$$

- ***dominant*** if for all $a \in A$ and $\theta_i \in \Theta_i$

$$p_i(s_i(\theta_i), a_{-i}, \theta_i) \geq p_i(a_i, a_{-i}, \theta_i),$$

Then a joint strategy $s(\cdot)$ is called an ***ex-post equilibrium*** if each $s_i(\cdot)$ is a best response to $s_{-i}(\cdot)$. Alternatively, $s(\cdot) := (s_1(\cdot), \ldots, s_n(\cdot))$ is an ex-post equilibrium if

$$\forall\theta \in \Theta \; \forall i \in \{1, \ldots, n\} \; \forall a_i \in A_i \; p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq p_i(a_i, s_{-i}(\theta_{-i}), \theta_i),$$

where $s_{-i}(\theta_{-i})$ is an abbreviation for the sequence of actions $(s_j(\theta_j))_{j \neq i}$.

So $s(\cdot)$ is an ex-post equilibrium iff for every joint type $\theta \in \Theta$ the sequence of actions $(s_1(\theta_1), \ldots, s_n(\theta_n))$ is a Nash-equilibrium in the corresponding θ-game. Further, $s_i(\cdot)$ is a dominant strategy of player i iff for every type $\theta_i \in \Theta_i$, $s_i(\theta_i)$ is a dominant strategy of player i in every (θ_i, θ_{-i})-game.

We also have the following immediate observation.

Note 56 (Dominant Strategy) *Consider a pre-Bayesian game G. Suppose that $s(\cdot)$ is a joint strategy such that each $s_i(\cdot)$ is a dominant strategy. Then it is an ex-post equilibrium of G.* □

Example 26 As an example of a pre-Bayesian game, suppose that

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$,

and consider the pre-Bayesian game uniquely determined by the following four θ-games. Here and below we marked the payoffs in Nash equilibria in these θ-games in bold.

L

U		F	B
	F	$\mathbf{2, 1}$	$2, 0$
	B	$0, 1$	$2, 1$

R

	F	B
F	$2, 0$	$\mathbf{2, 1}$
B	$0, 0$	$2, 1$

		F	B
D	F	$3,1$	$2,0$
	B	$\mathbf{5,1}$	$4,1$

	F	B
F	$3,0$	$2,1$
B	$5,0$	$\mathbf{4,1}$

This shows that the strategies $s_1(\cdot)$ and $s_2(\cdot)$ such that

$$s_1(U) := F,\ s_1(D) := B,\ s_2(L) = F,\ s_2(R) = B$$

form here an ex-post equilibrium. □

However, there is a crucial difference between strategic games and pre-Bayesian games. We call a pre-Bayesian game ***finite*** if each set of actions and each set of types is finite. By the ***mixed extension*** of a finite pre-Bayesian game

$$(A_1, \ldots, A_n, \Theta_1, \ldots, \Theta_n, p_1, \ldots, p_n)$$

we mean below the pre-Bayesian game

$$(\Delta A_1, \ldots, \Delta A_n, \Theta_1, \ldots, \Theta_n, p_1, \ldots, p_n).$$

Example 27 Consider the following pre-Bayesian game:

- $\Theta_1 = \{U, B\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{C, D\}$,

L

		C	D
U	C	$2,2$	$0,0$
	D	$3,0$	$\mathbf{1,1}$

		C	D
B	C	$\mathbf{1,2}$	$3,0$
	D	$0,0$	$2,1$

R

	C	D
C	$2,1$	$0,0$
D	$3,0$	$\mathbf{1,2}$

	C	D
C	$\mathbf{1,1}$	$3,0$
D	$0,0$	$2,2$

Even though each θ-game has a Nash equilibrium, they are so 'positioned' that the pre-Bayesian game has no ex-post equilibrium. Even more, if we consider a mixed extension of this game, then the situation does not change. The reason is that no new Nash equilibria are then added to the original θ-games.

Indeed, each of these original θ-games is solved by IESDS and hence by the IESDMS Theorem 39(ii) has a unique Nash equilibrium. This shows that a mixed extension of a finite pre-Bayesian game does not need to have an ex-post equilibrium, which contrasts with the existence of Nash equilibria in mixed extensions of finite strategic games. $\Box$

This motivates the introduction of a new notion of an equilibrium. A strategy $s_i(\cdot)$ for player i is called ***safety-level best response*** to the joint strategy $s_{-i}(\cdot)$ of the opponents of i if for all strategies $s'_i(\cdot)$ of player i and all $\theta_i \in \Theta_i$

$$\min_{\theta_{-i} \in \Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq \min_{\theta_{-i} \in \Theta_{-i}} p_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

Then a joint strategy $s(\cdot)$ is called a ***safety-level equilibrium*** if each $s_i(\cdot)$ is a safety-level best response to $s_{-i}(\cdot)$.

The following theorem was established by Monderer and Tennenholz.

Theorem 57 *Every mixed extension of a finite pre-Bayesian game has a safety-level equilibrium.* $\Box$

We now relate pre-Bayesian games to mechanism design. To this end we need one more notion. We say that a pre-Bayesian game is of a ***revelation-type*** if $A_i = \Theta_i$ for all $i \in \{1, \ldots, n\}$. So in a revelation-type pre-Bayesian game the strategies of a player are the functions on his set of types. A strategy for player i is called then ***truth-telling*** if it is the identity function $\pi_i(\cdot)$ on Θ_i.

Now mechanism design can be viewed as an instance of the revelation-type pre-Bayesian games. Indeed, we have the following immediate, yet revealing observation.

Theorem 58 *Given a direct mechanism*

$$(D \times \mathbb{R}^n, \Theta_1, \ldots, \Theta_n, u_1, \ldots, u_n, (f, t))$$

associate with it a revelation-type pre-Bayesian game, in which each payoff function p_i is defined by

$$p_i((\theta'_i, \theta_{-i}), \theta_i) := u_i((f, t)(\theta'_i, \theta_{-i}), \theta_i).$$

Then the mechanism is incentive compatible iff in the associated pre-Bayesian game for each player truth-telling is a dominant strategy.

By Groves Theorem 49 we conclude that in the pre-Bayesian game associated with a Groves mechanism, $(\pi_1(\cdot), \ldots, \pi_n(\cdot))$ is a dominant strategy ex-post equilibrium.