Chapter 2

Nash Equilibria and Pareto Efficient Outcomes

To discuss strategic games in a meaningful way we need to introduce further, natural, concepts. Fix a strategic game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.

We call a joint strategy s a ***Pareto efficient outcome*** if for no joint strategy s'

$$\forall i \in \{1, \ldots, n\}\ p_i(s') \geq p_i(s) \text{ and } \exists i \in \{1, \ldots, n\}\ p_i(s') > p_i(s).$$

That is, a joint strategy is a Pareto efficient outcome if no joint strategy is both a weakly better outcome for all players and a strictly better outcome for some player.

Further, given a joint strategy s we call the sum $\sum_{j=1}^{n} p_j(s)$ the ***social welfare*** of s. Next, we call a joint strategy s a ***social optimum*** if the social welfare of s is maximal.

Clearly, if s is a social optimum, then s is Pareto efficient. The converse obviously does not hold. Indeed, in the Prisoner's Dilemma game the joint strategis (C, D) and (D, C) are both Pareto efficient, but their social welfare is not maximal. Note that (D, D) is the only outcome that is not Pareto efficient. The social optimum is reached in the strategy profile (C, C). In contrast, the social welfare is smallest in the Nash equlibrium (D, D).

This discrepancy between Nash equilibria and Pareto efficient outcomes is absent in the Battle of Sexes game. Indeed, here both concepts coincide.

The tension between Nash equlibria and Pareto efficient outcomes present in the Prisoner's Dilemma game occurs in several other natural games. It forms one of the fundamental topics in the theory of strategic games. In this chapter we shall illustrate this phenomenon by a number of examples.

Example 3 (Prisoner's Dilemma for n players)
First, the Prisoner's Dilemma game can be easily generalized to n players as follows. It is convenient to assume that each player has two strategies, 1, representing cooperation, (formerly C) and 0, representing defection, (formerly D). Then, given a joint strategy s_{-i} of the opponents of player i, $\sum_{j \neq i} s_j$ denotes the number of 1 strategies in s_{-i}. Denote by $\mathbf{1}$ the joint strategy in which each strategy equals 1 and similarly with $\mathbf{0}$.

We put

$$p_i(s) := \begin{cases} 2\sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\ 2\sum_{j \neq i} s_j & \text{if } s_i = 1 \end{cases}$$

Note that for $n = 2$ we get the original Prisoner's Dilemma game.

It is easy to check that the strategy profile $\mathbf{0}$ is the unique Nash equilibrium in this game. Indeed, in each other strategy profile a player who chose 1 (cooperate) gets a higher payoff when he switches to 0 (defect).

Finally, note that the social welfare in $\mathbf{1}$ is $2n(n-1)$, which is strictly more than n, the social welfare in $\mathbf{0}$. We now show that $2n(n-1)$ is the social optimum. To this end it suffices to note that if a single player switches from 0 to 1, then his payoff decreases by 1 but the payoff of each other player increases by 2, and hence the social welfare increases. □

The next example deals with the depletion of ***common resources***, which in economics are goods that are not *excludable* (people cannot be prevented from using them) but are *rival* (one person's use of them diminishes another person's enjoyment of it). Examples are congested toll-free roads, fish in the ocean, or the environment. The overuse of such common resources leads to their destruction. This phenomenon is called the ***tragedy of the commons***.

One way to model it is as a Prisoner's dilemma game for n players. But such a modeling is too crude as it does not reflect the essential characteristics of the problem. We provide two more adequate modeling of it, one for the case of a binary decision (for instance, whether to use a congested road or not), and another one for the case when one decides about the intensity of using the resource (for instance on what fraction of a lake should one fish).

Example 4 (Tragedy of the commons I)

Assume $n > 1$ players, each having to its disposal two strategies, 1 and 0 reflecting, respectively, that the player decides to use the common resource or not. If he does not use the resource, he gets a fixed payoff. Further, the users of the resource get the same payoff. Finally, the more users of the common resource the smaller payoff for each of them gets, and when the number of users exceeds a certain threshold it is better for the other players not to use the resource.

The following payoff function realizes these assumptions:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0 \\ F(m)/m & \text{otherwise} \end{cases}$$

where $m = \sum_{j=1}^{n} s_j$ and

$$F(m) := 1.1m - 0.1m^2.$$

Indeed, the function $F(m)/m$ is strictly decreasing. Moreover, $F(9)/9 = 0.2$, $F(10)/10 = 0.1$ and $F(11)/11 = 0$. So when there are already ten or more users of the resource it is indeed better for other players not to use the resource.

To find a Nash equilibrium of this game, note that given a strategy profile s with $m = \sum_{j=1}^{n} s_j$ player i profits from switching from s_i to another strategy in precisely two circumstances:

- $s_i = 0$ and $F(m+1)/(m+1) > 0.1$,
- $s_i = 1$ and $F(m)/m < 0.1$.

In the first case we have $m + 1 < 10$ and in the second case $m > 10$.

Hence when $n < 10$ the only Nash equilibrium is when all players use the common resource and when $n \geq 10$ then s is a Nash equilibrium when either 9 or 10 players use the common resource.

Assume now that $n \geq 10$. Then in a Nash equilibrium s the players who use the resource receive the payoff 0.2 (when $m = 9$) or 0.1 (when $m = 10$). So the maximum social welfare that can be achieved in a Nash equilibrium is $0.1(n - 9) + 1.8 = 0.1n + 0.9$.

To find a strategy profile in which social optimum is reached with the largest social welfare we need to find m for which the function $0.1(n - m) + F(m)$ reaches the maximum. Now, $0.1(n - m) + F(m) = 0.1n + m - 0.1m^2$

and by elementary calculus we find that $m = 5$ for which $0.1(n-m)+F(m) = 0.1n+2.5$. So the social optimum is achieved when 5 players use the common resource. □

Example 5 (Tragedy of the commons II)
Assume $n > 1$ players, each having to its disposal an infinite set of strategies that consists of the real interval $[0, 1]$. View player's strategy as its chosen fraction of the common resource. Then the following payoff function reflects the fact that player's enjoyment of the common resource depends positively from his chosen fraction of the resource and negatively from the total fraction of the common resource used by all players:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The second alternative reflects the phenomenon that if the total fraction of the common resource by all players exceeds a feasible level, here 1, then player's enjoyment of the resource becomes zero. We can write the payoff function in a more compact way as

$$p_i(s) := \max(0, s_i(1 - \sum_{j=1}^{n} s_j)).$$

To find a Nash equilibrium of this game, fix $i \in \{1, \ldots, n\}$ and s_{-i} and denote $\sum_{j \neq i} s_j$ by t. Then $p_i(s_i, s_{-i}) = \max(0, s_i(1 - t - s_i))$.

By elementary calculus player's i payoff becomes maximal when $s_i = \frac{1-t}{2}$. This implies that when for all $i \in \{1, \ldots, n\}$ we have

$$s_i = \frac{1 - \sum_{j \neq i} s_j}{2},$$

then s is a Nash equilibrium. This system of n linear equations has a unique solution $s_i = \frac{1}{n+1}$ for $i \in \{1, \ldots, n\}$. In this strategy profile each player's payoff is $\frac{1-n/(n+1)}{n+1} = \frac{1}{(n+1)^2}$, so its social welfare is $\frac{n}{(n+1)^2}$.

There are other Nash equilibria. Indeed, suppose that for all $i \in \{1, \ldots, n\}$ we have $\sum_{j \neq i} s_j \geq 1$, which is the case for instance when $s_i = \frac{1}{n-1}$ for $i \in \{1, \ldots, n\}$. It is straightforward to check that each such strategy profile is a Nash equilibrium in which each player's payoff is 0 and hence the social welfare is also 0. It is easy to check that no other Nash equilibria exist.

To find a strategy profile in which social optimum is reached fix a strategy profile s and let $t := \sum_{j=1}^{n} s_j$. First note that if $t > 1$, then the social welfare is 0. So assume that $t \leq 1$. Then $\sum_{j=1}^{n} p_j(s_j) = t(1-t)$. By elementary calculus this expression becomes maximal precisely when $t = \frac{1}{2}$ and then it equals $\frac{1}{4}$.

Now, for all $n > 1$ we have $\frac{n}{(n+1)^2} < \frac{1}{4}$. So the social welfare of each solution for which $\sum_{j=1}^{n} s_j = \frac{1}{2}$ is superior to the social welfare of the Nash equilibria. In particular, no such strategy profile is a Nash equilibrium.

In conclusion, the social welfare is maximal, and equals $\frac{1}{4}$, when precisely half of the common resource is used. In contrast, in the 'best' Nash equilibrium the social welfare is $\frac{n}{(n+1)^2}$ and the fraction $\frac{n}{n+1}$ of the common resource is used. So when the number of players increases, the social welfare of the best Nash equilibrium becomes arbitrarily small, while the fraction of the common resource being used becomes arbitrarily large. □

The analysis carried out in the above two examples reveals that for the adopted payoff functions the common resource will be overused, to the detriment of the players (society). The same conclusion can be drawn for a much larger of class payoff functions that properly reflect the characteristics of using a common resource.

Example 6 (Cournot competition)
This example deals with a situation in which n companies independently decide their production levels of a given product. The price of the product is a linear function that depends negatively on the total output.

We model it by means of the following strategic game. We assume that for each player i:

- his strategy set is $\mathbb{R}_+$,
- his payoff function is defined by

$$p_i(s) := s_i(a - b\sum_{j=1}^{n} s_j) - cs_i$$

for some given a, b, c, where $a > c$ and $b > 0$.

Let us explain this payoff function. The price of the product is represented by the expression $a - b\sum_{j=1}^{n} s_j$, which, thanks to the assumption $b > 0$, indeed depends negatively on the total output, $\sum_{j=1}^{n} s_j$. Further, cs_i is the production cost corresponding to the production level s_i. So we assume for simplicity that the production cost functions are the same for all companies.

Further, note that if $a \leq c$, then the payoffs would be always negative or zero, since $p_i(s) = (a-c)s_i - bs_i \sum_{j=1}^{n} s_j$. This explains the assumption that $a > c$. For simplicity we do allow a possibility that the prices are negative, but see Exercise 4. The assumption $c > 0$ is obviously meaningful but not needed.

To find a Nash equilibrium of this game fix $i \in \{1, \ldots, n\}$ and s_{-i} and denote $\sum_{j \neq i} s_j$ by t. Then $p_i(s_i, s_{-i}) = s_i(a - c - bt - bs_i)$. By elementary calculus player's i payoff becomes maximal iff

$$s_i = \frac{a-c}{2b} - \frac{t}{2}.$$

This implies that s is a Nash equilibrium iff for all $i \in \{1, \ldots, n\}$

$$s_i = \frac{a-c}{2b} - \frac{\sum_{j \neq i} s_j}{2}.$$

One can check that this system of n linear equations has a unique solution, $s_i = \frac{a-c}{(n+1)b}$ for $i \in \{1, \ldots, n\}$. So this is a unique Nash equilibrium of this game.

Note that for these values of s_is the price of the product is

$$a - b\sum_{j=1}^{n} s_j = a - b\frac{n(a-c)}{(n+1)b} = \frac{a+nc}{n+1}.$$

To find the social optimum let $t := \sum_{j=1}^{n} s_j$. Then $\sum_{j=1}^{n} p_j(s) = t(a - c - bt)$. By elementary calculus this expression becomes maximal precisely when $t = \frac{a-c}{2b}$. So s is a social optimum iff $\sum_{j=1}^{n} s_j = \frac{a-c}{2b}$. The price of the product in a social optimum is $a - b\frac{a-c}{2b} = \frac{a+c}{2}$.

Now, the assumption $a > c$ implies that $\frac{a+c}{2} > \frac{a+nc}{n+1}$. So we see that the price in the social optimum is strictly higher than in the Nash equilibrium. This can be interpreted as a statement that the competition between the producers of the product drives its price down, or alternatively, that the cartel between the producers leads to higher profits for them (i.e., higher social welfare), at the cost of a higher price. So in this example reaching the social optimum is not a desirable state of affairs. The reason is that in our analysis we focussed only on the profits of the producers and omitted the customers.

As an aside also notice that when n, so the number of companies, increases, the price $\frac{a+nc}{n+1}$ in the Nash equilibrium decreases. This corresponds to the intuition that increased competition is beneficial for the customers. Note also that in the limit the price in the Nash equilibrium converges to the production cost c. □

While the last two examples refer to completely different scenarios, their mathematical analysis is very similar. Their common characteristics is that in both examples the payoff functions can be written as $f(s_i, \sum_{j=1}^{n} s_j)$, where f is increasing in the first argument and decreasing in the second argument.

Exercise 3 Prove that in the game discussed in Example 5 indeed no other Nash equilibria exist apart of the mentioned ones. □

Exercise 4 Modify the game from Example 6 by considering the following payoff functions:

$$p_i(s) := s_i \max(0, a - b \sum_{j=1}^{n} s_j) - cs_i.$$

Compute the Nash equilibria of this game.
Hint. Proceed as in Example 5. □