Chapter 4

Weak Dominance and Never Best Responses

Let us return now to our analysis of an arbitrary strategic game $G := (S_1, \ldots, S_n,\ p_1, \ldots, p_n)$. Let s_i, s'_i be strategies of player i. We say that s_i ***weakly dominates*** s'_i (or equivalently, that s'_i is ***weakly dominated by*** s_i) if

$$\forall s_{-i} \in S_{-i}\ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}) \text{ and } \exists s_{-i} \in S_{-i}\ p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}).$$

Further, we say that s_i is ***weakly dominant*** if it weakly dominates all other strategies of player i.

4.1 Elimination of weakly dominated strategies

Analogous considerations to the ones concerning strict dominance can be carried out for the elimination of weakly dominated strategies. To this end we consider the reduction relation $\rightarrow_W$ on the restrictions of G, defined by

$$R \rightarrow_W R'$$

when $R \neq R'$, $R' \subseteq R$ and

$$\forall i \in \{1, \ldots, n\}\ \forall s_i \in R_i \setminus R'_i\ \exists s'_i \in R_i\ s_i \text{ is weakly dominated in } R \text{ by } s'_i.$$

Below we abbreviate iterated elimination of weakly dominated strategies to ***IEWDS***.

However, in the case of IEWDS some complications arise. To illustrate them consider the following game that results from equipping each player in the Matching Pennies game with a third strategy E (for Edge):

	H	T	E
H	$1, -1$	$-1, \ \ 1$	$-1, -1$
T	$-1, \ \ 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

Note that

- (E, E) is its only Nash equilibrium,
- for each player E is the only strategy that is weakly dominated.

Any form of elimination of these two E strategies, simultaneous or iterated, yields the same outcome, namely the Matching Pennies game, that, as we have already noticed, has no Nash equilibrium. So during this eliminating process we 'lost' the only Nash equilibrium. In other words, part (i) of the IESDS Theorem 2 does not hold when reformulated for weak dominance.

On the other hand, some partial results are still valid here. As before we prove first a lemma that clarifies the situation.

Lemma 8 (Weak Elimination) *Given a finite strategic game G consider two restrictions R and R' of G such that $R \rightarrow_W R'$. Then if s is a Nash equilibrium of R', then it is a Nash equilibrium of R.*

Proof. Suppose s is a Nash equilibrium of R' but not a Nash equilibrium of R. Then for some $i \in \{1, \ldots, n\}$ the set

$$A := \{s'_i \in R_i \mid p_i(s'_i, s_{-i}) > p_i(s)\}$$

is non-empty.

Weak dominance is a strict partial ordering (i.e. an irreflexive transitive relation) and A is finite, so some strategy s'_i in A is not weakly dominated in R by any strategy in A. But each strategy in A is eliminated in the reduction

$R \rightarrow_W R'$ since s is a Nash equilibrium of R'. So some strategy $s_i^* \in R_i$ weakly dominates s'_i in R. Consequently

$$p_i(s_i^*, s_{-i}) \geq p_i(s'_i, s_{-i})$$

and as a result $s_i^* \in A$. But this contradicts the choice of s'_i. □

This brings us directly to the following result.

Theorem 9 (IEWDS) *Suppose that G is a finite strategic game.*

(i) If G' is an outcome of IEWDS from G and s is a Nash equilibrium of G', then s is a Nash equilibrium of G.

(ii) If G is solved by IEWDS, then the resulting joint strategy is a Nash equilibrium of G.

Proof. By the Weak Elimination Lemma 8. □

Corollary 10 (Weak Dominance) *Consider a finite strategic game G.*

Suppose that s is a joint strategy such that each s_i is a weakly dominant strategy. Then it is a Nash equilibrium of G.

Proof. By the IEWDS Theorem 9(*ii*). □

Note that in contrast to the Strict Dominance Corollary 3 we do not claim here that s is a unique Nash equilibrium of G. In fact, such a stronger claim does not hold. Indeed, consider the game

	L	R
T	$1,1$	$1,1$
B	$1,1$	$0,0$

Here T is a weakly dominant strategy for the player 1, L is a weakly dominant strategy for player 2 and, as prescribed by the above Note, (T, L), is a Nash equilibrium. However, this game has two other Nash equilibria, (T, R) and (B, L).

Example 10 Let us return to the beauty contest game introduced in Example 2 of Chapter 1. One can check that this game is solved by IEWDS and results in the joint strategy $(1, \ldots, 1)$. Hence, we can conclude by the IEWDS Theorem 9 this joint strategy is a (not necessarily unique; we shall return to this question in a later chapter) Nash equilibrium. □

Note that in contrast to the IESDS Theorem 2 we do not claim in part (*ii*) of the IEWDS Theorem 9 that the resulting joint strategy is a *unique* Nash equilibrium. In fact, such a stronger claim does not hold. Further, in contrast to strict dominance, an iterated elimination of weakly dominated strategies can yield several outcomes.

The following example reveals even more peculiarities of this procedure.

Example 11 Consider the following game:

	L	M	R
T	$0,1$	$1,0$	$0,0$
B	$0,0$	$0,0$	$1,0$

It has three Nash equlibria, (T, L), (B, L) and (B, R). This game can be solved by IEWDS but only if in the first round we do not eliminate all weakly dominated strategies, which are M and R. If we eliminate only R, then we reach the game

	L	M
T	$0,1$	$1,0$
B	$0,0$	$0,0$

that is solved by IEWDS by eliminating B and M. This yields

	L
T	$0,1$

So not only IEWDS is not order independent; in some games it is advantageous *not* to proceed with the deletion of the weakly dominated strategies 'at full speed'. One can also check that the second Nash equilibrium, (B, L), can be found using IEWDS, as well, but not the third one, (B, R). □

It is instructive to see where the proof of order independence given in the Appendix of the previous chapter breaks down in the case of weak dominance. This proof crucially relied on the fact that the relation of being strictly dominated is hereditary. In contrast, the relation of being weakly dominated is not hereditary.

To summarize, the iterated elimination of weakly dominated strategies

- can lead to a deletion of Nash equilibria,

- does not need to yield a unique outcome,
- can be too restrictive if we stipulate that in each round all weakly dominated strategies are eliminated.

Finally, note that the above IEWDS Theorem 9 does not hold for infinite games. Indeed, Example 9 applies here, as well.

4.2 Elimination of never best responses

Iterated elimination of strictly or weakly dominated strategies allow us to solve various games. However, several games cannot be solved using them.

For example, consider the following game:

	X	Y
A	$2,1$	$0,0$
B	$0,1$	$2,0$
C	$1,1$	$1,2$

Here no strategy is strictly or weakly dominated. On the other hand C is a ***never best response***, that is, it is not a best response to any strategy of the opponent. Indeed, A is a unique best response to X and B is a unique best response to Y. Clearly, the above game is solved by an iterated elimination of never best responses. So this procedure can be stronger than IESDS and IEWDS.

Formally, we introduce the following reduction notion between the restrictions R and R' of a given strategic game G:

$$R \rightarrow_N R'$$

when $R \neq R'$, $R' \subseteq R$ and

$$\forall i \in \{1, \ldots, n\}\ \forall s_i \in R_i \setminus R'_i\ \neg \exists s_{-i} \in R_{-i}\ s_i \text{ is a best response to } s_{-i} \text{ in } R.$$

That is, $R \rightarrow_N R'$ when R' results from R by removing from it some strategies that are never best responses. Note that in contrast to strict and weak dominance there is now no 'witness' strategy that acounts for a removal of a strategy.

We now focus on the iterated elimination of never best responses, in short ***IENBR***, obtained by using the $\rightarrow_N^*$ relation. The following counterpart of the IESDS Theorem 2 holds.

Theorem 11 (IENBR) *Suppose that G' is an outcome of IENBR from a strategic game G.*

(i) If s is a Nash equilibrium of G, then it is a Nash equilibrium of G'.

(ii) If G is finite and s is a Nash equilibrium of G', then it is a Nash equilibrium of G.

(iii) If G is finite and solved by IENBR, then the resulting joint strategy is a unique Nash equilibrium.

Proof. Analogous to the proof of the IESDS Theorem 2 and omitted. □

Further, we have the following analogue of the Hereditarity I Lemma 7.

Lemma 12 (Hereditarity II) *The relation of never being a best response is hereditary on the set of restrictions of a given finite game.*

Proof. Suppose a strategy $s_i \in R'_i$ is a never best response in R and $R \rightarrow_N R'$. Assume by contradiction that for some $s_{-i} \in R'_{-i}$, s_i is a best response to s_{-i} in R', i.e.,

$$\forall s'_i \in R'_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

The initial game is finite, so there exists a best response s'_i to s_{-i} in R. Then s'_i is not eliminated in the step $R \rightarrow_N R'$ and hence is a strategy in R'_i. But by the choice of s_i

$$p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}),$$

so we reached a contradiction. □

This leads us to the following analogue of the Order Independence I Theorem 4.

Theorem 13 (Order Independence II) *Given a finite strategic game all iterated eliminations of never best responses yield the same outcome.*

Proof. By Theorem 5 and the Hereditarity II Lemma 12. □

In the case of infinite games we encounter the same problems as in the case of IESDS. Indeed, Example 9 readily applies to IENBR, as well, since in this game no strategy is a best response. In particular, this example shows that if we solve an infinite game by IENBR we cannot claim that we obtained a Nash equilibrium. Still, IENBR can be useful in such cases.

Example 12 Consider the following infinite variant of the location game considered in Example 8. We assume that the players choose their strategies from the open interval $(0, 100)$ and that at each real in $(0, 100)$ there resides one customer. We have then the following payoffs that correspond to the intuition that the customers choose the closest vendor:

$$p_i(s_i, s_{-i}) := \begin{cases} \dfrac{s_i + s_{-i}}{2} & \text{if } s_i < s_{-i} \\ 100 - \dfrac{s_i + s_{-i}}{2} & \text{if } s_i > s_{-i} \\ 50 & \text{if } s_i = s_{-i} \end{cases}$$

It is easy to check that in this game no strategy strictly or weakly dominates another one. On the other hand each strategy 50 is a best response (namely to strategy 50 of the opponent) and no other strategies are best responses. So this game is solved by IENBR, in one step.

We cannot claim automatically that the resulting joint strategy $(50, 50)$ is a Nash equilibrium, but it is clearly so since each strategy 50 is a best response to the 'other' strategy 50. Moreover, by the IENBR Theorem 11(i) we know that this is a unique Nash equilibrium. □

Exercise 6 Show that the beauty contest game from Example 2 is indeed solved by IEWDS. □

Exercise 7 Show that in the location game from Example 12 indeed no strategy is strictly or weakly dominated. □

Exercise 8 Given a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ we say that that a strategy s_i of player i is ***dominant*** if for all strategies s'_i of player i

$$p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

Suppose that s is a joint strategy such that each s_i is a dominant strategy. Prove that it is a Nash equilibrium of G. □