Chapter 6

Strictly Competitive Games

In this chapter we discuss a special class of two-player games for which stronger results concerning Nash equilibria can be established. To study them we shall crucially rely on the notions introduced in Section 5.2, namely security strategies and $maxmin_i$ and $minmax_i$.

More specifically, we introduce a natural class of two-player games for which the equalities between the $maxmin_i$ and $minmax_i$ values for $i = 1, 2$ constitute a necessary and sufficient condition for the existence of a Nash equilibrium. In these games any Nash equilibrium consists of a pair of security strategies.

A ***strictly competitive game*** is a two-player strategic game (S_1, S_2, p_1, p_2) in which for $i = 1, 2$ and any two joint strategies s and s'

$$p_i(s) \geq p_i(s') \text{ iff } p_{-i}(s) \leq p_{-i}(s').$$

That is, a joint strategy that is better for one player is worse for the other player. This formalizes the intuition that the interests of both players are diametrically opposed and explains the terminology.

By negating both sides of the above equivalence we get

$$p_i(s) < p_i(s') \text{ iff } p_{-i}(s) > p_{-i}(s').$$

So an alternative way of defining a strictly competitive game is by stating that this is a two-player game in which every joint strategy is a Pareto efficient outcome.

To illustrate this concept let us fill in the game considered in Example 14 the payoffs for the column player in such a way that the game becomes strictly competitive:

	L	M	R
T	$3,4$	$4,3$	$5,2$
B	$6,0$	$2,5$	$1,6$

Canonic examples of strictly competitive games are ***zero-sum games***. These are two-player games in which for each joint strategy s we have

$$p_1(s) + p_2(s) = 0.$$

So a zero-sum game is an extreme form of a strictly competitive game in which whatever one player 'wins', the other one 'loses'. A simple example is the Matching Pennies game from Chapter 1.

Another well-known zero-sum game is the Rock, Paper, Scissors game. In this game, often played by children, both players simultaneously make a sign with a hand that identifies one of these three objects. If both players make the same sign, the game is a draw. Otherwise one player wins, say, 1 Euro from the other player according to the following rules:

- the rock defeats (breaks) scissors,
- scissors defeat (cut) the paper,
- the paper defeats (wraps) the rock.

Since in a zero-sum game the payoff for the second player is just the negative of the payoff for the first player, each zero-sum game can be represented in a simplified form, called ***reward matrix***. It is simply the matrix that represents only the payoffs for the first player. So the reward matrix for the Rock, Paper, Scissors game looks as follows:

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

For the strictly competitive games, so a fortiori the zero-sum games, the following counterpart of the Lower Bound Lemma 16 holds.

Lemma 17 (Upper Bound) *Consider a strictly competitive game $G := (S_1, S_2, p_1, p_2)$. If (s_1^*, s_2^*) is a Nash equilibrium of G, then for $i = 1, 2$*

(i) $p_i(s_i^*, s_{-i}^*) \leq \min_{s_{-i}} p_i(s_i^*, s_{-i})$,

(ii) $p_i(s_i^*, s_{-i}^*) \leq maxmin_i$.

Both items provide an upper bound on the payoff in each Nash equilibrium, which explains the name of the lemma.

Proof.

(i) Fix i. Suppose that (s_i^*, s_{-i}^*) is a Nash equilibrium of G. Fix s_{-i}. By the definition of Nash equilibrium

$$p_{-i}(s_i^*, s_{-i}^*) \geq p_{-i}(s_i^*, s_{-i}),$$

so, since G is strictly competitive,

$$p_i(s_i^*, s_{-i}^*) \leq p_i(s_i^*, s_{-i}).$$

But s_{-i} was arbitrary, so

$$p_i(s_i^*, s_{-i}^*) \leq \min_{s_{-i}} p_i(s_i^*, s_{-i}).$$

(ii) By definition

$$\min_{s_{-i}} p_i(s_i^*, s_{-i}) \leq \max_{s_i} \min_{s_{-i}} p_i(s_i, s_{-i}),$$

so by (i)

$$p_i(s_i^*, s_{-i}^*) \leq \max_{s_i} \min_{s_{-i}} p_i(s_i, s_{-i}).$$

$\square$

Combining the Lower Bound Lemma 16 and the Upper Bound Lemma 17 we can draw the following conclusions about strictly competitive games.

Theorem 18 (Strictly Competitive Games) *Consider a strictly competitive game G.*

(i) If for $i = 1, 2$ we have $maxmin_i = minmax_i$, then G has a Nash equilibrium.

(ii) If G has a Nash equilibrium, then for $i = 1, 2$ we have $maxmin_i = minmax_i$.

(iii) All Nash equilibria of G yield the same payoff, namely $maxmin_i$ for player i.

(iv) All Nash equilibria of G are of the form (s_1^*, s_2^*) where each s_i^* is a security strategy for player i.

Proof. Suppose $G = (S_1, S_2, p_1, p_2)$.
(i) Fix i. Let s_i^* be a security strategy for player i, i.e., such that $\min_{s_{-i}} p_i(s_i^*, s_{-i}) = maxmin_i$, and let s_{-i}^* be such that $\max_{s_i} p_i(s_i, s_{-i}^*) = minmax_i$. We show that (s_i^*, s_{-i}^*) is a Nash equilibrium of G.

We already noted in the proof of the Lower Bound Lemma 16(i) that

$$maxmin_i = \min_{s_{-i}} p_i(s_i^*, s_{-i}) \leq p_i(s_i^*, s_{-i}^*) \leq \max_{s_i} p_i(s_i, s_{-i}^*) = minmax_i.$$

But now $maxmin_i = minmax_i$, so all these values are equal. In particular

$$p_i(s_i^*, s_{-i}^*) = \max_{s_i} p_i(s_i, s_{-i}^*) \tag{6.1}$$

and

$$p_i(s_i^*, s_{-i}^*) = \min_{s_{-i}} p_i(s_i^*, s_{-i}).$$

Fix now s_{-i}. By the last equality

$$p_i(s_i^*, s_{-i}^*) \leq p_i(s_i^*, s_{-i}),$$

so, since G is strictly competitive,

$$p_{-i}(s_i^*, s_{-i}^*) \geq p_{-i}(s_i^*, s_{-i}).$$

But s_{-i} was arbitrary, so

$$p_{-i}(s_i^*, s_{-i}^*) = \max_{s_{-i}} p_{-i}(s_i^*, s_{-i}). \tag{6.2}$$

Now (6.1) and (6.2) mean that indeed (s_i^*, s_{-i}^*) is a Nash equilibrium of G.

(ii) and (iii) If s is a Nash equilibrium of G, by the Lower Bound Lemma 16(i) and (ii) and the Upper Bound Lemma 17(ii) we have for $i = 1, 2$

$$maxmin_i \leq minmax_i \leq p_i(s) \leq maxmin_i.$$

So all these values are equal.

(iv) Fix i. Take a Nash equilibrium (s_i^*, s_{-i}^*) of G. We always have

$$\min_{s_{-i}} p_i(s_i^*, s_{-i}) \leq p_i(s_i^*, s_{-i}^*)$$

and by the Upper Bound Lemma 17(i) we also have

$$p_i(s_i^*, s_{-i}^*) \leq \min_{s_{-i}} p_i(s_i^*, s_{-i}).$$

So

$$\min_{s_{-i}} p_i(s_i^*, s_{-i}) = p_i(s_i^*, s_{-i}^*) = maxmin_i,$$

where the last equality holds by (iii). So s_i^* is a security strategy for player i. □

Combining (i) and (ii) we see that a strictly competitive game has a Nash equilibrium iff for $i = 1, 2$ we have $maxmin_i = minmax_i$. So in a strictly competitive game each player can determine whether a Nash equilibrium exists without knowing the payoff of the other player. All what he needs to know is that the game is strictly competitive. Indeed, each player i then just needs to check whether his $maxmin_i$ and $minmax_i$ values are equal.

Morever, by (iv), each player can select on his own a strategy that forms a part of a Nash equilibrium: it is simply any of his security strategies.

6.1 Zero-sum games

Let us focus now on the special case of zero-sum games. We first show that for zero-sum games the $maxmin_i$ and $minmax_i$ values for one player can be directly computed from the corresponding values for the other player.

Theorem 19 (Zero-sum) *Consider a zero-sum game* (S_1, S_2, p_1, p_2). *For* $i = 1, 2$ *we have*

$$maxmin_i = -minmax_{-i}$$

and

$$minmax_i = -maxmin_{-i}.$$

Proof. Fix i. For each joint strategy (s_i, s_{-i})

$$p_i(s_i, s_{-i}) = -p_{-i}(s_i, s_{-i}),$$

so

$$\max_{s_i} \min_{s_{-i}} p_i(s_i, s_{-i}) = \max_{s_i}(\min_{s_{-i}} -p_{-i}(s_i, s_{-i})) = -\min_{s_i} \max_{s_{-i}} p_{-i}(s_i, s_{-i}).$$

This proves the first equality. By interchanging i and $-i$ we get the second equality. □

It follows by the Strictly Competitive Games Theorem 18(i) that for zero-sum games a Nash equilibrium exists iff $maxmin_1 = minmax_1$. When this condition holds in a zero-sum game, any pair of security strategies for both players is called a ***saddle point*** of the game and the common value of $maxmin_1$ and $minmax_1$ is called the ***value*** of the game.

Example 15 To illustrate the introduced concepts consider the zero-sum game represented by the following reward matrix:

	L	M	R
T	4	3	5
B	6	2	1

To compute $maxmin_1$ and $minmax_1$, as in Example 14, we extend the matrix with an additional row and column and fill in the minima of the rows and the maxima of the columns:

	L	M	R	f_1
T	4	3	5	3
B	6	2	1	1
F_1	6	3	5	

We see that $maxmin_1 = minmax_1 = 3$. So 3 is the value of this game. Moreover, (T, M) is the only pair of the security strategies of the row and column players, i.e., the only saddle point in this game. □

The above result does not hold for arbitrary strictly competitive games. To see it notice that in any two-player game a multiplication of the payoffs of player i by 2 leads to the doubling of the value of $maxmin_i$ and it does not affect the value of $minmax_{-i}$. Moreover, this multiplication procedure does not affect the property that a game is strictly competitive.

In an arbitrary strategic game with multiple Nash equilibria, for example the Battle of the Sexes game, the players face the following coordination problem. Suppose that each of them chooses a strategy from a Nash equilibrium. Then it can happen that this way they selected a joint strategy that is not a Nash equilibrium. For instance, in the Battle of the Sexes game the players can choose respectively F and B. The following result shows that in a zero-sum game such a coordination problem does not exist.

Theorem 20 (Interchangeability) *Consider a zero-sum game G.*

(i) Suppose that a Nash equilibrium of G exists. Then any joint strategy (s_1^*, s_2^*) such that each s_i^* is a security strategy for player i is a Nash equilibrium of G.

(ii) Suppose that (s_1^*, s_2^*) and (t_1^*, t_2^*) are Nash equilibria of G. Then so are (s_1^*, t_2^*) and (t_1^*, s_2^*).

Proof.
(i) Let (s_1^*, s_2^*) be a pair of security strategies for players 1 and 2. Fix i. By definition

$$\min_{s_i} p_{-i}(s_i, s_{-i}^*) = maxmin_{-i}. \tag{6.3}$$

But

$$\min_{s_i} p_{-i}(s_i, s_{-i}^*) = \min_{s_i} -p_i(s_i, s_{-i}^*) = -\max_{s_i} p_i(s_i, s_{-i}^*)$$

and by the Zero-sum Theorem 19

$$maxmin_{-i} = -minmax_i.$$

So (6.3) implies

$$\max_{s_i} p_i(s_i, s_{-i}^*) = minmax_i. \tag{6.4}$$

We now rely on the Strictly Competitive Games Theorem 18. By item (ii) for $j = 1, 2$ we have $maxmin_j = minmax_j$, so by the proof of item (i) and (6.4) we conclude that (s_i^*, s_{-i}^*) is a Nash equilibrium.

(ii) By (i) and the Strictly Competitive Games Theorem 18(iv). $\square$

The assumption that a Nash equilibrium exists is obviously necessary in item (i) of the above theorem. Indeed, in the finite zero-sum games security strategies always exist, in contrast to the Nash equilibrium.

Finally, recall that throughout this chapter we assumed the existence of various minima and maxima. So the results of this chapter apply only to a specific class of strictly competitive and zero-sum games. This class includes finite games. We shall return to this matter in a later chapter.