Chapter 7

Sealed-bid Auctions

An ***auction*** is a procedure used for selling and buying items by offering them up for bid. Auctions are often used to sell objects that have a variable price (for example oil) or an undetermined price (for example radio frequencies). There are several types of auctions. In its most general form they can involve multiple buyers and multiple sellers with multiple items being offered for sale, possibly in succession. Moreover, some items can be sold in fractions, for example oil.

Here we shall limit our attention to a simple situation in which only one seller exists and offers one object for sale that has to be sold in its entirety (for example a painting). So in this case an auction is a procedure that involves

- one seller who offers an object for sale,
- n bidders, each bidder i having a valuation $v_i \geq 0$ of the object.

The procedure we discuss here involves submission of ***sealed bids***. More precisely, the bidders simultaneously submit their bids in closed envelopes and the object is allocated, in exchange for a payment, to the bidder who submitted the highest bid (the ***winner***). Such an auction is called a ***sealed-bid auction***. To keep things simple we assume that when more than one bidder submitted the highest bid the object is allocated to the highest bidder with the lowest index.

To formulate a sealed-bid auction as a strategic game we consider each bidder as a player. Then we view each bid of player i as his possible strategy. We allow any nonnegative real as a bid.

We assume that the valuations v_i are fixed and publicly known. This is an unrealistic assumption to which we shall return in a later chapter. However, this assumption is necessary, since the valuations are used in the definition of the payoff functions and by assumption the players have common knowledge of the game and hence of each others' payoff functions. When defining the payoff functions we consider two options, each being determined by the underlying payment procedure.

Given a sequence $b := (b_1, \ldots, b_n)$ of reals, we denote the least l such that $b_l = \max_{k \in \{1,\ldots,n\}} b_k$ by $\text{argsmax}\, b$. That is, $\text{argsmax}\, b$ is the smallest index l such that b_l is a largest element in the sequence b. For example, $\text{argsmax}\,(6, 7, 7, 5) = 2$.

7.1 First-price auction

The most commonly used rule in a sealed-bid auction is that the winner i pays to the seller the amount equal to his bid. The resulting mechanism is called the ***first-price auction***.

Assume the winner is bidder i, whose bid is b_i. Since his value for the sold object is v_i, his payoff (profit) is $v_i - b_i$. For the other players the payoff (profit) is 0. Note that the winner's profit can be negative. This happens when he wins the object by ***overbidding***, i.e., submitting a bid higher than his valuation of the object being sold. Such a situation is called the ***winner's curse***.

To summarize, the payoff function p_i of player i in the game associated with the first-price auction is defined as follows, where b is the vector of the submitted bids:

$$p_i(b) := \begin{cases} v_i - b_i & \text{if } i = \text{argsmax}\, b \\ 0 & \text{otherwise} \end{cases}$$

Let us now analyze the resulting game. The following theorem provides a complete characterization of its Nash equilibria.

Theorem 21 (Characterization I) *Consider the game associated with the first-price auction with the players' valuations v. Then b is a Nash equilibrium iff for $i = \text{argsmax}\, b$*

(i) $b_i \leq v_i$

(the winner does not suffer from the winner's curse),

(ii) $\max_{j \neq i} v_j \leq b_i$

(the winner submitted a sufficiently high bid),

(iii) $b_i = \max_{j \neq i} b_j$

(another player submitted the same bid as player i).

These three conditions can be compressed into the single statement

$$\max_{j \neq i} v_j \leq \max_{j \neq i} b_j = b_i \leq v_i,$$

where $i = \text{argsmax} b$. Also note that (i) and (ii) imply that $v_i = \max v$, which means that in every Nash equilibrium a player with the highest valuation is the winner.

Proof.

$(\Rightarrow)$

(i) If $b_i > v_i$, then player's i payoff is negative and it increases to 0 if he submits the bid equal to v_i.

(ii) If $\max_{j \neq i} v_j > b_i$, then player j such that $v_j > b_i$ can win the object by submitting a bid in the open interval (b_i, v_j), say $v_j - \epsilon$. Then his payoff increases from 0 to ϵ.

(iii) If $b_i > \max_{j \neq i} b_j$, then player i can increase his payoff by submitting a bid in the open interval $(\max_{j \neq i} b_j, b_i)$, say $b_i - \epsilon$. Then his payoff increases from $v_i - b_i$ to $v_i - b_i + \epsilon$.

So if any of the conditions $(i) - (iii)$ is violated, then b is not a Nash equilibrium.

$(\Leftarrow)$ Suppose that a vector of bids b satisfies $(i) - (iii)$. Player i is the winner and by (i) his payoff is non-negative. His payoff can increase only if he bids less, but then by (iii) another player (the one who initially submitted the same bid as player i) becomes the winner, while player's i payoff becomes 0.

The payoff of any other player j is 0 and can increase only if he bids more and becomes the winner. But then by (ii), $\max_{j \neq i} v_j < b_j$, so his payoff becomes negative.

So b is a Nash equilibrium. $\square$

As an illustration of the above theorem suppose that the vector of the valuations is $(1, 6, 5, 2)$. Then the vectors of bids $(1, 5, 5, 2)$ and $(1, 5, 2, 5)$ satisfy the above three conditions and are both Nash equilibria. The first vector of bids shows that player 2 can secure the object by bidding the second highest valuation. In the second vector of bids player 4 overbids but his payoff is 0 since he is not the winner.

By the ***truthful bidding*** we mean the vector b of bids, such that for each player i we have $b_i = v_i$, i.e., each player bids his own valuation. Note that by the Characterization Theorem 21 truthful bidding, i.e., v, is a Nash equilibrium iff the two highest valuations coincide.

Further, note that for no player i such that $v_i > 0$ his truthful bidding is a dominant strategy. Indeed, truthful bidding by player i always results in payoff 0. However, if all other players bid 0, then player i can increase his payoff by submitting a lower, positive bid.

Observe that the above analysis does not allow us to conclude that in each Nash equlibrium the winner is the player who wins in the case of truthful bidding. Indeed, suppose that the vector of valuations is $(0, 5, 5, 5)$, so that in the case of truthful bidding by all players player 2 is the winner. Then the vector of bids $(0, 4, 5, 5)$ is a Nash equilibrium with player 3 being the winner.

Finally, notice the following strange consequence of the above theorem: in no Nash equilibrium the last player, n, is a winner. The reason is that we resolved the ties in the favour of a bidder with the lowest index. Indeed, by item (*iii*) in every Nash equilibrium b we have $\text{argsmax}\, b < n$.

7.2 Second-price auction

We consider now an auction with the following payment rule. As before the winner is the bidder who submitted the highest bid (with a tie broken, as before, to the advantage of the bidder with the smallest index), but now he pays to the seller the amount equal to the *second* highest bid. This sealed-bid auction is called the ***second-price auction***. It was proposed by W. Vickrey and is alternatively called ***Vickrey auction***. So in this auction in the absence of ties the winner pays to the seller a lower price than in the first-price auction.

Let us formalize this auction as a game. The payoffs are now defined as follows:

$$p_i(b) := \begin{cases} v_i - \max_{j \neq i} b_j & \text{if } i = \text{argsmax}\, b \\ 0 & \text{otherwise} \end{cases}$$

Note that bidding v_i always yields a non-negative payoff but can now lead to a strictly positive payoff, which happens when v_i is a unique winning bid. However, when the highest two bids coincide the payoffs are still the same as in the first-price auction, since then for $i = \text{argsmax} b$ we have $b_i = \max_{j \neq i} b_j$. Finally, note that the winner's curse still can take place here, namely when $v_i < b_i$ and some other bid is in the open interval (v_i, b_i).

The analysis of the second-price auction as a game leads to different conclusions that for the first-price auction. The following theorem provides a complete characterization of the Nash equilibria of the corresponding game.

Theorem 22 (Characterization II) *Consider the game associated with the second-price auction with the players' valuations v. Then b is a Nash equilibrium iff for $i = argsmax\, b$*

(i) $\max_{j \neq i} v_j \leq b_i$

(the winner submitted a sufficiently high bid),

(ii) $\max_{j \neq i} b_j \leq v_i$

(the winner's valuation is sufficiently high).

Proof.

$(\Rightarrow)$

(i) If $\max_{j \neq i} v_j > b_i$, then player j such that $v_j > b_i$ can win the object by submitting a bid in the open interval (b_i, v_j). Then his payoff increases from 0 to $v_j - b_i$.

(ii) If $\max_{j \neq i} b_j > v_i$, then player's i payoff is negative, namely $v_i - \max_{j \neq i} b_j$, and can increase to 0 if player i submits a losing bid.

So if condition (i) or (ii) is violated, then b is not a Nash equilibrium.

$(\Leftarrow)$ Suppose that a vector of bids b satisfies (i) and (ii). Player i is the winner and by (ii) his payoff is non-negative. By submitting another bid he either remains a winner, with the same payoff, or becomes a loser with the payoff 0.

The payoff of any other player j is 0 and can increase only if he bids more and becomes the winner. But then his payoff becomes $v_j - b_i$, so by (i) becomes negative.

So b is a Nash equilibrium. $\square$

This characterization result shows that several Nash equilibria exist. We now exhibit three specific ones that are of particular interest. In each case it is straightforward to check that conditions (i) and (ii) of the above theorem hold.

Truthful bidding

Recall that in the case of the first-price auction truthful bidding is a Nash equilibrium iff for the considered sequence of valuations the auction coincides with the second-price auction. Now truthful bidding, so v, is always a Nash equilibrium. Below we prove another property of truthful bidding in second-price auction.

Wolf and sheep Nash equilibrium

Suppose that $i = \text{argsmax}\, v$, i.e., player i is the winner in the case of truthful bidding. Consider the strategy profile in which player i bids v_i and everybody else bids 0. This Nash equilibrium is called ***wolf and sheep***, where player i plays the role of a wolf by bidding aggressively and scaring the sheep being the other players who submit their minimal bids.

Yet another Nash equilibrium

Finally, we exhibit a Nash equilibrium in which the player with the uniquely highest valuation is not a winner. This is in contrast with what we observed in the case of the first-price auction. Suppose that the two highest bids are v_j and v_i, where $i < j$ and $v_j > v_i > 0$. Then the strategy profile in which player i bids v_j, player j bids v_i and everybody else bids 0 is a Nash equilibrium.

In both the first-price and the second-price auctions overbidding, i.e., submitting a bid above one's valuation of the object looks risky and therefore not credible. Note that the bids that do not exceed one's valuation are exactly the security strategies.

So when we add the following additional condition to each characterization theorem:

- for all $j \in \{1, \ldots, n\}$, $b_j \leq v_j$,

we characterize in each case Nash equilibria in the security strategies.

7.3 Incentive compatibility

So far we discussed two examples of sealed-bid auctions. A general form of such an auction is determined by fixing for each bidder i the payment procedure pay_i which given a sequence b of bids such that bidder i is the winner yields his payment.

In the resulting game, that we denote by $G_{pay,v}$, the payoff function is defined by

$$p_i(b) := \begin{cases} v_i - pay_i(b) & \text{if } i = \text{argsmax}\, b \\ 0 & \text{otherwise} \end{cases}$$

Intuitively, bidding 0 means that the bidder is not interested in the object. So if all players bid 0 then none of them is interested in the object. According to our definition the object is then allocated to the first bidder. We assume that then his payment is 0. That is, we stipulate that $pay_1(0, \ldots, 0) = 0$.

When designing a sealed-bid auction it is natural to try to induce the bidders to bid their valuations. This leads to the following notion.

We call a sealed-bid auction with the payment procedures $pay_1, \ldots, pay_n$ ***incentive compatible*** if for all sequences v of players' valuations for each bidder i his valuation v_i is a dominant strategy in the corresponding game $G_{pay,v}$.

While dominance of a strategy does not guarantee that a player will choose it, it ensures that deviating from it is not profitable. So dominance of each valuation v_i can be viewed as a statement that in the considered auction lying does not pay off.

We now show that the condition of incentive compatibility fully characterizes the corresponding auction. More precisely, the following result holds.

Theorem 23 (Second-price auction) *A sealed-bid auction is incentive compatible iff it is the second-price auction.*

Proof. Fix a sequence of the payment procedures $pay_1, \ldots, pay_n$ that determines the considered sealed-bid auction.

$(\Rightarrow)$ Choose an arbitrary sequence of bids that for the clarity of the argument we denote by (v_i, b_{-i}). Suppose that $i = \text{argsmax}\,(v_i, b_{-i})$. We establish the following four claims.

Claim 1. $pay_i(v_i, b_{-i}) \leq v_i$.

Proof. Suppose by contradiction that $pay_i(v_i, b_{-i}) > v_i$. Then in the corresponding game $G_{pay,v}$ we have $p_i(v_i, b_{-i}) < 0$. On the other hand $p_i(0, b_{-i}) \geq 0$. Indeed, if $i \neq \text{argsmax}\,(0, b_{-i})$, then $p_i(0, b_{-i}) = 0$. Otherwise all bids in b_{-i} are 0 and $i = 1$, and hence $p_i(0, b_{-i}) = v_i$, since by assumption $pay_1(0, \ldots, 0) = 0$.

This contradicts the assumption that v_i is a dominant strategy in the corresponding game $G_{pay,v}$.

Claim 2. For all $b_i \in (\max_{j \neq i} b_j, v_i)$ we have $pay_i(v_i, b_{-i}) \leq pay_i(b_i, b_{-i})$.

Proof. Suppose by contradiction that for some $b_i \in (\max_{j \neq i} b_j, v_i)$ we have $pay_i(v_i, b_{-i}) > pay_i(b_i, b_{-i})$. Then $i = \text{argsmax}\,(b_i, b_{-i})$ so

$$p_i(v_i, b_{-i}) = v_i - pay_i(v_i, b_{-i}) < v_i - pay_i(b_i, b_{-i}) = p_i(b_i, b_{-i}).$$

This contradicts the assumption that v_i is a dominant strategy in the corresponding game $G_{pay,v}$.

Claim 3. $pay_i(v_i, b_{-i}) \leq \max_{j \neq i} b_j$.

Proof. Suppose by contradiction that $pay_i(v_i, b_{-i}) > \max_{j \neq i} b_j$. Take some $v'_i \in (\max_{j \neq i} b_j, pay_i(v_i, b_{-i}))$. By Claim 1 $v'_i < v_i$, so by Claim 2 $pay_i(v_i, b_{-i}) \leq pay_i(v'_i, b_{-i})$. Further, by Claim 1 for the sequence (v'_i, v_{-i}) of valuations we have $pay_i(v'_i, b_{-i}) \leq v'_i$.

So $pay_i(v_i, b_{-i}) \leq v'_i$, which contradicts the choice of v'_i.

Claim 4. $pay_i(v_i, b_{-i}) \geq \max_{j \neq i} b_j$.

Proof. Suppose by contradiction that $pay_i(v_i, b_{-i}) < \max_{j \neq i} b_j$. Take an arbitrary $v'_i \in (pay_i(v_i, b_{-i}), \max_{j \neq i} b_j)$. Then $p_i(v'_i, b_{-i}) = 0$, while

$$p_i(v_i, b_{-i}) = v_i - pay_i(v_i, b_{-i}) > v_i - \max_{j \neq i} b_j \geq 0.$$

This contradicts the assumption that v'_i is a dominant strategy in the corresponding game $G_{pay,(v'_i, v_{-i})}$.

So we proved that for $i = \text{argsmax}\,(v_i, b_{-i})$ we have $pay_i(v_i, b_{-i}) = \max_{j \neq i} b_j$, which shows that the considered sealed-bid auction is second price.

($\Leftarrow$) We actually prove a stronger claim, namely that all sequences of valuations v each v_i is a weakly dominant strategy for player i.

To this end take a vector b of bids. By definition $p_i(b_i, b_{-i}) = 0$ or $p_i(b_i, b_{-i}) = v_i - \max_{j \neq i} b_j \leq p_i(v_i, b_{-i})$. But $0 \leq p_i(v_i, b_{-i})$, so

$$p_i(b_i, b_{-i}) \leq p_i(v_i, b_{-i}).$$

Consider now a bid $b_i \neq v_i$. If $b_i < v_i$, then take b_{-i} such that each element of it lies in the open interval (b_i, v_i). Then b_i is a losing bid and v_i is a winning bid and

$$p_i(b_i, b_{-i}) = 0 < v_i - \max_{j \neq i} b_j = p_i(v_i, b_{-i}).$$

If $b_i > v_i$, then take b_{-i} such that each element of it lies in the open interval (v_i, b_i). Then b_i is a winning bid and v_i is a losing bid and

$$p_i(b_i, b_{-i}) = v_i - \max_{j \neq i} b_j < 0 = p_i(v_i, b_{-i}).$$

So we proved that each strategy $b_i \neq v_i$ is weakly dominated by v_i, i.e., that v_i is a weakly dominant strategy. As an aside, recall that each weakly dominant strategy is unique, so we characterized bidding one's valuation in the second-price auction in game theoretic terms. $\square$

Exercise 9 Prove that the game associated with the first-price auction with the players' valuations v has no Nash equilibrium iff v_n is the unique highest valuation. $\square$