Chapter 9

Mixed Extensions

We now study a special case of infinite strategic games that are obtained in a canonic way from the finite games, by allowing mixed strategies. Below $[0, 1]$ stands for the real interval $\{r \in \mathbb{R} \mid 0 \leq r \leq 1\}$. By a ***probability distribution*** over a finite non-empty set A we mean a function

$$\pi : A \rightarrow [0, 1]$$

such that $\sum_{a \in A} \pi(a) = 1$. We denote the set of probability distributions over A by ΔA.

9.1 Mixed strategies

Consider now a finite strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$. By a ***mixed strategy*** of player i in G we mean a probability distribution over S_i. So ΔS_i is the set of mixed strategies available to player i. In what follows, we denote a mixed strategy of player i by m_i and a joint mixed strategy of the players by m.

Given a mixed strategy m_i of player i we define

$$support(m_i) := \{a \in S_i \mid m_i(a) > 0\}$$

and call this set the ***support*** of m_i. In specific examples we write a mixed strategy m_i as the sum $\sum_{a \in A} m_i(a) \cdot a$, where A is the support of m_i.

Note that in contrast to S_i the set ΔS_i is infinite. When referring to the mixed strategies, as in the previous chapters, we use the '$_{-i}$' notation. So for $m \in \Delta S_1 \times \ldots \times \Delta S_n$ we have $m_{-i} = (m_j)_{j \neq i}$, etc.

We can identify each strategy $s_i \in S_i$ with the mixed strategy that puts 'all the weight' on the strategy s_i. In this context s_i will be called a ***pure strategy***. Consequently we can view S_i as a subset of ΔS_i and S_{-i} as a subset of $\times_{j \neq i} \Delta S_j$.

By a ***mixed extension*** of $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ we mean the strategic game

$$(\Delta S_1, \ldots, \Delta S_n, p_1, \ldots, p_n),$$

where each function p_i is extended in a canonic way from $S := S_1 \times \ldots \times S_n$ to $M := \Delta S_1 \times \ldots \times \Delta S_n$ by first viewing each joint mixed strategy $m = (m_1, \ldots, m_n) \in M$ as a probability distribution over S, by putting for $s \in S$

$$m(s) := m_1(s_1) \cdot \ldots \cdot m_n(s_n),$$

and then by putting

$$p_i(m) := \sum_{s \in S} m(s) \cdot p_i(s).$$

Example 18 Reconsider the Battle of the Sexes game from Chapter 1. Suppose that player 1 (man) chooses the mixed strategy $\frac{1}{2}F + \frac{1}{2}B$, while player 2 (woman) chooses the mixed strategy $\frac{1}{4}F + \frac{3}{4}B$. This pair m of the mixed strategies determines a probability distribution over the set of joint strategies, that we list to the left of the bimatrix of the game:

	F	B
F	$\frac{1}{8}$	$\frac{3}{8}$
B	$\frac{1}{8}$	$\frac{3}{8}$

	F	B
F	$2,1$	$0,0$
B	$0,0$	$1,2$

To compute the payoff of player 1 for this mixed strategy m we multiply each of his payoffs for a joint strategy by its probability and sum it up:

$$p_1(m) = \frac{1}{8}2 + \frac{3}{8}0 + \frac{1}{8}0 + \frac{3}{8}1 = \frac{5}{8}.$$

Analogously

$$p_2(m) = \frac{1}{8}1 + \frac{3}{8}0 + \frac{1}{8}0 + \frac{3}{8}2 = \frac{7}{8}.$$

□

This example suggests the computation of the payoffs in two-player games using matrix multiplication. First, we view each bimatrix of such a game as a pair of matrices $(\mathbf{A}, \mathbf{B})$. The first matrix represents the payoffs to player 1 and the second one to player 1. Assume now that player 1 has k strategies and player 2 has ℓ strategies. Then both $\mathbf{A}$ and $\mathbf{B}$ are $k \times \ell$ matrices. Further, each mixed strategy of player 1 can be viewed as a row vector $\mathbf{p}$ of length k (i.e., a $1 \times k$ matrix) and each mixed strategy of player 2 as a row vector $\mathbf{q}$ of length ℓ (i.e., a $1 \times \ell$ matrix). Since $\mathbf{p}$ and $\mathbf{q}$ represent mixed strategies, we have $\mathbf{p} \in \Delta^{k-1}$ and $\mathbf{q} \in \Delta^{\ell-1}$, where for all $m \geq 0$

$$\Delta^{m-1} := \{(x_1, \ldots, x_m) \mid \sum_{i=1}^{m} x_i = 1 \text{ and } \forall i \in \{1, \ldots, m\}\ x_i \geq 0\}.$$

Δ^{m-1} is called the $(m-1)$-***dimensional unit simplex***.

In the case of our example we have

$$\mathbf{p} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \mathbf{q} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Now, the payoff functions can be defined as follows:

$$p_1(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \mathbf{p}_i \mathbf{q}_j \mathbf{A}_{ij} = \mathbf{p}\mathbf{A}\mathbf{q}^T$$

and

$$p_2(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \mathbf{p}_i \mathbf{q}_j \mathbf{B}_{ij} = \mathbf{p}\mathbf{B}\mathbf{q}^T.$$

9.2 Nash equilibria in mixed strategies

In the context of a mixed extension we talk about a ***pure Nash equlibrium***, when each of the constituent strategies is pure, and refer to an arbitrary Nash equilibrium of the mixed extension as a ***Nash equilibrium in mixed strategies*** of the initial finite game. In what follows, when we use the letter m we implicitly refer to the latter Nash equilibrium.

Below we shall need the following notion. Given a probability distribution π over a finite non-empty multiset[1] A of reals, we call

$$\sum_{r \in A} \pi(r) \cdot r$$

[1]This reference to a multiset is relevant.

a ***convex combination of the elements of*** A. For instance, given the multiset $A := \{\!\{4, 2, 2\}\!\}$, $\frac{1}{3}4 + \frac{1}{3}2 + \frac{1}{3}2$, so $\frac{8}{3}$, is a convex combination of the elements of A.

To see the use of this notion when discussing mixed strategies note that for every joint mixed strategy m we have

$$p_i(m) = \sum_{s_i \in support(m_i)} m_i(s_i) \cdot p_i(s_i, m_{-i}).$$

That is, $p_i(m)$ is a convex combination of the elements of the multiset

$$\{\!\{p_i(s_i, m_{-i}) \mid s_i \in support(m_i)\}\!\}.$$

We shall employ the following simple observations on convex combinations.

Note 27 (Convex Combination) *Consider a convex combination*

$$cc := \sum_{r \in A} \pi(r) \cdot r$$

of the elements of a finite multiset A of reals. Then

(i) $max\ A \geq cc$,

(ii) $cc \geq max\ A$ iff

- *$cc = r$ for all $r \in A$ such that $\pi(r) > 0$,*
- *$cc \geq r$ for all $r \in A$ such that $\pi(r) = 0$.*

□

Lemma 28 (Characterization) *Consider a finite strategic game*

$$(S_1, \ldots, S_n, p_1, \ldots, p_n).$$

The following statements are equivalent:

(i) m is a Nash equilibrium in mixed strategies, i.e.,

$$p_i(m) \geq p_i(m'_i, m_{-i})$$

for all $i \in \{1, \ldots, n\}$ and all $m'_i \in \Delta S_i$, *(ii)* *for all* $i \in \{1, \ldots, n\}$ *and all* $s_i \in S_i$

$$p_i(m) \geq p_i(s_i, m_{-i}),$$

(iii) *for all* $i \in \{1, \ldots, n\}$ *and all* $s_i \in support(m_i)$

$$p_i(m) = p_i(s_i, m_{-i})$$

and for all $i \in \{1, \ldots, n\}$ *and all* $s_i \notin support(m_i)$

$$p_i(m) \geq p_i(s_i, m_{-i}).$$

Note that the equivalence between *(i)* and *(ii)* implies that each Nash equilibrium of the initial game is a pure Nash equilibrium of the mixed extension. In turn, the equivalence between *(i)* and *(iii)* provides us with a straightforward way of testing whether a joint mixed strategy is a Nash equilibrium.

Proof.

(i) ⇒ *(ii)* Immediate.

(ii) ⇒ *(iii)* We noticed already that $p_i(m)$ is a convex combination of the elements of the multiset

$$A := \{\!\{ p_i(s_i, m_{-i}) \mid s_i \in support(m_i) \}\!\}.$$

So this implication is a consequence of part *(ii)* of the Convex Combination Note 27.

(iii) ⇒ *(i)* Consider the multiset

$$A := \{\!\{ p_i(s_i, m_{-i}) \mid s_i \in S_i \}\!\}.$$

But for all $m'_i \in \Delta S_i$, in particular m_i, the payoff $p_i(m'_i, m_{-i})$ is a convex combination of the elements of the multiset A.

So by the assumptions and part *(ii)* of the Convex Combination Note 27

$$p_i(m) \geq max\ A,$$

and by part *(i)* of the above Note

$$max\ A \geq p_i(m'_i, m_{-i}).$$

Hence $p_i(m) \geq p_i(m'_i, m_{-i})$. □

We now illustrate the use of the above theorem by finding in the Battle of the Sexes game a Nash equilibrium in mixed strategies, in addition to the two pure ones exhibited in Chapter 3. Take

$$\begin{aligned} m_1 &:= r_1 \cdot F + (1 - r_1) \cdot B, \\ m_2 &:= r_2 \cdot F + (1 - r_2) \cdot B, \end{aligned}$$

where $0 < r_1, r_2 < 1$. By definition

$$\begin{aligned} p_1(m_1, m_2) &= 2 \cdot r_1 \cdot r_2 + (1 - r_1) \cdot (1 - r_2), \\ p_2(m_1, m_2) &= r_1 \cdot r_2 + 2 \cdot (1 - r_1) \cdot (1 - r_2). \end{aligned}$$

Suppose now that (m_1, m_2) is a Nash equilibrium in mixed strategies. By the equivalence between *(i)* and *(iii)* of the Characterization Lemma 28 $p_1(F, m_2) = p_1(B, m_2)$, i.e., (using $r_1 = 1$ and $r_1 = 0$ in the above formula for $p_1(\cdot)$) $2 \cdot r_2 = 1 - r_2$, and $p_2(m_1, F) = p_2(m_1, B)$, i.e., (using $r_2 = 1$ and $r_2 = 0$ in the above formula for $p_2(\cdot)$) $r_1 = 2 \cdot (1 - r_1)$. So $r_2 = \frac{1}{3}$ and $r_1 = \frac{2}{3}$.

This implies that for these values of r_1 and r_2, (m_1, m_2) is a Nash equilibrium in mixed strategies and we have

$$p_1(m_1, m_2) = p_2(m_1, m_2) = \tfrac{2}{3}.$$

9.3 Nash theorem

We now establish a fundamental result about games that are mixed extensions. In what follows we shall use the following result from the calculus.

Theorem 29 (Extreme Value Theorem) *Suppose that A is a non-empty compact subset of $\mathbb{R}^n$ and*

$$f : A \rightarrow \mathbb{R}$$

is a continuous function. Then f attains a minimum and a maximum. □

The example of the Matching Pennies game illustrated that some strategic games do not have a Nash equilibrium. In the case of mixed extensions the situation changes and we have the following fundamental result established by J. Nash in 1950.

Theorem 30 (Nash) *Every mixed extension of a finite strategic game has a Nash equilibrium.*

In other words, every finite strategic game has a Nash equilibrium in mixed strategies. In the case of the Matching Pennies game it is straightforward to check that $(\frac{1}{2} \cdot H + \frac{1}{2} \cdot T, \frac{1}{2} \cdot H + \frac{1}{2} \cdot T)$ is such a Nash equilibrium. In this equilibrium the payoffs to each player are 0.

Nash Theorem follows directly from the following result.[2]

Theorem 31 (Kakutani) *Suppose that A is a non-empty compact and convex subset of $\mathbb{R}^n$ and*

$$\Phi : A \to \mathcal{P}(A)$$

such that

- *$\Phi(x)$ is non-empty and convex for all $x \in A$,*
- *the **graph** of Φ, so the set $\{(x, y) \mid y \in \Phi(x)\}$, is closed.*

Then $x^* \in A$ exists such that $x^* \in \Phi(x^*)$. □

Proof of Nash Theorem. Fix a finite strategic game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. Define the function $best_i : \times_{j \neq i} \Delta S_j \to \mathcal{P}(\Delta S_i)$ by

$$best_i(m_{-i}) := \{m_i \in \Delta S_i \mid m_i \text{ is a best response to } m_{-i}\}.$$

Then define the function $best : \Delta S_1 \times \ldots \times \Delta S_n \to \mathcal{P}(\Delta S_1 \times \ldots \times \Delta S_n)$ by

$$best(m) := best_1(m_{-1}) \times \ldots \times best_n(m_{-n}).$$

It is now straightforward to check that m is a Nash equilibrium iff $m \in best(m)$. Moreover, one easily can check that the function $best(\cdot)$ satisfies the conditions of Kakutani Theorem. The fact that for every joint mixed strategy m, $best(m)$ is non-empty is a direct consequence of the Extreme Value Theorem 29. □

Ever since Nash established his celebrated Theorem, a search has continued to generalize his result to a larger class of games. A motivation for this endevour has been existence of natural infinite games that are not mixed extensions of finite games. As an example of such an early result let us mention the following theorem stablished independently in 1952 by Debreu, Fan and Glickstein.

[2]Recall that a subset A of $\mathbb{R}^n$ is called ***compact*** if it is closed and bounded.

Theorem 32 *Consider a strategic game such that*

- *each strategy set is a non-empty compact convex subset of* $\mathbb{R}^n$,
- *each payoff function* p_i *is continuous and quasi-concave in the ith argument.*[3]

Then a Nash equilibrium exists.

More recent work in this area focused on existence of Nash equilibria in games with non-continuous payoff functions.

9.4 Minimax theorem

Let us return now to strictly competitive games that we studied in Chapter 6. First note the following lemma.

Lemma 33 *Consider a strategic game* $(S_1, \ldots, S_n, p_1, \ldots, p_n)$ *that is a mixed extension. Then*

(i) For all $s_i \in S_i$, $\min_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i})$ *exists.*

(ii) $\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i})$ *exists.*

(iii) For all $s_{-i} \in S_{-i}$, $\max_{s_i \in S_i} p_i(s_i, s_{-i})$ *exists.*

(iv) $\min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} p_i(s_i, s_{-i})$ *exists.*

Proof. It is a direct consequence of the Extreme Value Theorem 29. □

This lemma implies that we can apply the results of Chapter 6 to each strictly competitive game that is a mixed extension. Indeed, it ensures that the minima and maxima the existence of which we assumed in the proofs given there always exist. However, equipped with the knowledge that each such game has a Nash equilibrium we can now draw additional conclusions.

Theorem 34 *Consider a strictly competitive game that is a mixed extension. For* $i = 1, 2$ *we have* $maxmin_i = minmax_i$.

[3]Recall that the function $p_i : S \to \mathbb{R}$ is ***quasi-concave in the*** i***th argument*** if the set $\{s'_i \in S_i \mid p_i(s'_i, s_{-i}) \geq p_i(s)\}$ is convex for all $s \in S$.

Proof. By the Nash Theorem 30 and the Strictly Competitive Games Theorem 18(ii). □

The formulation 'a strictly competitive game that is a mixed extension' is rather awkward and it is tempting to write instead 'the mixed extension of a strictly competitive game'. However, one can show that the mixed extension of a strictly competitive game does not need to be a strictly competitive game, see Exercise 11.

On the other hand we have the following simple observation.

Note 35 (Mixed Extension) *The mixed extension of a zero-sum game is a zero-sum game.*

Proof. Fix a finite zero-sum game (S_1, S_2, p_1, p_2). For each joint strategy m we have

$$p_1(m)+p_2(m) = \sum_{s \in S} m(s)p_1(s)+\sum_{s \in S} m(s)p_2(s) = \sum_{s \in S} m(s)(p_1(s)+p_2(s)) = 0.$$

□

This means that for finite zero-sum games we have the following result, originally established by von Neumann in 1928.

Theorem 36 (Minimax) *Consider a finite zero-sum game* $G := (S_1, S_2, p_1, p_2)$. *Then for* $i = 1, 2$

$$\max_{m_i \in M_i} \min_{m_{-i} \in M_{-i}} p_i(m_i, m_{-i}) = \min_{m_{-i} \in M_{-i}} \max_{m_i \in M_i} p_i(m_i, m_{-i}).$$

Proof. By the Mixed Extension Note 35 the mixed extension of G is zero-sum, so strictly competitive. It suffices to use Theorem 34 and expand the definitions of $minmax_i$ and $maxmin_i$. □

Finally, note that using the matrix notation we can rewrite the above equalities as follows, where $\mathbf{A}$ is an arbitrary $k \times \ell$ matrix (that is the reward matrix of a zero-sum game):

$$\max_{\mathbf{p} \in \Delta^{k-1}} \min_{\mathbf{q} \in \Delta^{\ell-1}} \mathbf{pAq}^T = \min_{\mathbf{q} \in \Delta^{\ell-1}} \max_{\mathbf{p} \in \Delta^{k-1}} \mathbf{pAq}^T.$$

So the Minimax Theorem can be alternatively viewed as a theorem about matrices and unit simplices. This formulation of the Minimax Theorem has been generalized in many ways to a statement

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y),$$

where X and Y are appropriate sets replacing the unit simplices and $f : X \times Y \rightarrow \mathbb{R}$ is an appropriate function replacing the payoff function. Such theorems are called Minimax theorems.

Exercise 11 Find a 2×2 strictly competitive game such that its mixed extension is not a strictly competitive game.

Exercise 12 Prove that the Matching Pennies game has exactly one Nash equilibrium in mixed strategies. □