Mechanism Design

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Overview

- Decision problems.
- Direct mechanisms.
- Groves mechanisms.
- Examples.

Intelligent Design

Finance & Economics

Economics focus

Intelligent design

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A theory of an intelligently guided invisible hand wins the Nobel prize

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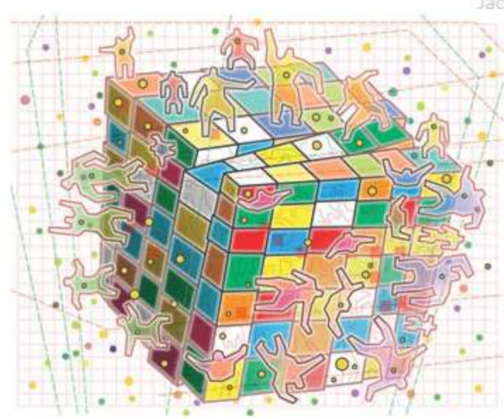

"WHAT on earth is mechanism design?" was the typical reaction to this year's Nobel prize in economics, announced on October 15th. In this era of "Freakonomics", in which everyone is discovering their inner economist, economics has become unexpectedly sexy. So what possessed the Nobel committee to honour a subject that sounds so thoroughly dismal? Why didn't they follow the lead of the peace-prize judges, who know not to let technicalities about being true to the meaning of the award get in the way of good headlines?

In fact, despite its dreary name, mechanism design is a hugely important area of economics, and underpins much of what dismal scientists do today. It goes to the heart of one of the biggest challenges in economics: how to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like. The word "mechanism" refers to the institutions and the rules of the game that govern our economic activities, which can range from a Ministry of Planning in a command economy to the internal organisation of a company to trading in a market.

Intelligent Design

A theory of an intelligently guided invisible hand wins the Nobel prize

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(The Economist, Oct. 18th, 2007)

Decision Problems

Decision problem for n players:

- set D of decisions,
- for each player i a set of (private) types Θ_i
- and a utility function

$$v_i : D \times \Theta_i \rightarrow \mathcal{R}.$$

- Intuitions
  - Type is some private information known only to the player (e.g., player's valuation of the item for sale),
  - $v_i(d, \theta_i)$ represents the benefit to player i of type θ_i from the decision $d \in D$.
- Assume the individual types are $\theta_1, \ldots, \theta_n$. Then $\sum_{i=1}^{n} v_i(d, \theta_i)$ is the social welfare from $d \in D$.

Decision Rules

- Decision rule is a function

$$f : \Theta_1 \times \ldots \times \Theta_n \rightarrow D.$$

- Decision rule f is efficient if

$$\sum_{i=1}^{n} v_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} v_i(d, \theta_i)$$

  for all $\theta \in \Theta$ and $d \in D$.

- Intuition f is efficient if it always maximizes the social welfare.

Set up

- Each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ'_i,
- the central authority computes decision

$$d := f(\theta'_1, \ldots, \theta'_n),$$

and communicates it to each player i.

Basic problem How to ensure that $\theta'_i = \theta_i$.

Example 1: Sealed-Bid Auction

Set up There is a single object for sale. Each player is a buyer. The decision is taken by means of a sealed-bid auction. The object is sold to the highest bidder.

- $D = \{1, \ldots, n\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(d, \theta_i) := \begin{cases} \theta_i & \text{if } d = i \\ 0 & \text{otherwise} \end{cases}$
- Let argsmax $\theta := \mu i(\theta_i = \max_{j \in \{1,\ldots,n\}} \theta_j)$.
- $f(\theta) :=$ argsmax θ.
- Note f is efficient.
- Payments will be treated later.

Example 2: Public Project Problem

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Each person is asked to report his or
her willingness to pay for the
project, and the project is undertaken
if and only if the aggregate reported
willingness to pay exceeds the cost of
the project.
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(15 October 2007, The Royal Swedish Academy of Sciences, Press Release, Scientific Background)

Public Project Problem, formally

- c: cost of the public project (e.g., building a bridge),
- $D = \{0, 1\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(d, \theta_i) := d(\theta_i - \frac{c}{n})$,
- $f(\theta) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$
- Note f is efficient.

Example 3: Reversed Sealed-bid Auction

Set up Each player offers the same service. The decision is taken by means of a sealed-bid auction. The service is purchased from the lowest bidder.

- $D = \{1, \ldots, n\}$,
- each Θ_i is $\mathbb{R}_-$;
  $-\theta_i$ is the price player i offers,
- $$v_i(d, \theta_i) := \begin{cases} \theta_i & \text{if } d = i \\ 0 & \text{otherwise} \end{cases}$$
- $f(\theta) :=$ argsmax θ.

Example $f(-8, -5, -4, -6) = 3$. That is, given the offers $8, 5, 4, 6$, the service is bought from player 3.

Example 4: Buying a Path in a Network

Set up Given a graph $G := (V, E)$.

- Each edge $e \in E$ is owned by player e.
- Two distinguished vertices $s, t \in V$.
- Each player e submits the cost θ_e of using the edge e.
- The central authority selects the shortest $s - t$ path in G.

- $D = \{p \mid p$ is a $s - t$ path in $G\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(p, \theta_i) := \begin{cases} -\theta_i & \text{if } i \in p \\ 0 & \text{otherwise} \end{cases}$
- $f(\theta) := p$, where p is the shortest $s - t$ path in G.

Manipulations

Example An optimal strategy for player i in public project problem:

- if $\theta_i \geq \frac{c}{n}$ submit $\theta_i' = c$.
- if $\theta_i < \frac{c}{n}$ submit $\theta_i' = 0$.

For example, assume $c = 30$.

player	type
A	6
B	7
C	25

Players A and B should submit 0. Player c should submit 30.

Revised Set-up: Direct Mechanisms

- Each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ'_i,
- the central authority computes decision

$$d := f(\theta'_1, \ldots, \theta'_n),$$

and taxes

$$(t_1, \ldots, t_n) := g(\theta'_1, \ldots, \theta'_n) \in \mathbb{R}^n,$$

and communicates to each player i both d and t_i.

- final utility function for player i:

$$u_i(d, \theta_i) := v_i(d, \theta_i) + t_i.$$

Direct Mechanisms, ctd

- Direct mechanism (f, t) is incentive compatible if for all $\theta \in \Theta$, $i \in \{1, \ldots, n\}$ and $\theta'_i \in \Theta_i$

$$u_i((f,t)(\theta_i, \theta_{-i}), \theta_i) \geq u_i((f,t)(\theta'_i, \theta_{-i}), \theta_i).$$

- Intuition Submitting false type (so $\theta'_i \neq \theta_i$) does not pay off.
- Direct mechanism (f, t) is feasible if $\sum_{i=1}^{n} t_i(\theta) \leq 0$ for all θ.
- Intuition External financing is never needed.

Groves Mechanisms

- $t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) + h_i(\theta_{-i})$, where

  $h_i : \Theta_{-i} \rightarrow \mathbb{R}$ is an arbitrary function.

- Note

$$u_i((f,t)(\theta), \theta_i) = \sum_{j=1}^{n} v_j(f(\theta), \theta_j) + h_i(\theta_{-i}).$$

- Intuitions
  - Player i cannot manipulate the value of $h_i(\theta_{-i})$.
  - Suppose $h_i(\theta_{-i}) = 0$.
    When the individual types are $\theta_1, \ldots, \theta_n$
    $u_i((f,t)(\theta), \theta_i)$ is the social welfare from $f(\theta)$.

Groves Theorem

Theorem (Groves '73)
Suppose f is efficient. Then each Groves mechanism is incentive compatible.

Proof.
For all $\theta \in \Theta$, $i \in \{1, \ldots, n\}$ and $\theta'_i \in \Theta_i$

$$u_i((f,t)(\theta_i, \theta_{-i}), \theta_i) = \sum_{j=1}^{n} v_j(f(\theta_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i})$$

$$(f \text{ is efficient}) \geq \sum_{j=1}^{n} v_j(f(\theta'_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i})$$

$$= u_i((f,t)(\theta'_i, \theta_{-i}), \theta_i).$$

Special Case: Pivotal Mechanism

- $h_i(\theta_{-i}) := -\max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j)$.
- Then

$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j) \leq 0.$$

- Note Pivotal mechanism is feasible.

Re: Sealed-Bid Auction

Note In the pivotal mechanism

$$t_i(\theta) = \begin{cases} -\max_{j \neq i} \theta_j & \text{if } i = \text{argsmax}\, \theta. \\ 0 & \text{otherwise} \end{cases}$$

So the pivotal mechanism is Vickrey auction:
the winner pays the 2nd highest bid.

Example

player	bid	tax to authority	util.
A	18	0	0
B	24	-21	3
C	21	0	0

Social welfare: 0 + 0 + 3 = 3.

Maximizing Social Welfare

Question: Does Vickrey auction maximize social welfare?

Notation θ^*: the reordering of θ is descending order.

Example For $\theta = (1, 4, 2, 3, 1)$ we have
$\theta_{-2} = (1, 2, 3, 0)$,
$(\theta_{-2})^* = (3, 2, 1, 0)$,
so $(\theta_{-2})^*_2 = 2$.

Intuition $(\theta_{-2})^*_2$ is the second highest bid among other bids.

Bailey-Cavallo Mechanism

Bailey-Cavallo mechanism ($n \geq 3$):

$$t_i(\theta) := t_i^p(\theta) + \frac{(\theta_{-i})_2^*}{n}$$

Note Bailey-Cavallo mechanism is a Groves mechanism.

Example

player	bid	tax to authority	util.	why?
A	18	0	7	(= $1/3$ of 21)
B	24	-2	9	(= $24 - 2 - 7 - 6$)
C	21	0	6	(= $1/3$ of 18)

Bailey-Cavallo Mechanism, ctd

Note Bailey-Cavallo mechanism is feasible.

Proof. For all i and θ, $(\theta_{-i})^*_2 \leq \theta^*_2$, so

$$\sum_{i=1}^{n} t_i(\theta) = -\theta^*_2 + \sum_{i=1}^{n} \frac{(\theta_{-i})^*_2}{n} = \sum_{i=1}^{n} \frac{-\theta^*_2 + (\theta_{-i})^*_2}{n} \leq 0.$$

Theorem In the case of sealed-bid auctions Bailey-Cavallo mechanism maximizes social welfare.

Re: Public Project Problem

Assume the pivotal mechanism.
Examples Suppose $c = 30$ and $n = 3$.

player	type	tax	u_i
A	6	0	-4
B	7	0	-3
C	25	-7	8

Social welfare can be negative.

player	type	tax	u_i
A	4	-5	-5
B	3	-6	-6
C	22	0	0

Formally

Note In the pivotal mechanism

$$t_i(\theta) = \begin{cases} 0 & \text{if } \sum_{j \neq i} \theta_j \geq \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j \geq c \\ \sum_{j \neq i} \theta_j - \frac{n-1}{n} c & \text{if } \sum_{j \neq i} \theta_j < \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j \geq c \\ 0 & \text{if } \sum_{j \neq i} \theta_j \leq \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j < c \\ \frac{n-1}{n} c - \sum_{j \neq i} \theta_j & \text{if } \sum_{j \neq i} \theta_j > \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j < c \end{cases}$$

Theorem In the case of the public project problem the pivotal maximizes social welfare.

Re: Reversed Sealed-Bid Auction

Take

$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{d \in D \setminus \{i\}} \sum_{j \neq i} v_j(d, \theta_j).$$

Note

$$t_i(\theta) = \begin{cases} -\max_{j \neq i} \theta_j & \text{if } i = \text{argsmax}\, \theta. \\ 0 & \text{otherwise} \end{cases}$$

So in this mechanism the winner receives the amount equal to the 2nd lowest offer.

Example Consider $\Theta = (-8, -5, -4, -6)$. The service is bought from player 3 who receives for it 5.

Re: Buying a Path in a Network

Take

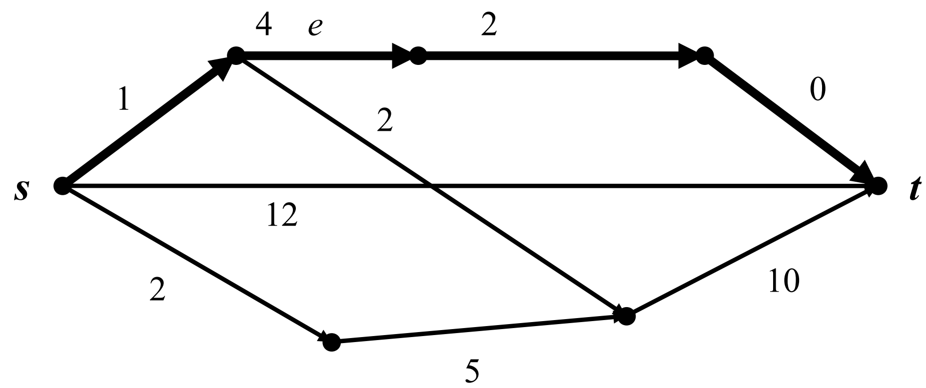
$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) - \max_{p \in D(G \setminus \{i\})} \sum_{j \neq i} v_j(p, \theta_j).$$

Note

$$t_i(\theta) = \begin{cases} cost(p_2) - cost(p_1 - \{i\}) & \text{if } i \in p_1 \\ 0 & \text{otherwise} \end{cases}$$

where

p_1 is the shortest $s - t$ path in $G(\theta)$,
p_2 is the shortest $s - t$ path in $(G \setminus \{i\})(\theta_{-i})$.



Consider the player owning the edge e.
To compute the payment he receives

- determine the shortest $s - t$ path. Its length is 7. It contains e.
- determine the shortest $s - t$ path that does not include e. Its length is 12.
- So player e receives $12 - (7 - 4) = 9$.
  His final utility is $9 - 4 = 5$.