Mixed Strategies

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Mixed Extension of a Finite Game

- Probability distribution over a finite non-empty set A:

$$\pi : A \rightarrow [0, 1]$$

  such that $\sum_{a \in A} \pi(a) = 1$.
- Notation: ΔA.

Fix a finite strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.

- Mixed strategy of player i in G: $m_i \in \Delta S_i$.
- Joint mixed strategy: $m = (m_1, \ldots, m_n)$.

Mixed Extension of a Finite Game (2)

- Mixed extension of G:

$$(\Delta S_1, \ldots, \Delta S_n, p_1, \ldots, p_n),$$

where

$$m(s) := m_1(s_1) \cdot \ldots \cdot m_n(s_n)$$

and

$$p_i(m) := \sum_{s \in S} m(s) \cdot p_i(s).$$

- Theorem (Nash '50) Every mixed extension of a finite strategic game has a Nash equilibrium.

Kakutani's Fixed Point Theorem

Theorem (Kakutani '41)
Suppose A is a compact and convex subset of $\mathbb{R}^n$ and

$$\Phi : A \rightarrow \mathcal{P}(A)$$

is such that

- $\Phi(x)$ is non-empty and convex for all $x \in A$,
- for all sequences (x_i, y_i) converging to (x, y)

  $$y_i \in \Phi(x_i) \text{ for all } i \geq 0,$$

  implies that

  $$y \in \Phi(x).$$

Then $x^* \in A$ exists such that $x^* \in \Phi(x^*)$.

Proof of Nash Theorem

Fix $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. Define

$$best_i : \Pi_{j \neq i} \Delta S_j \rightarrow \mathcal{P}(\Delta S_i)$$

by

$best_i(m_{-i}) := \{m'_i \in \Delta S_i \mid p_i(m'_i, m_{-i})$ attains the maximum$\}$.

Then define

$$best : \Delta S_1 \times \ldots \Delta S_n \rightarrow \mathcal{P}(\Delta S_1 \times \ldots \times \Delta S_n)$$

by

$$best(m) := best_1(m_{-1}) \times \ldots \times best_1(m_{-n}).$$

Note m is a Nash equilibrium iff $m \in best(m)$.
$best(\cdot)$ satisfies the conditions of Kakutani's Theorem.

Comments

- First special case of Nash theorem: Cournot (1838).
- Nash theorem generalizes von Neumann's Minimax Theorem ('28).
- An alternative proof (also by Nash) uses Brouwer's Fixed Point Theorem.
- Search for conditions ensuring existence of Nash equilibrium.

2 Examples

Matching Pennies

	H	T
H	$1, -1$	$-1, \quad 1$
T	$-1, \quad 1$	$1, -1$

- $(\frac{1}{2} \cdot H + \frac{1}{2} \cdot T, \frac{1}{2} \cdot H + \frac{1}{2} \cdot T)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: 0.

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

- $(2/3 \cdot F + 1/3 \cdot B, 1/3 \cdot F + 1/3 \cdot B)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: $2/3$.

Dominance by a Mixed Strategy

Example

	X	Y	Z
A	$2, -$	$0, -$	$1, -$
B	$0, -$	$2, -$	$1, -$
C	$1, -$	$1, -$	$0, -$
D	$1, -$	$0, -$	$0, -$

- D is weakly dominated by A,
- C is weakly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot B$,
- D is strictly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot C$.

Iterated Elimination of Strategies

Consider weak dominance by a mixed strategy.

	X	Y	Z
A	$2, 1$	$0, 1$	$1, 0$
B	$0, 1$	$2, 1$	$1, 0$
C	$1, 1$	$1, 0$	$0, 0$
D	$1, 0$	$0, 1$	$0, 0$

- D is weakly dominated by A,
- Z is weakly dominated by X,
- C is weakly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot B$.

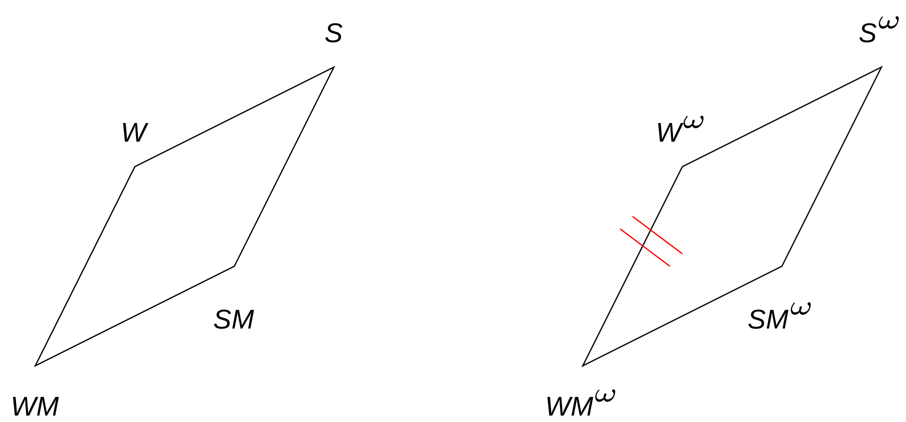
By eliminating them we get the final outcome:

	X	Y
A	$2, 1$	$0, 1$
B	$0, 1$	$2, 1$

Relative Strength of Strategy Elimination

- Weak dominance by a pure strategy is less powerful than weak dominance by a mixed strategy, but
- iterated elimination using weak dominance by a pure strategy (W^ω) can be more powerful than iterated elimination using weak dominance by a mixed strategy (MW^ω).

In general (Apt '07):



Best responses to Mixed Strategies

- s_i is a best response to m_{-i} if

$$\forall s'_i \in S_i \; p_i(s_i, m_{-i}) \geq p_i(s'_i, m_{-i}).$$

- $support(m_i) := \{a \in S_i \mid m_i(a) > 0\}.$
- Theorem (Pearce '84) In a 2-player finite game
  - s_i is strictly dominated by a mixed strategy iff it is not a best response to a mixed strategy.
  - s_i is weakly dominated by a mixed strategy iff it is not a best response to a mixed strategy with full support.

IESDMS

Theorem

- If G' is an outcome of IESDMS starting from G, then m is a Nash equilibrium of G' iff it is a Nash equilibrium of G.
- If G is solved by IESDMS, then the resulting joint strategy is a unique Nash equilibrium of G.
- (Osborne, Rubinstein, '94) Outcome of IESDMS is unique (order independence).

IESDMS: Example

Beauty-contest game

- each set of strategies = $\{1, \ldots, 100\}$,
- payoff to each player:
  1 is split equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average.
- This game is solved by IESDMS, in 99 steps.
- Hence it has a unique Nash equilibrium, $(1, \ldots, 1)$.

IEWDMS

Theorem

- If G' is an outcome of IEWDMS starting from G and m is a Nash equilibrium of G', then m is a Nash equilibrium of G.
- If G is solved by IEWDMS, then the resulting joint strategy is a Nash equilibrium of G.
- Outcome of IEWDS does not need to be unique (no order independence).
- Every mixed extension of a finite strategic game has a Nash equilibrium in which no pure strategy is weakly dominated by a mixed strategy.

Rationalizable Strategies

- Introduced in Bernheim '84 and Pearce '84.
- Strategies in the outcome of IENBRM.
- Subtleties in the definition ...

Theorem

- (Bernheim '84) If G' is an outcome of IENBRM starting from G, then m is a Nash equilibrium of G' iff it is a Nash equilibrium of G.
- If G is solved by IESDMS, then the resulting joint strategy is a unique Nash equilibrium of G.
- (Apt '05) Outcome of IENBRM is unique (order independence).