Nash Equilibria and Pareto Efficient Outcomes

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Basic Concepts

Overview

- Best response.
- Nash equilibrium.
- Pareto efficient outcomes.
- Social welfare.
- Social optima.
- Examples.

Strategic Games: Definition

Strategic game for $n \geq 2$ players:

- (possibly infinite) set S_i of strategies,
- payoff function $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$,

for each player i.

Basic assumptions:

- players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others' rationality.

Three Examples

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Main Concepts

- Notation: $s_i, s'_i \in S_i$, $s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n$.
- s_i is a best response to s_{-i} if

$$\forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- s is a Nash equilibrium if $\forall i \;\; s_i$ is a best response to s_{-i}:

$$\forall i \in \{1, \ldots, n\} \; \forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- s is Pareto efficient if for no s'

$$\forall i \in \{1, \ldots, n\} \; p_i(s') \geq p_i(s),$$
$$\exists i \in \{1, \ldots, n\} \; p_i(s') > p_i(s).$$

- Social welfare of s: $\sum_{j=1}^{n} p_j(s)$.
- s is a social optimum if $\sum_{j=1}^{n} p_j(s)$ is maximal.

Nash Equlibrium

Prisoner's Dilemma: 1 Nash equilibrium

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

The Battle of the Sexes: 2 Nash equilibria

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies: no Nash equlibrium

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

- 1 Nash equilibrium: (D, D),
- 3 Pareto efficient outcomes: (C, C), (C, D),(D, C),
- 1 social optimum: (C, C).

Prisoner's Dilemma for n Players

- $n > 1$ players,
- two strategies:
  1 (formerly C),
  0 (formerly D).

$$p_i(s) := \begin{cases} 2\sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\ 2\sum_{j \neq i} s_j & \text{if } s_i = 1 \end{cases}$$

- For $n = 2$ we get the original Prisoner's Dilemma game.
- Let **1** $= (1, \ldots, 1)$ and **0** $= (0, \ldots, 0)$.
- **0** is the unique Nash equilibrium, with social welfare n.
- Social optimum: **1**, with social welfare $2n(n-1)$.

Tragedy of the Commons

- Common resources: goods that are are not *excludable* (people cannot be prevented from using them) but are *rival* (one person's use of them diminishes another person's enjoyment of it).
- Examples: congested toll-free roads, fish in the ocean, the environment, . . .,
- Problem: Overuse of such common resources leads to their destruction.
- This phenomenon is called the tragedy of the commons (Hardin '81).

Tragedy of the Commons I

(Gardner '95)

- $n > 1$ players,
- two strategies:
  1 (use the resource),
  0 (don't use),
- payoff function:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0 \\ F(m)/m & \text{otherwise} \end{cases}$$

where $m = \sum_{j=1}^{n} s_j$ and

$$F(m) := 1.1m - 0.1m^2.$$

Tragedy of the Commons I, ctd

- payoff function:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0 \\ F(m)/m & \text{otherwise} \end{cases}$$

  where $m = \sum_{j=1}^{n} s_j$ and $F(m) := 1.1m - 0.1m^2$.

- Note: $F(m)/m$ is strictly decreasing,
  $F(9)/9 = 0.2$, $F(10)/10 = 0.1$, $F(11)/11 = 0$.

- Nash equilibria:
  $n < 10$: all players use the resource,
  $n \geq 10$: 9 or 10 players use the resource,

- Social optimum: 5 players use the resource.

Tragedy of the Commons II

(Osborne '04)

- $n > 1$ players,
- strategies: $[0, 1]$,
- payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Tragedy of the Commons II, ctd

- payoff function:

$$p_i(s) := \begin{cases} s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- 'Best' Nash equilibrium:
  when each $s_i = \frac{1}{n+1}$,
  with social welfare $\frac{n}{(n+1)^2}$ and $\sum_{j=1}^{n} s_j = \frac{n}{n+1}$.
- Social optimum, when $\sum_{j=1}^{n} s_j = \frac{1}{2}$,
  with social welfare $\frac{1}{4}$.
- For all $n > 1$, $\frac{n}{(n+1)^2} < \frac{1}{4}$.
- $\lim_{n \to \infty} \frac{n}{(n+1)^2} = 0$ and $\lim_{n \to \infty} \frac{n}{n+1} = 1$.