Nash Equilibria and Pareto Efficient Outcomes

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Basic Concepts
Overview

- Best response.
- Nash equilibrium.
- Pareto efficient outcomes.
- Social welfare.
- Social optima.
- Examples.
Strategic Games: Definition

Strategic game for $n \geq 2$ players:

- (possibly infinite) set $S_i$ of strategies,
- payoff function $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$, for each player $i$.

Basic assumptions:

- players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others’ rationality.
Three Examples

Prisoner’s Dilemma

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The Battle of the Sexes

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Matching Pennies

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Main Concepts

- **Notation**: \( s_i, s'_i \in S_i, s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n. \)

- \( s_i \) is a **best response** to \( s_{-i} \) if

  \[
  \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).
  \]

- \( s \) is a **Nash equilibrium** if \( \forall i \ s_i \) is a best response to \( s_{-i} \):

  \[
  \forall i \in \{1, \ldots, n\} \ \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).
  \]

- \( s \) is **Pareto efficient** if for no \( s' \)

  \[
  \forall i \in \{1, \ldots, n\} \ p_i(s') \geq p_i(s), \quad \exists i \in \{1, \ldots, n\} \ p_i(s') > p_i(s).
  \]

- **Social welfare** of \( s \): \( \sum_{j=1}^{n} p_j(s) \).

- \( s \) is a **social optimum** if \( \sum_{j=1}^{n} p_j(s) \) is maximal.
Nash Equilibrium

Prisoner’s Dilemma: 1 Nash equilibrium

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The Battle of the Sexes: 2 Nash equilibria

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Matching Pennies: no Nash equilibrium

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Prisoner’s Dilemma

- 1 Nash equilibrium: \((D, D)\),
- 3 Pareto efficient outcomes: \((C, C)\), \((C, D)\), \((D, C)\),
- 1 social optimum: \((C, C)\).
Prisoner’s Dilemma for $n$ Players

- $n > 1$ players,
- two strategies: $1$ (formerly $C$), $0$ (formerly $D$).

$$p_i(s) := \begin{cases} 
2 \sum_{j \neq i} s_j + 1 & \text{if } s_i = 0 \\
2 \sum_{j \neq i} s_j & \text{if } s_i = 1 
\end{cases}$$

For $n = 2$ we get the original Prisoner’s Dilemma game.

Let $1 = (1, \ldots, 1)$ and $0 = (0, \ldots, 0)$.

$0$ is the unique Nash equilibrium, with social welfare $n$.

Social optimum: $1$, with social welfare $2n(n - 1)$. 

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Common resources: goods that are not *excludable* (people cannot be prevented from using them) but are *rival* (one person’s use of them diminishes another person’s enjoyment of it).

Examples: congested toll-free roads, fish in the ocean, the environment, . . . ,

Problem: Overuse of such common resources leads to their destruction.

This phenomenon is called the *tragedy of the commons* (Hardin ’81).
Tragedy of the Commons I

(Gardner '95)

- $n > 1$ players,
- two strategies: 1 (use the resource), 0 (don't use),
- payoff function:

$$p_i(s) := \begin{cases} 0.1 & \text{if } s_i = 0 \\ F(m)/m & \text{otherwise} \end{cases}$$

where $m = \sum_{j=1}^{n} s_j$ and

$$F(m) := 1.1m - 0.1m^2.$$
Tragedy of the Commons I, ctd

payoff function:

\[ p_i(s) := \begin{cases} 
0.1 & \text{if } s_i = 0 \\
F(m)/m & \text{otherwise} 
\end{cases} \]

where \( m = \sum_{j=1}^{n} s_j \) and \( F(m) := 1.1m - 0.1m^2 \).

Note: \( F(m)/m \) is strictly decreasing, \( F(9)/9 = 0.2, F(10)/10 = 0.1, F(11)/11 = 0 \).

Nash equilibria:
\( n < 10 \): all players use the resource,
\( n \geq 10 \): 9 or 10 players use the resource,
Social optimum: 5 players use the resource.
Tragedy of the Commons II

(Osborne ’04)

- $n > 1$ players,
- strategies: $[0, 1]$,
- payoff function:

\[ p_i(s) := \begin{cases} 
  s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\
  0 & \text{otherwise}
\end{cases} \]
payoff function:

\[ p_i(s) := \begin{cases} 
  s_i(1 - \sum_{j=1}^{n} s_j) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\
  0 & \text{otherwise}
\end{cases} \]

‘Best’ Nash equilibrium: when each \( s_i = \frac{1}{n+1} \),
with social welfare \( \frac{n}{(n+1)^2} \) and \( \sum_{j=1}^{n} s_j = \frac{n}{n+1} \).

Social optimum, when \( \sum_{j=1}^{n} s_j = \frac{1}{2} \),
with social welfare \( \frac{1}{4} \).

For all \( n > 1 \), \( \frac{n}{(n+1)^2} < \frac{1}{4} \).

\[ \lim_{n \to \infty} \frac{n}{(n+1)^2} = 0 \text{ and } \lim_{n \to \infty} \frac{n}{n+1} = 1. \]