# Chapter 8

# Potential games

These games are studied in D. Monderer and L.S. Shapley (1996). See also Voorneveld (1999). Let $\Gamma = \langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p \rangle$ be a $p$-person game and let $P : \prod_{i=1}^{p} X_i \to \mathbb{R}$ be a real-valued function on the Cartesian product of the strategy spaces of $\Gamma$. Then $P$ is called a *potential* of $\Gamma$ if for each $i \in \{1, 2, \ldots, p\}$, each $x^{-i} \in \prod_{k \neq i} X_k$ and all $x_i$, $x_i' \in X_i$ we have

$$K_i(x^{-i}, x_i') - K_i(x^{-i}, x_i) = P(x^{-i}, x_i') - P(x^{-i}, x_i).$$

A game for which a potential exists is called a *potential game.*

**Example 8.1** Consider the following situation, where two players are involved. Player 1 has to travel from $A$ to $C$ and player 2 from $B$ to $D$. See figure 14.

Player 1 can travel via $B$ or via $D$ and player 2 via $A$ or via $C$. The costs of using $AB$ is 2 (5) if one player (both players) uses



(use) $AB$ etc. This situation reduces to the bimatrix cost game

$$\begin{array}{cc} & \begin{array}{cc} \text{via } A & \text{via } C \end{array} \\ \begin{array}{c} \text{via } B \\ \text{via } D \end{array} & \begin{bmatrix} (8,9) & (8,7) \\ (11,12) & (7,6)^* \end{bmatrix} \end{array}$$

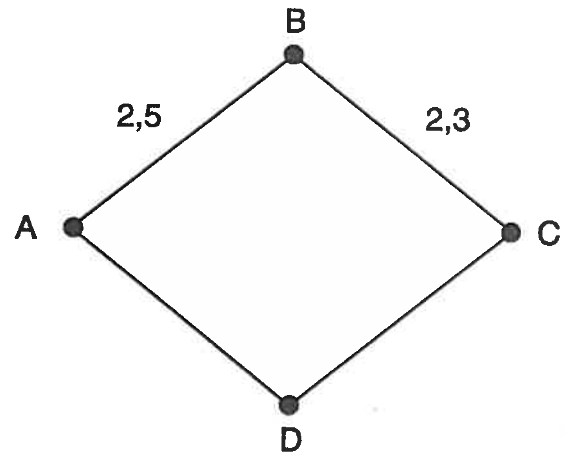


Figure 14.

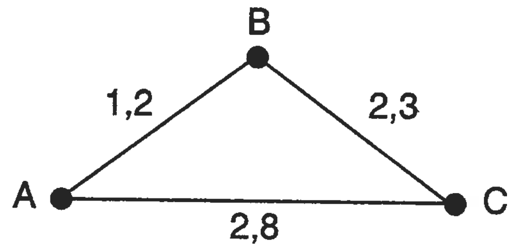


Figure 15.

in which (via $D$, via $C$) with costs 7 and 6 for player 1 and 2, is a pure Nash equilibrium. A cost potential for this game is given by
$P = \begin{bmatrix} 14 & 12 \\ 17 & 11 \end{bmatrix}$.

Not all bimatrix games possess a potential as we can conclude from

**Theorem 8.2** Let $\Gamma$ be a finite $p$-person game and suppose that $P$ is a potential of $\Gamma$. Then

(i) $NE\langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p\rangle = NE\langle X_1, X_2, \ldots, X_p, P, P, \ldots, P\rangle$,

(ii) $\Gamma$ has at least one pure Nash equilibrium.

**Proof.**

(i) follows directly from the definition of a potential.

(ii) follows from (i) and the remark that $\hat{x}$ is a Nash equilibrium for $\langle X_1, X_2, \ldots X_p, P, P, \ldots, P\rangle$ if $P(\hat{x}) = \max\{P(x) \mid x \in \prod_{i=1}^{p} X_i\}$. □

**Exercise 8.1** Let $\Gamma$ be a $p$-person game with a potential $P$ and suppose that all payoff functions are bounded. Prove that for each $\varepsilon > 0$, $\Gamma$ has an $\varepsilon$-Nash equilibrium. [$\hat{x}$ is called and $\varepsilon$-Nash equilibrium if for all $i \in \{1, 2, \ldots, p\}$ and all $x_i \in X_i$:

$$K_i(\hat{x}^{-i}, x_i) \leq K_i(\hat{x}) + \varepsilon$$

which means that unilateral deviation from $\hat{x}$ pays at most $\varepsilon$.]



**Exercise 8.2** Let $\langle X_1, X_2, \ldots, X_p, K_1, K_2, \ldots, K_p\rangle$ be a game corresponding to an oligopoly situation with linear price function and cost functions $c_1, c_2, \ldots, c_p$ with continuous derivatives. So $X_i = [0, \infty)$, $K_i(x_1, \ldots, x_n) = x_i(a - b\sum_{j=1}^{p} x_j) - c_i(x_i)$ for each $(x_1, \ldots, x_p) \in \mathbb{R}_+^p$.
Prove that $P : \mathbb{R}_+^p \to \mathbb{R}$, defined by

$$P(x_1, x_2, \ldots, x_p) = a\sum_{j=1}^{p} x_j - b\sum_{j=1}^{p} x_j^2 - b\sum_{1\leq i<j\leq p} x_i x_j - \sum_{j=1}^{p} c_j(x_j)$$

is a potential for this game.
[Hint: It is sufficient to show that $\frac{\partial P}{\partial x_i} = \frac{\partial K_i}{\partial x_i}$ for all $i \in \{1, \ldots, p\}$.]

Interesting classes of cost games, arising from congestion models with a potential were introduced by R.W. Rosenthal (1973). A *congestion model* can be described as a 4-tuple $\langle N, M, (X_i)_{i\in N}, (c_j)_{j\in M}\rangle$ where

(i) $N$ is the set of players involved (drivers on roads, producers),

(ii) $M$ is the set of facilities $\{1, 2, \ldots, m\}$ involved (such as road segments, primary production factors),

(iii) $X_i$ is the set of strategies of player $i$, where strategies of player $i$ consist of suitable non-empty subsets of $M$,

(iv) $c_j : \{1, 2, \ldots, n\} \to \mathbb{R}$ is for facility $j$ the cost function, where $c_j(k)$ denotes the costs to each user of facility $j$ in case there are precisely $k$ users.

The *congestion game* corresponding to the congestion model is the cost game in strategic form $\langle X_1, X_2, \ldots, X_n, C_1, C_2, \ldots, C_n\rangle$, where the costs $C_i(B_1, B_2, \ldots, B_n)$ for player $i$ is equal to

$$\sum_{j \in B_i} c_j(t_j(B_1, B_2, \ldots, B_n))$$

with $t_j(B_1, B_2, \ldots, B_n) := |\{r \in N \mid j \in B_r\}|$, the number of users of facility $j$ if $B_1, B_2, \ldots, B_n$ are the chosen strategies. (The number of elements in a finite set $S$ is denoted by $|S|$.)

**Example 8.3** The situation in example 8.1 can be seen as a congestion situation, where $N = \{1, 2\}, M = \{1, 2, 3, 4\}$ if we identify $AB, BC, AD, DC$ with 1, 2, 3 and 4. Furthermore, $X_1 = \{\{1, 2\}, \{3, 4\}\}$ and $X_2 = \{\{1, 3\}, \{2, 4\}\}$. Moreover $c_1(1) = 2$, $c_1(2) = 5$, $c_2(1) = 3$, $c_2(2) = 6$, $c_3(1) = 4$, $c_3(2) = 10$, $c_4(1) = 1$ and $c_4(2) = 3$.

**Example 8.4** There are three machines 1, 2, 3 used by firms 1 and 2. Firm 1 can produce using machines 1 and 2 or 1 and 3. Firm 2 can produce using machines 1 and 2, 1 and 3 or 2 and 3. Costs for using machine 1 are 5 and 6 respectively, corresponding to one and two users respectively. For machine 2 the costs are 3 and 4, and for machine 3 the costs are 2 and 5 respectively. This corresponds to a congestion situation. The corresponding cost game is given by

$$\begin{array}{c} \\ \{1,2\} \\ \{1,3\} \end{array} \begin{array}{c} \begin{array}{ccc} \{1,2\} & \{1,3\} & \{2,3\} \end{array} \\ \left[ \begin{array}{ccc} (6+4, 6+4) & (6+3, 6+2) & (5+4, 4+2) \\ (6+2, 6+3) & (6+5, 6+5) & (5+5, 3+5) \end{array} \right] \end{array}$$

$$= \left[ \begin{array}{ccc} (10,10) & (9,8) & (9,6)^* \\ (8,9) & (11,11) & (10,8) \end{array} \right].$$



The unique pure Nash equilibrium corresponds to the situation where player 1 produces using machines 1 and 2 and player 2 produces using machines 2 and 3. A (cost) potential for this game is given by

$$P = \left[ \begin{array}{ccc} 18 & 16 & 14 \\ 16 & 18 & 15 \end{array} \right].$$

**Exercise 8.3** Let $N, M, (X_i)_{i \in N}, (c_j)_{j \in M}$ be a congestion situation and let $t_j(B_1, B_2, \ldots, B_n)$ be as above. Prove that $P : \prod_{i \in N} X_i \to \mathbb{R}$, defined by

$$P(B_1, B_2, \ldots, B_n) = \sum_{j \in \cup_{i \in N} B_i} \left( \sum_{k=1}^{t_j(B_1, \ldots, B_n)} c_j(k) \right)$$
for all $B_i \in X_i$ $(i \in N)$

is a cost potential for the corresponding cost game.

**Exercise 8.4** Find a potential for the (cost) congestion game, corresponding to the situation in figure 15 where player 1 has to go from $A$ to $C$, player 2 from $C$ to $A$ and where the costs of $AB$ is 1 (2) if one (two) player(s) use $AB$ etc. Note that this game has two pure $NE$.

(2) $z = (2+v(A))^{-1}(p,q,v(A)) \in \Delta^5$, $zB \geq 0 = v(B)1_5$.
So, $z \in O_1(B)$.
Note that $v(A) = 1\frac{1}{2}$, $O_1(A) = \{(\frac{3}{4},\frac{1}{4})\}$, $O_2(A) = \{(\frac{1}{2},\frac{1}{2})\}$.
So, $z = \frac{2}{7}(\frac{3}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{2}) \in O_1(B)$.

**7.1**

(i) Let $g(\hat{x}) = \hat{x}$. Then $\hat{x} = g(\hat{x}) \geq 0$, $f(\hat{x}) = 0 \geq 0$, $\hat{x}^T f(\hat{x}) = 0$. So, $\hat{x} \in O(f)$.

(ii) Let $\hat{x} \in O(f^*)$. Then $\hat{x} \geq 0$, $f^*(\hat{x}) \geq 0$, $\hat{x}^T f^*(\hat{x}) = 0$. This implies that $f_i^*(\hat{x}) = 0$ if $\hat{x}_i > 0$ and that $0 \leq f_i^*(\hat{x}) = \hat{x}_i - g_i(\hat{x}) = 0 - g_i(\hat{x}) \leq 0$ if $\hat{x}_i = 0$. So, $f^*(\hat{x}) = 0$, $\hat{x} = g(\hat{x})$.

**7.2** Equivalent are (1), (2), (3), (4) and (5) with
(1) $x$ is a solution of the primal problem and $y$ is a solution of the dual problem,
(2) $x^T A \geq b^T$, $x \geq 0$, $Ay \leq c$, $y \geq 0$, $x^T c = b^T y$,
(3) $-Ay + c \geq 0$, $A^T x - b \geq 0$, $x \geq 0$, $y \geq 0$, $(x^T, y^T)\begin{bmatrix} c \\ -b \end{bmatrix} = 0$,
(4) $\begin{bmatrix} 0 & -A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ -b \end{bmatrix} \geq 0$, $\begin{bmatrix} x \\ y \end{bmatrix} \geq 0$,

$$(x^T, y^T)\left(\begin{bmatrix} 0 & -A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ -b \end{bmatrix}\right) = 0,$$

(5) $\begin{bmatrix} x \\ y \end{bmatrix} \in O(q, M)$.

**8.1** Take $\varepsilon > 0$. It follows from the boundedness of all payoff functions that also $P$ is a bounded function. This implies that we can take $\hat{x} \in X$ such that $P(x) \leq P(\hat{x}) + \varepsilon$ for each $x \in X$. Then



for each $i \in \{1,2,\ldots,p\}$ and each $x_i \in X_i : K_i(\hat{x}^{-i}, x_i) - K_i(\hat{x}) = P(\hat{x}^{-i}, x_i) - P(\hat{x}) \leq \varepsilon$. So, $\hat{x}$ is an $\varepsilon$-Nash equilibrium of $\Gamma$.

**8.2** By the main theorem of calculus $P(x^{-i}, x_i') - P(x^{-i}, x_i) = \int_{x_i}^{x_i'} \frac{\partial P}{\partial x_i}(x^{-i}, t)dt$ and for each $i \in \{1,2,\ldots,p\} : K_i(x^{-i}, x_i') - K_i(x^{-i}, x_i) = \int_{x_i}^{x_i'} \frac{\partial K_i}{\partial x_i}(x^{-i}, t)dt$. So, to prove that the given $P$ is a potential for the oligopoly game, we have only to show that $\frac{\partial P}{\partial x_i} = \frac{\partial K_i}{\partial x_i}$ for $i \in \{1,2,\ldots,p\}$. Now, $\frac{\partial K_i}{\partial x_i}(x) = (a - b\sum_{j=1}^{p} x_j) - x_i b - c'(x_i) = a - 2bx_i - b\sum_{j \in P\setminus\{i\}} x_j - c'(x_i) = \frac{\partial P}{\partial x_i}(x)$.

**8.3** For mathematical convenience we extend $c_j$ to $\{0,1,\ldots,n\}$ with $c_j(0) = 0$. Also $B_i \subset M$ with $B_i = \emptyset$ we allow. Then

(i) $P(B_1,\ldots,B_i,\ldots,B_m) = \sum_{j \in M}\left(\sum_{k=0}^{t_j(B_1,\ldots,B_n)} c_j(k)\right)$. If we put $u_i(j) = 1$ if $j \in B_i$ and $u_i(j) = 0$ otherwise, then

(ii) $C_i(B_1,\ldots,B_i,\ldots,B_n) = \sum_{j \in M} c_j(t_j(B_1,\ldots,B_i,\ldots,B_n))u_i(j)$.

From (i) and (ii) follows

(iii) $P(B_1,\ldots,B_i,\ldots,B_m) - C_i(B_1,\ldots,B_i,\ldots,B_n) = \sum_{j \in M}\left(\sum_{k=0}^{t_j(B_1,\ldots,B_n)-u_i(j)} c_j(k)\right) = P(B_1,\ldots,B_{i-1},\emptyset, B_{i+1},\ldots,B_n)$.

In a similar way follows

(iv) $P(B_1,\ldots,B_i',\ldots,B_m) - C_i(B_1,\ldots,B_i',\ldots,B_n) = P(B_1,\ldots,B_{i-1},\emptyset,B_{i+1},\ldots,B_n)$.

From (iii) and (iv) follows that P is a potential for the congestion game.