Pre-Bayesian Games

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Pre-Bayesian Games

(Hyafil, Boutilier '04, Ashlagi, Monderer, Tennenholtz '06,)

- In a strategic game after each player selected his strategy each player knows all the payoffs (complete information).
- In a pre-Bayesian game after each player selected his strategy each player knows only his payoff (incomplete information).
- This is achieved by introducing (private) types.

Pre-Bayesian Games: Definition

Pre-Bayesian game for $n \geq 2$ players:

- (possibly infinite) set A_i of actions,
- (possibly infinite) set Θ_i of (private) types,
- payoff function $p_i : A_1 \times \ldots \times A_n \times \Theta_i \to \mathbb{R}$,

for each player i.

Basic assumptions:

- Nature moves first and provides each player i with a θ_i,
- players do not know the types received by other players,
- players choose their actions simultaneously,
- each player is rational (wants to maximize his payoff),
- players have common knowledge of the game and of each others' rationality.

Ex-post Equilibrium

- A strategy for player i:

$$s_i(\cdot) \in A_i^{\Theta_i}.$$

- Joint strategy $s_(\cdot)$ is an ex-post equilibrium if each $s_i(\cdot)$ is a best response to $s_{-i}(\cdot)$:

$$\forall \theta \in \Theta \ \forall i \in \{1, \ldots, n\} \ \forall s_i'(\cdot) \in A_i^{\Theta_i}$$
$$p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq p_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

- Note: For each $\theta \in \Theta$ we have one strategic game. $s_(\cdot)$ is an ex-post equilibrium if for each $\theta \in \Theta$ the joint action $(s_1(\theta_1), \ldots, s_n(\theta_n))$ is an ex-post equilibrium in the θ-game.

Quiz

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$.

L

U		F	B
	F	2, 1	2, 0
	B	0, 1	2, 1

D		F	B
	F	3, 1	2, 0
	B	5, 1	4, 1

R

U		F	B
	F	2, 0	2, 1
	B	0, 0	2, 1

D		F	B
	F	3, 0	2, 1
	B	5, 0	4, 1

Which strategies form an ex-post equilibrium?

Answer

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$.

L

U

U	F	B
F	2, 1	2, 0
B	0, 1	2, 1

D

D	F	B
F	3, 1	2, 0
B	5, 1	4, 1

R

U

U	F	B
F	2, 0	2, 1
B	0, 0	2, 1

D

D	F	B
F	3, 0	2, 1
B	5, 0	4, 1

- Strategies
  $s_1(U) = F, s_1(D) = B$,
  $s_2(L) = F, s_2(R) = B$
  form an ex-post equilibrium.

But . . .

Ex-post equilibrium does not need to exist in mixed extensions of finite pre-Bayesian games.

Example: Mixed extension of the following game.

- $\Theta_1 = \{U, B\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{C, D\}$.

L

		C	D
U	C	$2, 2$	$0, 0$
	D	$3, 0$	$1, 1$
B	C	$1, 2$	$3, 0$
	D	$0, 0$	$2, 1$

R

		C	D
U	C	$2, 1$	$0, 0$
	D	$3, 0$	$1, 2$
B	C	$1, 1$	$3, 0$
	D	$0, 0$	$2, 2$

Safety-level Equilibrium

- Strategy $s_i(\cdot)$ for player i is a safety-level best response to $s_{-i}(\cdot)$ if for all strategies $s_i'(\cdot)$ of player i and all $\theta_i \in \Theta_i$

$$\min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq \min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

- Intuition $\min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i)$ is the guaranteed payoff to player i when his type is θ_i and $s(\cdot)$ are the selected strategies.
- Joint strategy $s(\cdot)$ is a safety-level equilibrium if each $s_i(\cdot)$ is a safety-level best response to $s_{-i}(\cdot)$.
- Theorem
  Every mixed extension of a finite pre-Bayesian game has a safety-level equilibrium.

Direct Mechanisms

- each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ'_i,
- the central authority computes decision

$$d := f(\theta'_1, \ldots, \theta'_n),$$

and taxes

$$(t_1, \ldots, t_n) := g(\theta'_1, \ldots, \theta'_n) \in \mathbb{R}^n,$$

and communicates to each player i both d and t_i.

- final utility function for player i:

$$u_i(d, \theta_i) = v_i(d, \theta_i) + t_i.$$

Groves Mechanisms

- $t_i(\theta') := \sum_{j \neq i} v_j(f(\theta'), \theta'_j) + h_i(\theta'_{-i})$, where

  $h_i : \Theta_{-i} \rightarrow \mathbb{R}$ is an arbitrary function.

- Intuition:
  $\sum_{j \neq i} v_j(f(\theta'), \theta'_j)$
  is the social welfare with i excluded from decision $f(\theta')$.

- Note:

$$u_i((f, t)(\theta), \theta_i) = \sum_{j=1}^{n} v_j(f(\theta), \theta_j) + h_i(\theta_{-i}).$$

Groves Mechanisms, ctd

- Direct mechanism (f, t) is incentive compatible if for all $\theta \in \Theta$, $i \in \{1, \ldots, n\}$ and $\theta'_i \in \Theta_i$

$$u_i((f,t)(\theta_i, \theta_{-i}), \theta_i) \geq u_i((f,t)(\theta'_i, \theta_{-i}), \theta_i).$$

- Theorem (Groves '73)
  Suppose f is efficient. Then each Groves mechanism is incentive compatible.

Relation to pre-Bayesian Games

- Strategy $s_i(\cdot)$ is dominant if for all $a \in A$ and $\theta_i \in \Theta_i$

$$\forall a \in A \; p_i(s_i(\theta_i), a_{-i}, \theta_i) \geq p_i(a_i, a_{-i}, \theta_i).$$

- A pre-Bayesian game is of a revelation-type if $A_i = \Theta_i$ for all $i \in \{1, \ldots, n\}$.
- So in a revelation-type pre-Bayesian game the strategies of player i are the functions on Θ_i.
- A strategy for player i is called truth-telling if it is the identity function $\pi_i(\cdot)$.

Relation to pre-Bayesian Games, ctd

- Mechanism design (as discussed here) can be viewed as an instance of the revelation-type pre-Bayesian games.
- With each direct mechanism (f, t) we can associate a revelation-type pre-Bayesian game:
  - Each Θ_i as in the mechanism,
  - Each $A_i = \Theta_i$,
  - $p_i(\theta'_i, \theta_{-i}, \theta_i) := u_i((f,t)(\theta'_i, \theta_{-i}), \theta_i)$.
- Note Direct mechanism (f, t) is incentive compatible iff in the associated pre-Bayesian game for each player truth-telling is a dominant strategy.
- Conclusion In the pre-Bayesian game associated with a Groves mechanism, $(\pi_1(\cdot), \ldots, \pi_i(\cdot))$ is a dominant strategy ex-post equilibrium.