Pre-Bayesian Games

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Pre-Bayesian Games

(Hyafil, Boutilier '04, Ashlagi, Monderer, Tennenholtz '06,)

- In a strategic game after each player selected his strategy each player knows all the payoffs (complete information).
- In a pre-Bayesian game after each player selected his strategy each player knows only his payoff (incomplete information).
- This is achieved by introducing (private) types.

Pre-Bayesian Games: Definition

Pre-Bayesian game for $n \ge 2$ players:

- (possibly infinite) set A_i of actions,
- (possibly infinite) set Θ_i of (private) types,
- payoff function $p_i: A_1 \times \ldots \times A_n \times \Theta_i \to \mathbb{R}$,

for each player *i*.

Basic assumptions:

- Nature moves first and provides each player i with a θ_i ,
- players do not know the types received by other players,
- players choose their actions simultaneously,
- each player is rational (wants to maximize his payoff),
- players have common knowledge of the game and of each others' rationality.

Ex-post Equilibrium

A strategy for player i:

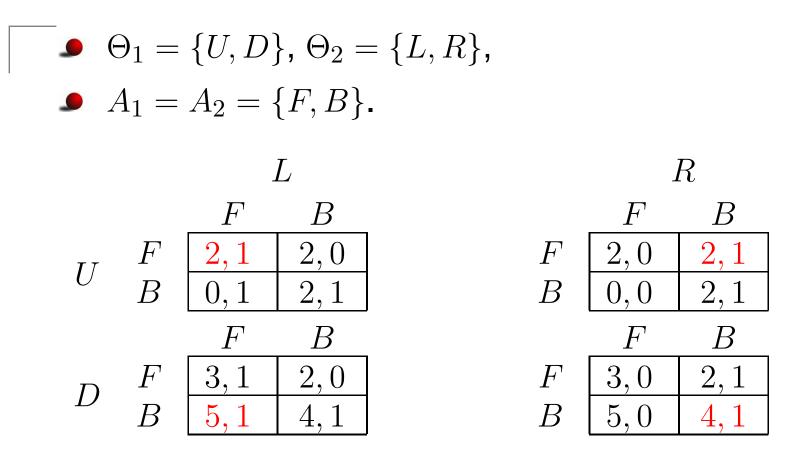
$$s_i(\cdot) \in A_i^{\Theta_i}$$

Joint strategy $s_{(·)}$ is an ex-post equilibrium if each $s_i(·)$ is a best response to $s_{-i}(·)$:

 $\forall \theta \in \Theta \ \forall i \in \{1, \dots, n\} \ \forall s'_i(\cdot) \in A_i^{\Theta_i}$ $p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \ge p_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$

• Note: For each $\theta \in \Theta$ we have one strategic game. $s_{(}\cdot)$ is an ex-post equilibrium if for each $\theta \in \Theta$ the joint action $(s_1(\theta_1), \ldots, s_n(\theta_n))$ is an ex-post equilibrium in the θ -game.

Quiz



Which strategies form an ex-post equilibrium?

Answer

R

B

2, 1

2, 1

B

2, 1

4,1

F

2, 0

0, 0

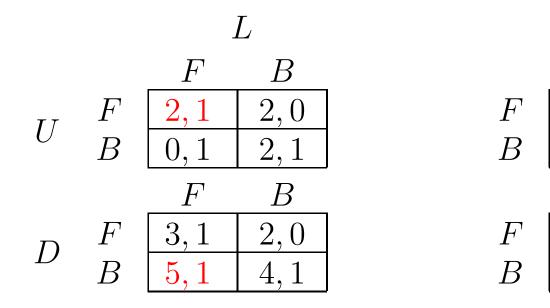
F

3, 0

5, 0

•
$$\Theta_1 = \{U, D\}, \ \Theta_2 = \{L, R\},$$

• $A_1 = A_2 = \{F, B\}.$



• Strategies $s_1(U) = F, s_1(D) = B,$ $s_2(L) = F, s_2(R) = B$ form an ex-post equilibrium.

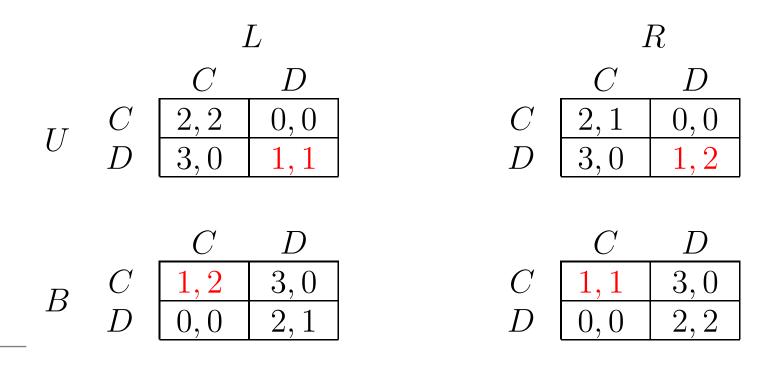
But ...

Ex-post equilibrium does not need to exist in mixed extensions of finite pre-Bayesian games.

Example: Mixed extension of the following game.

•
$$\Theta_1 = \{U, B\}, \ \Theta_2 = \{L, R\},$$

• $A_1 = A_2 = \{C, D\}.$



Safety-level Equilibrium

• Strategy $s_i(\cdot)$ for player *i* is a safety-level best response to $s_{-i}(\cdot)$ if for all strategies $s'_i(\cdot)$ of player *i* and all $\theta_i \in \Theta_i$

 $\min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \ge \min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$

- Intuition $\min_{\theta_{-i}\in\Theta_{-i}} p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i)$ is the guaranteed payoff to player *i* when his type is θ_i and $s(\cdot)$ are the selected strategies.
- Joint strategy $s(\cdot)$ is a safety-level equilibrium if each $s_i(\cdot)$ is a safety-level best response to $s_{-i}(\cdot)$.

Theorem

Every mixed extension of a finite pre-Bayesian game has a safety-level equilibrium.

Direct Mechanisms

- each player *i* receives/has a type θ_i ,
- each player i submits to the central authority a type θ'_i ,
- the central authority computes decision

$$d:=f(heta_1',\ldots, heta_n')$$
 ,

and taxes

$$(t_1,\ldots,t_n):=g(\theta'_1,\ldots,\theta'_n)\in\mathbb{R}^n$$
,

and communicates to each player i both d and t_i .

• final utility function for player i:

$$u_i(d,\theta_i) = v_i(d,\theta_i) + t_i.$$

Groves Mechanisms

• $t_i(\theta') := \sum_{j \neq i} v_j(f(\theta'), \theta'_j) + h_i(\theta'_{-i})$, where

 $h_i: \Theta_{-i} \to \mathbb{R}$ is an arbitrary function.

Intuition:

 $\sum_{j \neq i} v_j(f(\theta'), \theta'_j)$ is the social welfare with *i* excluded from decision $f(\theta')$.

Note:

$$u_i((f,t)(\theta),\theta_i) = \sum_{j=1}^n v_j(f(\theta),\theta_j) + h_i(\theta_{-i}).$$

Groves Mechanisms, ctd

• Direct mechanism (f,t) is incentive compatible if for all $\theta \in \Theta$, $i \in \{1, ..., n\}$ and $\theta'_i \in \Theta_i$

 $u_i((f,t)(\theta_i,\theta_{-i}),\theta_i) \ge u_i((f,t)(\theta'_i,\theta_{-i}),\theta_i).$

Theorem (Groves '73) Suppose f is efficient. Then each Groves mechanism is incentive compatible.

Relation to pre-Bayesian Games

• Strategy $s_i(\cdot)$ is dominant if for all $a \in A$ and $\theta_i \in \Theta_i$

$$\forall a \in A \ p_i(s_i(\theta_i), a_{-i}, \theta_i) \ge p_i(a_i, a_{-i}, \theta_i).$$

- A pre-Bayesian game is of a revelation-type if $A_i = \Theta_i$ for all $i \in \{1, ..., n\}$.
- So in a revelation-type pre-Bayesian game the strategies of player i are the functions on Θ_i .
- A strategy for player *i* is called truth-telling if it is the identity function $\pi_i(\cdot)$.

Relation to pre-Bayesian Games, ctd

- Mechanism design (as discussed here) can be viewed as an instance of the revelation-type pre-Bayesian games.
- With each direct mechanism (f, t) we can associate a revelation-type pre-Bayesian game:
 - Each Θ_i as in the mechanism,
 - Each $A_i = \Theta_i$,
 - $p_i(\theta'_i, \theta_{-i}, \theta_i) := u_i((f, t)(\theta'_i, \theta_{-i}), \theta_i).$
- Note Direct mechanism (f, t) is incentive compatible iff in the associated pre-Bayesian game for each player truth-telling is a dominant strategy.
- Conclusion In the pre-Bayesian game associated with a Groves mechanism, $(\pi_1(\cdot), \ldots, \pi_i(\cdot))$ is a dominant strategy ex-post equilibrium.