Chapter 9

Alternative Concepts

In the presentation until now we heavily relied on the definition of a strategic game and focused several times on the crucial notion of a Nash equilibrium. However, both the concept of an equilibrium and of a strategic game can be defined in alternative ways. Here we discuss some alternative definitions and explain their consequences.

9.1 Other equilibria notions

Nash equilibrium is a most popular and most widely used notion of an equilibrium. However, there are many other natural alternatives. In this section we briefly discuss three alternative equilibria notions. To define them fix a strategic game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.

Strict Nash equilibrium We call a joint strategy s a ***strict Nash equilibrium*** if

$$\forall i \in \{1, \ldots, n\} \ \forall s_i' \in S_i \setminus \{s_i\} \ p_i(s_i, s_{-i}) > p_i(s_i', s_{-i}).$$

So a joint strategy is a strict Nash equilibrium if each player achieves a *strictly lower* payoff by unilaterally switching to another strategy.

Obviously every strict Nash equilibrium is a Nash equilibrium and the converse does not need to hold.

Consider now the Battle of the Sexes game. Its pure Nash equilibria that we identified in Chapter 1 are clearly strict. However, its Nash equilibrium in mixed strategy we identified in Example 15 of Section 7.1 is not strict.

Indeed, the following simple observation holds.

Note 43 *Consider a mixed extension of a finite strategic game. Every strict Nash equilibrium is a Nash equilibrium in pure strategies.*

Proof. It is a direct consequence of the Characterization Lemma 22. □

Consequently each finite game with no Nash equilibrium in pure strategies, for instance the Matching Pennies game, has no strict Nash equilibrium in mixed strategies. So the analogue of Nash theorem does not hold for strict Nash equilibria, which makes this equilibrium notion less useful.

ϵ-Nash equilibrium The idea of an ϵ-Nash equilibrium formalizes the intuition that a joint strategy can be also be satisfactory for the players when each of them can gain only very little from deviating from his strategy.

Let $\epsilon > 0$ be a small positive real. We call a joint strategy s an ***ϵ-Nash equilibrium*** if

$$\forall i \in \{1, \ldots, n\}\ \forall s'_i \in S_i\ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}) - \epsilon.$$

So a joint strategy is an ϵ-Nash equilibrium if no player can gain more than ϵ by unilaterally switching to another strategy. In this context ϵ can be interpreted either as the amount of uncertainty about the payoffs or as the gain from switching to another strategy.

Clearly, a joint strategy is a Nash equilibrium iff it is an ϵ-Nash equilibrium for every $\epsilon > 0$. However, the payoffs in an ϵ-Nash equilibrium can be substantially lower than in a Nash equilibrium. Consider for example the following game:

	L	R
T	$1, \quad 1$	$0, \quad 0$
B	$1+\epsilon, \quad 1$	$100, 100$

This game has a unique Nash equilibrium (B, R), which obviously is also an ϵ-Nash equilibrium. However, (T, L) is also an ϵ-Nash equilibrium.

Strong Nash equilibrium Another variation of the notion of a Nash equilibrium focusses on the concept of a coalition, by which we mean a non-empty subset of all players.

Given a subset $K := \{k_1, \ldots, k_m\}$ of $N := \{1, \ldots, n\}$ we abbreviate the sequence $(s_{k_1}, \ldots, s_{k_m})$ of strategies to s_K and $S_{k_1} \times \ldots \times S_{k_m}$ to S_K.

We call a joint strategy s a ***strong Nash equilibrium*** if for all coalitions K there does not exist $s'_K \in S_K$ such that

$$p_i(s'_K, s_{N \setminus K}) > p_i(s_K, s_{N \setminus K}) \text{ for all } i \in K.$$

So a joint strategy is a strong Nash equilibrium if no coalition can profit from deviating from it, where by "profit from" we mean that each member of the coalition gets a strictly higher payoff. The notion of a strong Nash equilibrium generalizes the notion of a Nash equilibrium by considering possible deviations of coalitions instead of individual players.

Note that the unique Nash equilibrium of the Prisoner's Dilemma game is strict but not strong. For example, if both players deviate from D to C, then each of them gets a strictly higher payoff.

Correlated equilibrium The final concept of an equilibrium that we introduce is a generalization of Nash equilibrium in mixed strategies. Recall from Chapter 7 that given a finite strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ each joint mixed strategy $m = (m_1, \ldots, m_n)$ induces a probability distribution over S, defined by

$$m(s) := m_1(s_1) \cdot \ldots \cdot m_n(s_n),$$

where $s \in S$.

We have then the following observation.

Note 44 (Nash Equilibrium in Mixed Strategies) *Consider a finite strategic game* $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.

Then m is a Nash equilibrium in mixed strategies iff for all $i \in \{1, \ldots, n\}$ and all $s'_i \in S_i$

$$\sum_{s \in S} m(s) \cdot p_i(s_i, s_{-i}) \geq \sum_{s \in S} m(s) \cdot p_i(s'_i, s_{-i}).$$

Proof. Fix $i \in \{1, \dots, n\}$ and choose some $s_i' \in S_i$. Let

$$m_i'(s_i) := \begin{cases} 1 & \text{if } s_i = s_i' \\ 0 & \text{otherwise} \end{cases}$$

So m_i' is the mixed strategy that represents the pure strategy s_i'.

Let now $m' := (m_1, \dots, m_{i-1}, m_i', m_{i+1}, \dots, m_n)$. We have

$$p_i(m) = \sum_{s \in S} m(s) \cdot p_i(s_i, s_{-i})$$

and

$$p_i(s_i', m_{-i}) = \sum_{s \in S} m'(s) \cdot p_i(s_i, s_{-i}).$$

Further, one can check that

$$\sum_{s \in S} m'(s) \cdot p_i(s_i, s_{-i}) = \sum_{s \in S} m(s) \cdot p_i(s_i', s_{-i}).$$

So the claim is a direct consequence of the equivalence between items (i) and (ii) of the Characterization Lemma 22. □

We now generalize the above inequality to an arbitrary probability distribution over S. This yields the following equilibrium notion. We call a probability distribution π over S a ***correlated equilibrium*** if for all $i \in \{1, \dots, n\}$ and all $s_i' \in S_i$

$$\sum_{s \in S} \pi(s) \cdot p_i(s_i, s_{-i}) \geq \sum_{s \in S} \pi(s) \cdot p_i(s_i', s_{-i}).$$

By the above Note every Nash equilibrium in mixed strategies is a correlated equilibrium. To see that the converse is not true consider the Battle of the Sexes game:

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

It is easy to check that the following probability distribution forms a correlated equilibrium in this game:

$$
\begin{array}{c|c|c|}
 & F & B \\
\hline
F & \frac{1}{2} & 0 \\
\hline
B & 0 & \frac{1}{2} \\
\hline
\end{array}
$$

Intuitively, this equilibrium corresponds to a situation when an external observes flips a fair coin and gives each player a recommendation which strategy to choose.

Exercise 13 Check the above claim. □

9.2 Variations on the definition of strategic games

The notion of a strategic game is quantitative in the sense that it refers through payoffs to real numbers. A natural question to ask is: do the payoff values matter? The answer depends on which concepts we want to study. We mention here three qualitative variants of the definition of a strategic game in which the payoffs are replaced by preferences. By a ***preference relation*** on a set A we mean here a linear ordering on A.

In [14] a strategic game is defined as a sequence

$$(S_1, \ldots, S_n, \succeq_1, \ldots, \succeq_n),$$

where each $\succeq_i$ is player's i *preference relation* defined on the set $S_1 \times \ldots \times S_n$ of joint strategies.

In [1] another modification of strategic games is considered, called a ***strategic game with parametrized preferences***. In this approach each player i has a non-empty set of strategies S_i and a *preference relation* $\succeq_{s_{-i}}$ on S_i *parametrized* by a joint strategy s_{-i} of his opponents. In [1] only strict preferences were considered and so defined finite games with parametrized preferences were compared with the concept of ***CP-nets*** (Conditional Preference nets), a formalism used for representing conditional and qualitative preferences, see, e.g., [4].

Next, in [17] ***conversion/preference games*** are introduced. Such a game for n players consists of a set S of *situations* and for each player i a *preference relation* $\succeq_i$ on S and a *conversion relation* $\rightarrow_i$ on S. The definition is very general and no conditions are placed on the preference and conversion relations. These games are used to formalize gene regulation networks and some aspects of security.

Another generalization of strategic games, called ***graphical games***, introduced in [7]. These games stress the locality in taking decision. In a graphical game the payoff of each player depends only on the strategies of its neighbours in a given in advance graph structure over the set of players. Formally, such a game for n players with the corresponding strategy sets $S_1, \ldots, S_n$ is defined by assuming a neighbour function N that given a player i yields its set of neighbours $N(i)$. The payoff for player i is then a function p_i from $\times_{j \in N(i) \cup \{i\}} S_j$ to $\mathbb{R}$.

In all mentioned variants it is straightforward to define the notion of a Nash equilibrium. For example, in the conversion/preferences games it is defined as a situation s such that for all players i, if $s \rightarrow_i s'$, then $s' \not\succ_i s$. However, other introduced notions can be defined only for some variants. In particular, Pareto efficiency cannot be defined for strategic games with parametrized preferences since it requires a comparison of two arbitrary joint strategies. In turn, the notions of dominance cannot be defined for the conversion/preferences games, since they require the concept of a strategy for a player.

Various results concerning finite strategic games, for instance the IESDS Theorem 2, carry over directly to the the strategic games as defined in [14] or in [1]. On the other hand, in the variants of strategic games that rely on the notion of a preference we cannot consider mixed strategies, since the outcomes of playing different strategies by a player cannot be aggregated.